IN3070/4070 – Logic – Autumn 2020

Lecture 10: DPLL

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Today's Plan

- Motivation
- Simplification Rules
- Atomic Cut
- The DPLL Algorithm
- Other Tricks

Outline

- Motivation
- Simplification Rules
- Atomic Cut
- The DPLL Algorithm
- Other Tricks

Smullyan's categories: $\alpha/\beta/\gamma/\delta$

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α-rules

Propositional, one branch, e.g.

$$\frac{\Gamma, A, B \;\Rightarrow\; \Delta}{\Gamma, A \wedge B \;\Rightarrow\; \Delta}\ \wedge\text{-left}$$

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Propositional, splitting, e.g.

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Propositional, splitting, e.g.

$$\frac{\Gamma \;\Rightarrow\; A, \Delta \qquad \Gamma, B \;\Rightarrow\; \Delta}{\Gamma, A \rightarrow B \;\Rightarrow\; \Delta}$$

γ-rules

Apply for all terms t, e.g.

$$\frac{\Gamma, A[x\backslash t], \forall x\, A \;\Rightarrow\; \Delta}{\Gamma, \forall x\, A \;\Rightarrow\; \Delta}\ \forall\text{-left}$$

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Propositional, splitting, e.g.

$$\frac{\Gamma \;\Rightarrow\; A, \Delta \qquad \Gamma, B \;\Rightarrow\; \Delta}{\Gamma, A \to B \;\Rightarrow\; \Delta}$$

γ-rules

Apply for all terms t, e.g.

$$\frac{\Gamma, A[x\backslash t], \forall x\, A \;\Rightarrow\; \Delta}{\Gamma, \forall x\, A \;\Rightarrow\; \Delta} \forall\text{-left}$$

δ-rules

Introduce new constant c, e.g.

$$\frac{\Gamma, A[x\backslash c], \;\Rightarrow\; \Delta}{\Gamma, \exists x\, A \;\Rightarrow\; \Delta} \exists\text{-left}$$

A Naïve SAT Solver at Work

$$p \lor q, \neg p \lor q, p \lor \neg q, \neg p \lor \neg q \quad \Rightarrow$$

A Naïve SAT Solver at Work

$$p \vee q, \neg p \vee q, p \vee \neg q, \neg p \vee \neg q \;\Rightarrow$$

A Naïve SAT Solver at Work

$$p \vee q, \neg p \vee q, p \vee \neg q, \neg p \vee \neg q \quad \Rightarrow$$

A Naïve SAT Solver at Work

$$\dfrac{p \;\Rightarrow}{p \vee q, \neg p \vee q, p \vee \neg q, \neg p \vee \neg q \;\Rightarrow}$$

A Naïve SAT Solver at Work

$$\dfrac{\dfrac{\neg p, p \ \Rightarrow \qquad\qquad q, p \ \Rightarrow}{p \ \Rightarrow}}{p \vee q, \neg p \vee q, p \vee \neg q, \neg p \vee \neg q \ \Rightarrow}$$

A Naïve SAT Solver at Work

$$\dfrac{\dfrac{\overline{\neg p, p \;\Rightarrow}}{p \;\Rightarrow} \qquad q, p \;\Rightarrow}{p \lor q, \neg p \lor q, p \lor \neg q, \neg p \lor \neg q \;\Rightarrow}$$

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$$\dfrac{\dfrac{\overline{\neg p, p \;\Rightarrow}\qquad q, p \;\Rightarrow}{p \;\Rightarrow}}{p \vee q, \neg p \vee q, \color{red}{p \vee \neg q}, \neg p \vee \neg q \;\Rightarrow}$$

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$$\dfrac{\dfrac{\dfrac{}{\neg p, p \;\Rightarrow} \qquad \dfrac{p, q, p \;\Rightarrow \qquad \neg q, q, p \;\Rightarrow}{q, p \;\Rightarrow}}{p \;\Rightarrow}}{p \lor q, \neg p \lor q, p \lor \neg q, \neg p \lor \neg q \;\Rightarrow}$$

A Naïve SAT Solver at Work

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  \qquad
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    \dfrac{}{p, q, p \;\Rightarrow}
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    \dfrac{}{\neg q, q, p \;\Rightarrow}
  }{q, p \;\Rightarrow}
}{
  \dfrac{p \;\Rightarrow}{p \vee q, \neg p \vee q, p \vee \neg q, {\color{red}\neg p \vee \neg q} \;\Rightarrow}
}
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\dfrac{\dfrac{\neg p, p, q, p \;\Rightarrow \qquad \neg q, p, q, p \;\Rightarrow}{p, q, p \;\Rightarrow} \qquad \neg q, q, p \;\Rightarrow}{q, p \;\Rightarrow}}{p \;\Rightarrow}}{p \lor q, \neg p \lor q, p \lor \neg q, \neg p \lor \neg q \;\Rightarrow}
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$$\neg q, q, p \ \Rightarrow$$

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$\neg q, q, p \;\Rightarrow$ $\qquad$ $p, \neg p, q \;\Rightarrow$ $\qquad$ $\neg q, \neg p, q \;\Rightarrow$

$\neg p, q \;\Rightarrow$

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      \dfrac{}{\neg q, \neg p, q \;\Rightarrow}
    }{q \;\Rightarrow}
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      \qquad
      \neg q, q, q \;\Rightarrow
    }{q, q \;\Rightarrow}
  }{q \;\Rightarrow}
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Problems

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    \dfrac{}{\neg p, p \;\Rightarrow}
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        \dfrac{}{\neg p, p, q, p \;\Rightarrow}
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        \dfrac{}{\neg q, p, q, p \;\Rightarrow}
      }{p, q, p \;\Rightarrow}
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      \dfrac{}{\neg q, q, p \;\Rightarrow}
    }{q, p \;\Rightarrow}
  }{p \;\Rightarrow}
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  \dfrac{
    \dfrac{
      \dfrac{}{p, \neg p, q \;\Rightarrow}
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    }{\neg p, q \;\Rightarrow}
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        \dfrac{}{\neg p, p, q, q \;\Rightarrow}
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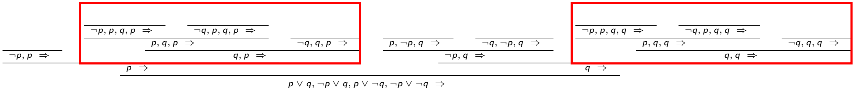
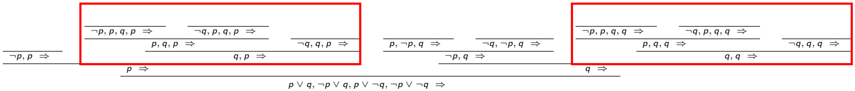
    }{q, q \;\Rightarrow}
  }{q \;\Rightarrow}
}{p \vee q, \neg p \vee q, p \vee \neg q, \neg p \vee \neg q \;\Rightarrow}
$$

Problems



- Costly repetitions of identical proof trees

Problems



- Costly repetitions of identical proof trees
- 9 Branches

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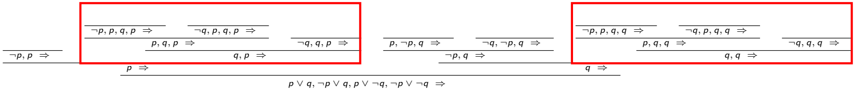

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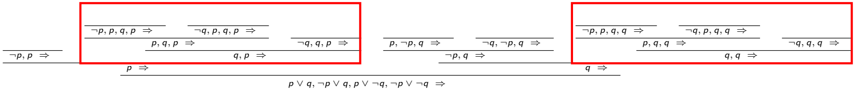

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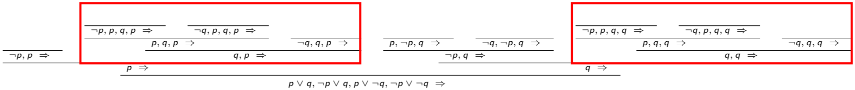

- Costly repetitions of identical proof trees
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- Can often be avoided by using β rules in the “right” order
- But finding the best order is harder (!) than finding a proof
- Better: avoid using β (i.e. splitting) rules

Outline

Simplification Rules: Motivation

- Given two formulas p and $q \wedge (r \rightarrow s \wedge p)$

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- *Simplifying subformulae as long as possible using the rewritings*

$$
\begin{array}{cc}
A \lor \mathit{true} \mapsto \mathit{true} & A \lor \mathit{false} \mapsto A \\
A \land \mathit{true} \mapsto A & A \land \mathit{false} \mapsto \mathit{false} \\
A \rightarrow \mathit{true} \mapsto \mathit{true} & A \rightarrow \mathit{false} \mapsto \neg A \\
\mathit{true} \rightarrow A \mapsto A & \mathit{false} \rightarrow A \mapsto \mathit{true} \\
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$$\begin{array}{cc}
A \lor true \mapsto true & A \lor false \mapsto A \\
A \land true \mapsto A & A \land false \mapsto false \\
A \to true \mapsto true & A \to false \mapsto \neg A \\
true \to A \mapsto A & false \to A \mapsto true \\
\neg true \mapsto false & \neg false \mapsto true
\end{array}$$

The simplification of A with $\neg B$, written $A[\neg B]$, is the result of

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$$

The simplification of A with $\neg B$, written $A[\neg B]$, is the result of

- *Replacing all occurrences of B in A by false, and*
- *Applying the same rewritings.*

Simplification Examples

- To compute $(q \wedge (r \rightarrow s \wedge p))[p]$

Simplification Examples

- To compute $(q \land (r \rightarrow s \land p))[p]$
  - Do the replacement: $q \land (r \rightarrow s \land \mathit{true})$

Simplification Examples

- To compute $(q \wedge (r \to s \wedge p))[p]$
  - Do the replacement: $q \wedge (r \to s \wedge \mathit{true})$
  - Then simplify $q \wedge (r \to s \wedge \mathit{true}) \mapsto$

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- So $(q \land (r \to s \land p))[p] \;=\; q \land (r \to s)$

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- To compute $(q \land (r \rightarrow s \land p))[\neg p]$
  - Do the replacement: $q \land (r \rightarrow s \land \mathit{false})$

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- To compute $(q \wedge (r \to s \wedge p))[\neg p]$
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- So $(q \land (r \to s \land p))[p] \;=\; q \land (r \to s)$

- To compute $(q \land (r \to s \land p))[\neg p]$
  - Do the replacement: $q \land (r \to s \land \mathit{false})$
  - Then simplify $q \land (r \to s \land \mathit{false}) \mapsto q \land (r \to \mathit{false}) \mapsto$

Simplification Examples

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  - Then simplify $q \land (r \to s \land \mathit{true}) \mapsto q \land (r \to s)$
- So $(q \land (r \to s \land p))[p] \; = \; q \land (r \to s)$
- To compute $(q \land (r \to s \land p))[\neg p]$
  - Do the replacement: $q \land (r \to s \land \mathit{false})$
  - Then simplify $q \land (r \to s \land \mathit{false}) \mapsto q \land (r \to \mathit{false}) \mapsto q \land \neg r$

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- To compute $(q \wedge (r \rightarrow s \wedge p))[p]$
  - Do the replacement: $q \wedge (r \rightarrow s \wedge \mathit{true})$
  - Then simplify $q \wedge (r \rightarrow s \wedge \mathit{true}) \mapsto q \wedge (r \rightarrow s)$
- So $(q \wedge (r \rightarrow s \wedge p))[p] \;=\; q \wedge (r \rightarrow s)$

- To compute $(q \wedge (r \rightarrow s \wedge p))[\neg p]$
  - Do the replacement: $q \wedge (r \rightarrow s \wedge \mathit{false})$
  - Then simplify $q \wedge (r \rightarrow s \wedge \mathit{false}) \mapsto q \wedge (r \rightarrow \mathit{false}) \mapsto q \wedge \neg r$
- So $(q \wedge (r \rightarrow s \wedge p))[\neg p] \;=\; q \wedge \neg r$

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Lemma 2.1.

Given a formula A that contains a subformula B, and let $B \equiv B'$. Then A is logically equivalent to the result of replacing B by B' in A.

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For the first step (replacing B by *true* or *false*), the proof is by structural induction on A. For the simplification steps, each formula is logicaly equivalent to the next, due to the preceding lemma. □

Simplification Rules

We add the following four “simplification rules” to LK:

$$\frac{B,\ A[B],\ \Gamma\ \Rightarrow\ \Delta}{B,\ A,\ \Gamma\ \Rightarrow\ \Delta} \qquad \frac{A[\neg B],\ \Gamma\ \Rightarrow\ B,\ \Delta}{A,\ \Gamma\ \Rightarrow\ B,\ \Delta}$$

$$\frac{B,\ \Gamma\ \Rightarrow\ A[B],\ \Delta}{B,\ \Gamma\ \Rightarrow\ A,\ \Delta} \qquad \frac{\Gamma\ \Rightarrow\ B,\ A[\neg B],\ \Delta}{\Gamma\ \Rightarrow\ B,\ A,\ \Delta}$$

Example: (one-sided) LK with Simplification Rules

$$p \lor q, \neg p \lor q, p \lor \neg q, \neg p \lor \neg q \;\; \Rightarrow$$

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$$\dfrac{\dfrac{p, (\neg p \lor q)[p], p \lor \neg q, \neg p \lor \neg q \;\Rightarrow}{p, \neg p \lor q, p \lor \neg q, \neg p \lor \neg q \;\Rightarrow} \qquad q, \neg p \lor q, p \lor \neg q, \neg p \lor \neg q \;\Rightarrow}{p \lor q, \neg p \lor q, p \lor \neg q, \neg p \lor \neg q \;\Rightarrow}$$

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Example: (one-sided) LK with Simplification Rules

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$$

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$$
\dfrac{\dfrac{\dfrac{\dfrac{p,\ q,\ true, (\neg p \vee \neg q)[p] \ \Rightarrow}{p,\ q,\qquad true, \neg p \vee \neg q \ \Rightarrow}}{p,\qquad q, p \vee \neg q, \neg p \vee \neg q \ \Rightarrow}}{p, \neg p \vee q, p \vee \neg q, \neg p \vee \neg q \ \Rightarrow} \qquad q, \neg p \vee q, p \vee \neg q, \neg p \vee \neg q \ \Rightarrow}{p \vee q, \neg p \vee q, p \vee \neg q, \neg p \vee \neg q \ \Rightarrow}
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Example: (one-sided) LK with Simplification Rules

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{p,\ {\color{red} q},\ true,\ {\color{red}\neg q}\ \Rightarrow}{p,\ q,\ true, \neg p \lor \neg q\ \Rightarrow}
    }{p,\ q, p \lor \neg q, \neg p \lor \neg q\ \Rightarrow}
  }{p, \neg p \lor q, p \lor \neg q, \neg p \lor \neg q\ \Rightarrow}
  \qquad
  q, \neg p \lor q, p \lor \neg q, \neg p \lor \neg q\ \Rightarrow
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$$

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$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{p,\ q,\ \mathit{true},\ \neg q \ \Rightarrow}
    }{p,\ q,\ \mathit{true}, \neg p \vee \neg q \ \Rightarrow}
  }{p,\ q, p \vee \neg q, \neg p \vee \neg q \ \Rightarrow}
}{p, \neg p \vee q, p \vee \neg q, \neg p \vee \neg q \ \Rightarrow}
\qquad
q, \neg p \vee q, p \vee \neg q, \neg p \vee \neg q \ \Rightarrow
$$

$$
\overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}
$$

$$
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$$

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$$
\dfrac{
  \dfrac{
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      \dfrac{
        \dfrac{}{p,\ q,\ \mathit{true},\ \neg q \ \Rightarrow}
      }{p,\ q,\ \mathit{true}, \neg p \vee \neg q \ \Rightarrow}
    }{p,\ q, p \vee \neg q, \neg p \vee \neg q \ \Rightarrow}
  }{p, \neg p \vee q, p \vee \neg q, \neg p \vee \neg q \ \Rightarrow}
  \qquad
  {\color{red} q, \neg p \vee q}, p \vee \neg q, \neg p \vee \neg q \ \Rightarrow
}{p \vee q, \neg p \vee q, p \vee \neg q, \neg p \vee \neg q \ \Rightarrow}
$$

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\dfrac{
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- In this case: from 9 branches down to 2.

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- Simplify a clause C with a literal L

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  - $C[L]$ is the resolvent of C and L!

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  - Replacing C by $A_1 \vee \cdots \vee A_k$ is called unit resolution
  - Note that C is subsumed by $A_1 \vee \cdots \vee A_k$
- Unit subsumption and unit resolution together: unit propagation
- Given a literal L, every clause can be either removed completely, or shortened by removing $\bar{L}$, unit propagation can be used to remove L from every other clause containing L or $\bar{L}$.

Outline

Atomic Cut: Motivation

- For a sequent with *n* different β-formulas,

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- each of them has to be expanded on every branch. . .

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Atomic Cut: Motivation

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- each of them has to be expanded on every branch...
- ...which gives 2^n branches...
- even though there might be only $k < n$ propositional variables,
- and therefore only 2^k different interpretations!

Atomic Cut: Motivation

- For a sequent with n different β-formulas,
- each of them has to be expanded on every branch…
- …which gives 2^n branches…
- even though there might be only $k < n$ propositional variables,
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- Idea: max. 1 split per propositional variable

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- The cut rule for LK:

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- The essence of human theorem proving: introducing the right lemmas!

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- But we don't need more atomic cuts than we have prop. variables :-)
- We can replace β rules in LK by atomic cut...
- ...if we add the simplification rules to deal with β formulas.

Atomic Cut + Unit Propagation

$$p \vee q, \neg p \vee q, p \vee \neg q, \neg p \vee \neg q \;\Rightarrow$$

Atomic Cut + Unit Propagation

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  - Pure Literal Elimination (exercise!)

Outline

- Motivation
- Simplification Rules
- Atomic Cut
- The DPLL Algorithm
- Other Tricks

Lemma Generation

- Remember the exercise sheet 2?

$$\frac{\Gamma \;\Rightarrow\; A,\, \Delta \qquad \Gamma,\, A \;\Rightarrow\; B,\, \Delta}{\Gamma \;\Rightarrow\; A \land B,\, \Delta}\;\land\text{-lg}$$

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- "Clause Learning"

Pruning

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Pruning

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Pruning

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Pruning

- Pruning ≡ Backjumping ≡ Intelligent Backtracking ≡ Non-chronological Backtracking
- Consider the following derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{needed B and G}}{p, q, \neg p, \neg r \;\Rightarrow} \qquad \dfrac{\text{needed G}}{p, q, r, \neg r \;\Rightarrow}}{p, q, \neg p \lor r, \neg r \;\Rightarrow}\,\text{G} \qquad p, s, \ldots \;\Rightarrow}{p, q \lor s, \neg p \lor r, \neg r \;\Rightarrow}\,\text{R} \qquad q, \ldots \;\Rightarrow}{p \lor q, q \lor s, \neg p \lor r, \neg r \;\Rightarrow}\,\text{B}$$

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- Could have closed the branch without applying R
- Pruning: after closing the left two branches, continue with

$$\dfrac{\dfrac{\dfrac{\text{needed B and G}}{p, \neg p, \neg r \;\Rightarrow} \qquad \dfrac{\text{needed G}}{p, r, \neg r \;\Rightarrow}}{p, \neg p \lor r, \neg r \;\Rightarrow}\;\text{G} \qquad q, \ldots \;\Rightarrow}{p \lor q, q \lor s, \neg p \lor r, \neg r \;\Rightarrow}\;\text{B}$$

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