IN3070/4070 – Logic – Autumn 2020

Lecture 10: DPLL

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Today's Plan

- Motivation
- Simplification Rules
- Atomic Cut
- The DPLL Algorithm
- Other Tricks

Outline

Smullyan's categories: $\alpha/\beta/\gamma/\delta$

- Many similar cases in proofs and implementations
- Categorise rules, rule applications, and formulae

α-rules

Propositional, one branch, e.g.

$$\frac{\Gamma, A, B \;\Rightarrow\; \Delta}{\Gamma, A \land B \;\Rightarrow\; \Delta}\ \land\text{-left}$$

β-rules

Propositional, splitting, e.g.

$$\frac{\Gamma \;\Rightarrow\; A, \Delta \qquad \Gamma, B \;\Rightarrow\; \Delta}{\Gamma, A \to B \;\Rightarrow\; \Delta}$$

γ-rules

Apply for all terms t, e.g.

$$\frac{\Gamma, A[x\backslash t], \forall x\, A \;\Rightarrow\; \Delta}{\Gamma, \forall x\, A \;\Rightarrow\; \Delta}\ \forall\text{-left}$$

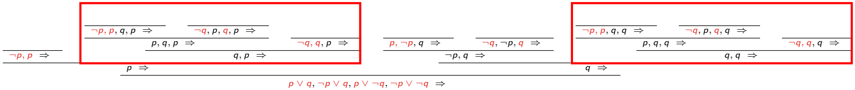
δ-rules

Introduce new constant c, e.g.

$$\frac{\Gamma, A[x\backslash c], \;\Rightarrow\; \Delta}{\Gamma, \exists x\, A \;\Rightarrow\; \Delta}\ \exists\text{-left}$$

A Naïve SAT Solver at Work

Problems



- Costly repetitions of identical proof trees
- 9 Branches
- Can often be avoided by using β rules in the "right" order
- But finding the best order is harder (!) than finding a proof
- Better: avoid using β (i.e. splitting) rules

Outline

Simplification Rules: Motivation

- Given two formulas p and $q \wedge (r \rightarrow s \wedge p)$
- And an interpretation $\mathcal{I}$ with $\mathcal{I} \models p$
- $v_{\mathcal{I}}(q \wedge (r \rightarrow s \wedge p)) = v_{\mathcal{I}}(q \wedge (r \rightarrow s \wedge \mathit{true})) = v_{\mathcal{I}}(q \wedge (r \rightarrow s))$

- An interpretation $\mathcal{I}$ falsifies a sequent

$$p,\ q \wedge (r \rightarrow s \wedge p), \Gamma \vdash \Delta$$

if and only if $\mathcal{I}$ falsifies the sequent

$$p,\ q \wedge (r \rightarrow s), \Gamma \vdash \Delta$$

Simplification

Definition 2.1 (Simplification).

Given two formulas A and B,where B does not have $\neg$ as top-symbol, the simplification of A with B, written $A[B]$, is the result of

- *Replacing all occurrences of B in A by true, and*
- *Simplifying subformulae as long as possible using the rewritings*

$$\begin{array}{cc}
A \lor true \mapsto true & A \lor false \mapsto A \\
A \land true \mapsto A & A \land false \mapsto false \\
A \to true \mapsto true & A \to false \mapsto \neg A \\
true \to A \mapsto A & false \to A \mapsto true \\
\neg true \mapsto false & \neg false \mapsto true
\end{array}$$

The simplification of A with $\neg B$, written $A[\neg B]$, is the result of

- *Replacing all occurrences of B in A by false, and*
- *Applying the same rewritings.*

Simplification Examples

- To compute $(q \land (r \rightarrow s \land p))[p]$
  - Do the replacement: $q \land (r \rightarrow s \land \mathit{true})$
  - Then simplify $q \land (r \rightarrow s \land \mathit{true}) \mapsto q \land (r \rightarrow s)$
- So $(q \land (r \rightarrow s \land p))[p] \ = \ q \land (r \rightarrow s)$

- To compute $(q \land (r \rightarrow s \land p))[\neg p]$
  - Do the replacement: $q \land (r \rightarrow s \land \mathit{false})$
  - Then simplify $q \land (r \rightarrow s \land \mathit{false}) \mapsto q \land (r \rightarrow \mathit{false}) \mapsto q \land \neg r$
- So $(q \land (r \rightarrow s \land p))[\neg p] \ = \ q \land \neg r$

Main property of Simplification

Lemma 2.1.

Given a formula A that contains a subformula B, and let $B \equiv B'$. Then A is logically equivalent to the result of replacing B by B' in A.

Proof.

Easily shown by structural induction over A. □

Theorem 2.1.

Given formulas A and B and an interpretation $\mathcal{I}$ with $\mathcal{I} \models B$.
Then $\mathcal{I} \models A$ if and only if $\mathcal{I} \models A[B]$

Proof.

For the first step (replacing B by *true* or *false*), the proof is by structural induction on A. For the simplification steps, each formula is logicaly equivalent to the next, due to the preceding lemma. □

Simplification Rules

We add the following four “simplification rules” to LK:

$$\frac{B, A[B], \Gamma \Rightarrow \Delta}{B, A, \Gamma \Rightarrow \Delta} \qquad \frac{A[\neg B], \Gamma \Rightarrow B, \Delta}{A, \Gamma \Rightarrow B, \Delta}$$

$$\frac{B, \Gamma \Rightarrow A[B], \Delta}{B, \Gamma \Rightarrow A, \Delta} \qquad \frac{\Gamma \Rightarrow B, A[\neg B], \Delta}{\Gamma \Rightarrow B, A, \Delta}$$

Example: (one-sided) LK with Simplification Rules

Left branch (top to bottom):

$$\overline{\qquad\qquad}$$
$$p,\ q,\ true, (\neg p \vee \neg q)[p]\ \neg q \quad \Rightarrow$$
$$\overline{\qquad\qquad}$$
$$p,\ q,\ (p \vee \neg q)[p]\ true, \neg p \vee \neg q \quad \Rightarrow$$
$$\overline{\qquad\qquad}$$
$$p, (\neg p \vee q)[p]\ q, p \vee \neg q, \neg p \vee \neg q \quad \Rightarrow$$
$$\overline{\qquad\qquad}$$
$$p, \neg p \vee q, p \vee \neg q, \neg p \vee \neg q \quad \Rightarrow$$

Right branch (top to bottom):

$$\overline{\qquad\qquad}$$
$$q,\ true, p, \neg p \quad \Rightarrow$$
$$\overline{\qquad\qquad}$$
$$q,\ true,\ p, \neg p \vee \neg q \quad \Rightarrow$$
$$\overline{\qquad\qquad}$$
$$q, true, p \vee \neg q, \neg p \vee \neg q \quad \Rightarrow$$
$$\overline{\qquad\qquad}$$
$$q, \neg p \vee q, p \vee \neg q, \neg p \vee \neg q \quad \Rightarrow$$

Root:

$$\overline{\qquad\qquad\qquad\qquad}$$
$$p \vee q, \neg p \vee q, p \vee \neg q, \neg p \vee \neg q \quad \Rightarrow$$

- Strategy: Apply simplification as much as possible, before β rules
- In this case: from 9 branches down to 2.

Simplification for Clauses

- Simplify a clause C with a literal L
- Case 1: C contains L, $C = A_1 \vee \cdots \vee A_k \vee L$
  - Then $C[L] = \mathit{true}$
  - In refutation (left of sequent, resolution), *true* is useless and can be removed
  - Removing C because $L \in C$ is called unit subsumption
- Case 2: C contains $\bar{L}$, $C = A_1 \vee \cdots \vee A_k \vee \bar{L}$
  - Then $C[L] = A_1 \vee \cdots \vee A_k$
  - $C[L]$ is the resolvent of C and L!
  - Replacing C by $A_1 \vee \cdots \vee A_k$ is called unit resolution
  - Note that C is subsumed by $A_1 \vee \cdots \vee A_k$
- Unit subsumption and unit resolution together: unit propagation
- Given a literal L, every clause can be either removed completely, or shortened by removing $\bar{L}$, unit propagation can be used to remove L from every other clause containing L or $\bar{L}$.

Outline

Atomic Cut: Motivation

- For a sequent with n different β-formulas,
- each of them has to be expanded on every branch...
- ...which gives 2^n branches...
- even though there might be only $k < n$ propositional variables,
- and therefore only 2^k different interpretations!

- E.g. in the motivating example:

  9 branches for 4 interpretations for 2 prop. variables.

- Idea: max. 1 split per propositional variable

The Cut Rule

- The cut rule for LK:

$$\frac{\Gamma \Rightarrow A, \Delta \qquad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

- The rule is sound (exercise) but not needed for completeness.
- It is a bit like proving a lemma A and then using it.
- Using cut can make proofs non-elementarily shorter (in first order logic)
- I.e. size $O(k)$ with cut but $O(\underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{k})$ without.
- Not useful for automated proof search, because A has to be guessed.
- The essence of human theorem proving: introducing the right lemmas!

Atomic Cut

- The atomic cut rule is just the cut rule

$$\frac{\Gamma \;\Rightarrow\; A, \Delta \qquad A, \Gamma \;\Rightarrow\; \Delta}{\Gamma \;\Rightarrow\; \Delta}$$

- Where A is resricted to be an atomic formula.
- No nonelementary speedup :-(
- But we don't need more atomic cuts than we have prop. variables :-)
- We can replace β rules in LK by atomic cut...
- ...if we add the simplification rules to deal with β formulas.

Atomic Cut + Unit Propagation

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q,\ p,\ \neg p \ \Rightarrow}}{q,\ p,\ \neg p \lor \neg q \ \Rightarrow}}{q, p \lor \neg q, \neg p \lor \neg q \ \Rightarrow}}{q, \neg p \lor q, p \lor \neg q, \neg p \lor \neg q \ \Rightarrow}}{q, p \lor q, \neg p \lor q, p \lor \neg q, \neg p \lor \neg q \ \Rightarrow}
\qquad
\dfrac{\dfrac{\dfrac{}{\neg q,\ p,\ \neg p, p \lor \neg q, \neg p \lor \neg q \ \Rightarrow}}{\neg q,\ p,\ \neg p \lor q, p \lor \neg q, \neg p \lor \neg q \ \Rightarrow}}{\neg q, p \lor q, \neg p \lor q, p \lor \neg q, \neg p \lor \neg q \ \Rightarrow}
}{p \lor q, \neg p \lor q, p \lor \neg q, \neg p \lor \neg q \ \Rightarrow}
$$

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The DPLL Algorithm

- DPLL stands for Davis-Putnam-Logemann-Loveland
- Introduced in 1962 by Martin Davis, George Logemann and Donald W. Loveland
- A refinement of an earlier algorithm, invented by Martin Davis and Hilary Putnam in (1960)
- Made propositional theorem proving ("SAT solving") practically viable
- After almost 60 years, still the basis of most efficient SAT solvers

- DPLL works on a set of propositional clauses
- DPLL Consists of
  - Atomic Cut (with a heuristic for choosing the atom)
  - Unit Propagation
  - Pure Literal Elimination (exercise!)

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Lemma Generation

- Remember the exercise sheet 2?

$$\frac{\Gamma \ \Rightarrow \ A, \Delta \qquad \Gamma, A \ \Rightarrow \ B, \Delta}{\Gamma \ \Rightarrow \ A \wedge B, \Delta} \wedge\text{-lg}$$

- Closing the left branch, we “learnt the Lemma A”
- With single-sided sequents:

$$\frac{A, \Gamma \ \Rightarrow \qquad \neg A, B, \Gamma \ \Rightarrow}{A \vee B, \Gamma \ \Rightarrow} \vee\text{-lg}$$

- We refuted A, so now we may assume $\neg A$.
- Whenever we close a branch, we learn that a certain combination of literals $L_1, \ldots, L_k$ leads to a contradiction
- We can add a clause $\overline{L_1} \vee \cdots \vee \overline{L_k}$ to caputre this.
- “Clause Learning”

Pruning

- Pruning ≡ Backjumping ≡ Intelligent Backtracking ≡ Non-chronological Backtracking
- Consider the following derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{needed B and G}}{p, q, \neg p, \neg r \;\Rightarrow} \qquad \dfrac{\text{needed G}}{p, q, r, \neg r \;\Rightarrow}}{p, q, \neg p \lor r, \neg r \;\Rightarrow}\,\text{G} \qquad p, s, \ldots \;\Rightarrow}{p, q \lor s, \neg p \lor r, \neg r \;\Rightarrow}\,\text{R} \qquad q, \ldots \;\Rightarrow}{p \lor q, q \lor s, \neg p \lor r, \neg r \;\Rightarrow}\,\text{B}$$

- No formulae introduced by R needed to close the two left branches
- Could have closed the branch without applying R
- Pruning: after closing the left two branches, continue with

$$\dfrac{\dfrac{\dfrac{\text{needed B and G}}{p, \neg p, \neg r \;\Rightarrow} \qquad \dfrac{\text{needed G}}{p, r, \neg r \;\Rightarrow}}{p, \neg p \lor r, \neg r \;\Rightarrow}\,\text{G} \qquad q, \ldots \;\Rightarrow}{p \lor q, q \lor s, \neg p \lor r, \neg r \;\Rightarrow}\,\text{B}$$

Conflict-driven clause learning (CDCL)

- The modern take on DPLL
- See e.g. the successful MiniSat implementation `http://minisat.se/`
- A combination of
  - Atomic cut
  - Unit propagation
  - Clause learning
  - Pruning

Stålmarck's Method

- Devised by Gunnar Stålmarck, applied for patent 1989
- The Dilemma Rule:

$$\dfrac{\dfrac{\Gamma_1 \cap \Gamma_2 \;\Rightarrow}{\begin{array}{ccc} \Gamma_1 \;\Rightarrow & & \Gamma_2 \;\Rightarrow \\ \vdots & & \vdots \\ A, \Gamma \;\Rightarrow & & \neg A, \Gamma \;\Rightarrow \end{array}}}{\Gamma \;\Rightarrow}$$

- After unit propagation, join branches generated by cut
- Stålmarck's discovery: often enough to consider max two branches
- Not always. Why?
- In general: nesting of Dilemma Rule.
- Still: deep nesting rarely needed.

Summary

- Efficient theorem provers *combine* formulas instead of just decomposing
  - The resolution rule is an example
  - The simplification rules are another
- For propositional logic, unit propagation is very effective
- Atomic cut and unit propagation are the main ingredients of DPLL
- DPLL has been refined to CDCL
- CDCL incorporates clause learning and pruning