IN3070/4070 – Logic – Autumn 2020

Lecture 8: First-order Resolution

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Today's Plan

- Reminder: Clausal Form Translations
- Reminder: Propositional Resolution
- Reminder: Unification
- First-Order Resolution
- Soundness and Completeness
- Compactness
- Summary

Outline

Translation into Clausal Form – Example

Example: $\forall x\, \exists y\, p(x, y) \rightarrow \exists y\, \forall x\, p(x, y)$

Try to prove this formula based on refutation in CNF

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- Eliminate implication $\to$: $\neg(\neg \forall x \, \exists y \, p(x, y) \lor \exists w \, \forall z \, p(z, w))$

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- Push negation inwards: $\forall x\, \exists y\, p(x, y) \wedge \forall w\, \exists z\, \neg p(z, w)$

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- Skolemize, i.e., replace $\exists$: $\forall x\, p(x, f(x)) \wedge \forall w\, \neg p(g(w), w)$

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- Write in clausal form : $\{\{p(x, f(x))\}, \{\neg p(g(w), w)\}\}$

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- Reminder: Propositional Resolution
- Reminder: Unification
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Reminder: The Resolution Rule

The resolution calculus is a refutation procedure.

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Let C_1, C_2 be clauses with $L \in C_1$ and $\overline{L} \in C_2$. The resolvent C' of C_1 and C_2 is $(C_1 \setminus \{L\}) \cup (C_2 \setminus \{\overline{L}\})$. C_1 and C_2 are the parents of C'.

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- the resolution rule maintains satisfiability: If $\mathcal{I} \models C_1$ and $\mathcal{I} \models C_2$ then $\mathcal{I} \models C'$
- if a set of clauses S is satisfiable and $C_1, C_2 \in S$, then $S \cup \{C'\}$ is satisfiable.

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Observations:

- if $\{a, b, \neg c\}$ and $\{b, c, \neg e\} \equiv (a \vee b \vee \neg c) \wedge (b \vee c \vee \neg e)$ are satisfiable, then $(a \vee b)$ is satisfiable (if c is true) or $(b \vee \neg e)$ is satisfiable (if c is false); hence $(a \vee b \vee \neg e)$ is satisfiable

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- the empty clauses $\{\,\}$ is unsatisfiable
- goal: derive empty clause $\{\,\}$

The Resolution Calculus

- a set of clauses is unsatisfiable iff the empty clause can be derived
- a clause C is true iff at least one of its literals is true; if there is no literal in C, then C is false and every set of clauses (in CNF) that contains C is false, i.e.unsatisfiable

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Given a set of clauses S.

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3. *if $C' = \{\}$, then output "unsatisfiable";*
   if all possible resolvents have been considered, then output "satisfiable"; otherwise continue with 1.

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- To terms are unifiable if they have a unifier.

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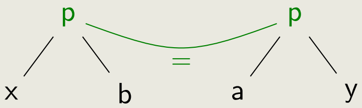

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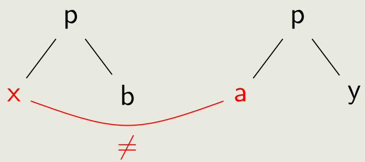

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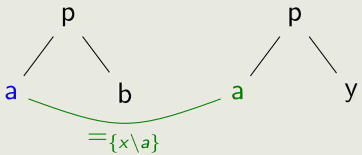

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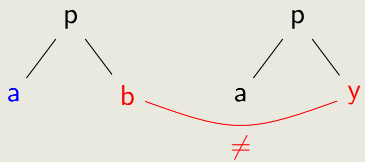

- The root symbols are the same.
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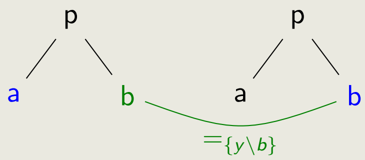

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- The right children are different, but can be unified with $\{y \backslash b\}$.

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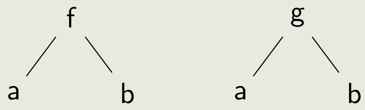

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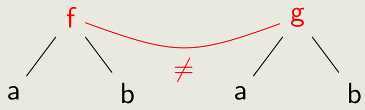

- The root symbols are different, and can *not* be unified!

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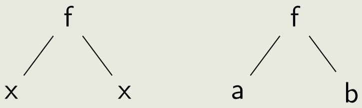

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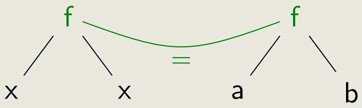

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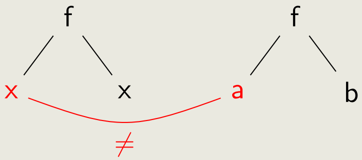

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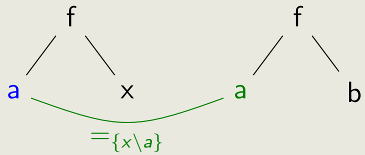

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Are $f(x, x)$ and $f(a, b)$ unifiable?

- The root symbols are equal.
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- We must apply $\{x\backslash a\}$ to x in both branches.

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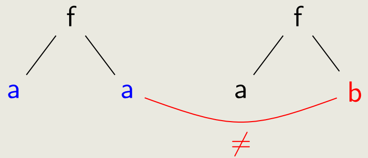

- The root symbols are equal.
- The left children are different, but can be unified with $\{x \backslash a\}$.
- We must apply $\{x \backslash a\}$ to x in both branches.
- The right children are now different, and can *not* be unified!

Are x and $f(x)$ unifiable?

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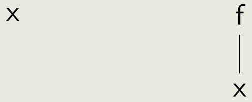

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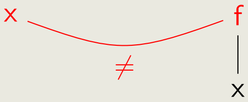

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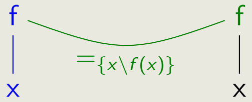

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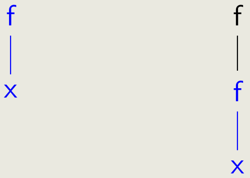

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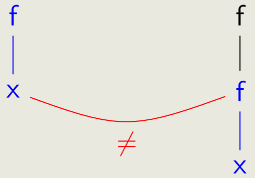

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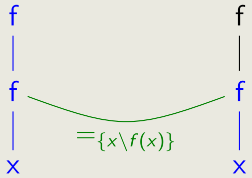

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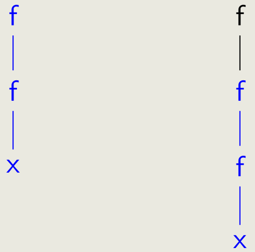

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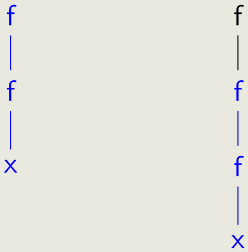

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- This continues indefinitely

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- Such a unifier is known as a most general unifier.

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Proposition 3.1.

If σ_1 and σ_2 are most general unifiers for two terms s and t, then there is a variable renaming η such that $\eta\sigma_1 = \sigma_2$.

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- We leave out the proof.

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- If t_1 and t_2 are not unifiable, the algorithm returns *"not unifiable"*.

Outline

- Reminder: Clausal Form Translations
- Reminder: Propositional Resolution
- Reminder: Unification
- First-Order Resolution
- Soundness and Completeness
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- Summary

The First-Order Resolution Calculus

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- a resolution proof for a set of clauses S is a derivation of S in the resolution calculus; the substitution σ is local for every rule application; variables in every clause C can be renamed

First-Order Resolution Calculus – Example

1. $\neg p(x), q(x), r(x, f(x))$
2. $\neg p(x), q(x), r'(f(x))$
3. $p'(a)$
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First-Order Resolution Calculus – Example

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2. $\neg p(x), q(x), r'(f(x))$
3. $p'(a)$
4. $p(a)$
5. $\neg r(a, y), p'(y)$
6. $\neg p'(x), \neg q(x)$
7. $\neg p'(x), \neg r'(x)$
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16. $\square$ — from 14 and 15

The Necessity of Factoring

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\begin{array}{llcl}
(1): & p(u) & \vee & p(f(u)) \\
(2): & \neg p(v) & \vee & p(f(w)) \\
(3): & \neg p(x) & \vee & \neg p(f(x))
\end{array}
$$

A possible resolution derivation:

The Necessity of Factoring

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$(4): \quad p(u) \vee p(f(w))$ by resolving (1) and (2), with $v = f(u)$

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\begin{array}{lll}
(4): & p(u) \lor p(f(w)) & \text{by resolving (1) and (2), with } v = f(u) \\
(5): & p(f(w)) & \text{by factoring (4), with } u = f(w)
\end{array}
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Outline

- Reminder: Clausal Form Translations
- Reminder: Propositional Resolution
- Reminder: Unification
- First-Order Resolution
- Soundness and Completeness
- Compactness
- Summary

Soundness and Completeness

Theorem 5.1 (Soundness and Completeness of Resolution).

The resolution calculus is sound and complete, i.e.

- *if A is provable in the resolution calculus, then A is valid* (*if* $\vdash A$ *then* $\models A$)
- *if A is valid, then A is provable in the resolution calculus* (*if* $\models A$ *then* $\vdash A$)

Proof.

See Ben-Ari, section 10.5, [Robinson 1965]. □

Soundness

Definition 5.1.

An interpretation $\mathcal{I}$ satisfies a clause C if for every variable assignment α, there is a $L \in C$ with $v_{\mathcal{I}}(\alpha, L) = T$.

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If a set of clauses S is satisfiable, then the result of adding the resolvent of two clauses $C_1, C_2 \in A$ to S is also satisfiable.

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Then $\mathcal{I} \models \sigma((C_1 \setminus \{L_1\}) \cup (C_2 \setminus \{\overline{L_2}\}))$ like for propositional logic. □

Completeness

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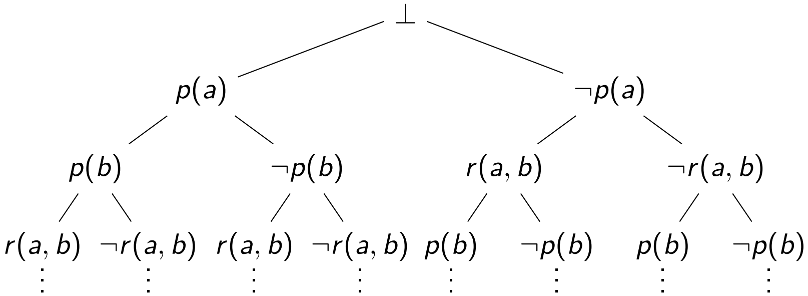

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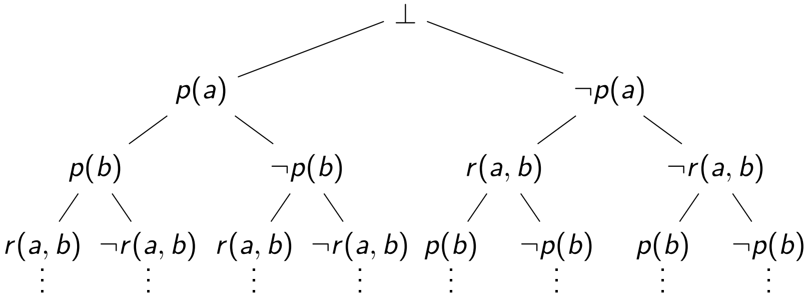

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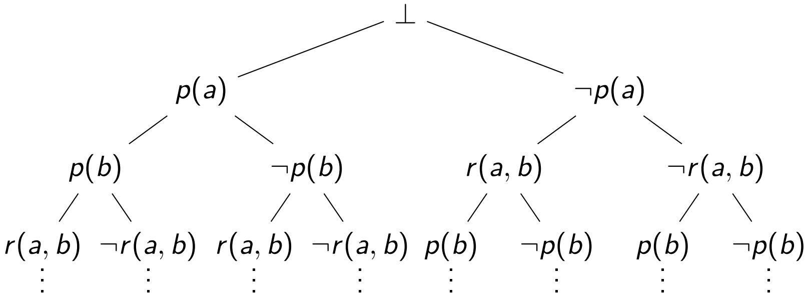

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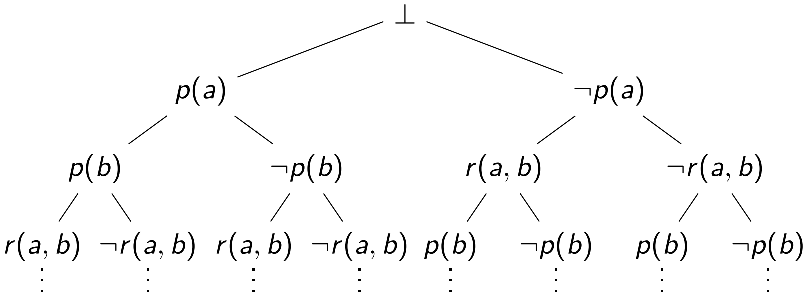

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- Lifting: There are corresponding steps using m.g.u.s

Outline

Compactness

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Nowhere in the definition of resolution do we need that S is finite.

- If S is unsatisfiable

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Theorem 6.1 (Compactness).

Every unsatisfiable set of clauses S has a finite unsatisfiable subset

Compactness: Example

$$\exists x\, \neg p(x),\; p(a),\; p(fa),\; p(ffa),\; p(fffa),\; \ldots$$

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- By compactness: The whole set is also satisfiable

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Theorem 6.2.

Satisfiability over the natural numbers is not compact.

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Reasoning about numbers involves more than just first-order logic.

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- resolution calculus is one of the most popular proof search calculi for (classical) first-order logic

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- resolution calculus is one of the most popular proof search calculi for (classical) first-order logic
- consists of:
  - one axiom
  - resolution rule
  - factorization rule

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- Next Week: logic programming and Prolog