IN3070/4070 – Logic – Autumn 2020

Lecture 11: Modal Logics

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Today's Plan

- Motivation
- Modal Logic
- Satisfiability & Validity
- Different Modal Logics
- A Modal Sequent Calculus

Outline

Motivation: Valid Argumentation

- Remember: "The subject in which nobody knows what one is talking about, nor whether what one is saying is true" [Bertrand Russell]
  - Logic is about the "shape" of valid argumentation
- "If it rains, then Peter knows that it rains. But Peter considers it possible that it doesn't rain. **Therefore** it doesn't rain."
  - Reasoning about knowledge
- "The light is green now. Whenever the light is green, it eventually turns red. Whenever the light is red, it eventually turns green. **Therefore**, at any point in time, the light will eventually turn from red to green."
  - Reasoning about time
- "A medical doctor has a doctoral degree in medicine. A doctor of law ahas a doctoral degree in law. It is not possible to have a doctoral degree in more than one subject. **Therefore** nobody is both a medical doctor and a doctor of law."
  - Reasoning about concepts and relationships

Motivation: Decidability

- Propositional Validity is undecidable (NP-hard)
- First-order validity is undecidable
- Question: are there more expressive decidable logics than propositional logic?
- Yes. E.g. the Bernays-Schönfinkel fragment.
- Also the two-variable fragment
- And quite a few more

- Turns out: many of the reasoning patterns from the previous slide can be turned into decidable logics.

Outline

Modal Logic: Syntax

Definition 2.1.

The formulae of propositional modal logic, are inductively defined as follows:

1. *Every atom $A \in \mathcal{P}$ is a formula.*
2. *If A and B are formulae, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$ and $(A \rightarrow B)$ are formulae.*
3. *If A is a formula, then $\Box A$ and $\Diamond A$ are formulae.*

E.g.

- $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ is a formula.
- $\Diamond(\Box p \vee q) \wedge \neg(\Diamond p \vee \Box\Diamond q)$ is a formula.
- $\Diamond\Box$ is not a formula.

Modal Logic: Intuition

- $\Box$ and $\Diamond$ are always dual:

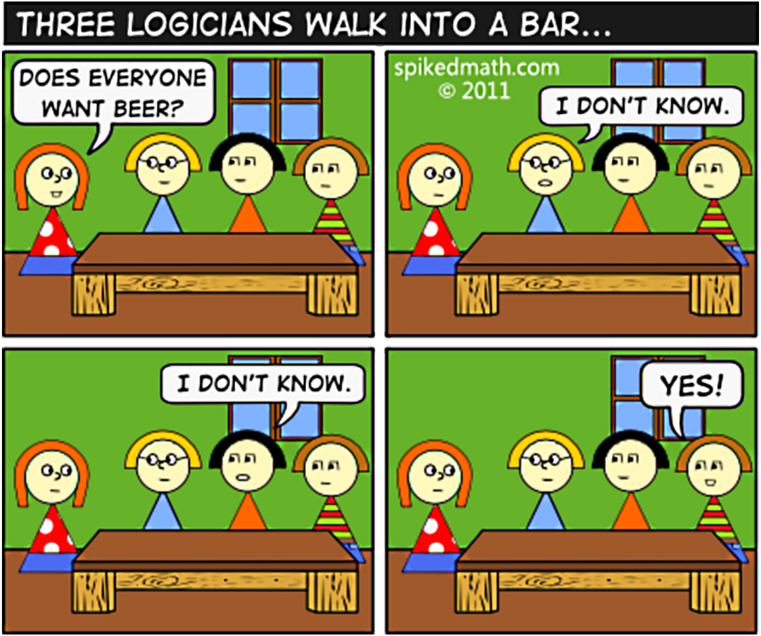
$$\Diamond A \equiv \neg\Box\neg A$$
$$\neg\Diamond A \equiv \Box\neg A$$

- Knowledge logic ("epistemic")
  - $\Box A$: the actor *knows* that A is the case
  - $\Diamond A$: the actor *considers it possible* that A is the case (based on their state of knowledge)
- Temporal logic:
  - $\Box A$: A is true at all future points in time
  - $\Diamond A$: A is true at some future point in time
- Deontic logic:
  - $\Box A$: A is obligatory (under law, morals, etc.)
  - $\Diamond A$: A is permitted

If it rains. . .

- "If it rains, then Peter knows that it rains. But Peter considers it possible that it doesn't rain. **Therefore** it doesn't rain."
- Let p stand for "It rains"
- If it rains, Peter knows that it rains: $p \rightarrow \Box p$
- Peter considers it possible that it doesn't rain: $\Diamond \neg p$
- It doesn't rain: $\neg p$
- For epistemic logic, we want:

$$\{p \rightarrow \Box p, \Diamond \neg p\} \models \neg p$$


THREE LOGICIANS WALK INTO A BAR...
DOES EVERYONE WANT BEER?
spikedmath.com © 2011
I DON'T KNOW.
I DON'T KNOW.
YES!

Three Logicians...

- Use three operators $\Box_1$, $\Box_2$, $\Box_3$ for the knowledge of three actors.
- Logican 1 knows whether she wants beer:
  $p_1 \rightarrow \Box_1 p_1$ and $\neg p_1 \rightarrow \Box_1 \neg p_1$
- Logican 1 does not know whether everybody wants beer:
  $\neg\Box_1(p_1 \wedge p_2 \wedge p_3)$ and $\neg\Box_1\neg(p_1 \wedge p_2 \wedge p_3)$
- From this, we can infer p_1
- ...

Red and green...

- "The light is green now. Whenever the light is green, it eventually turns red. Whenever the light is red, it eventually turns green. **Therefore**, at any point in time, the light will eventually turn from red to green."
- Let p stand for "The light is green" and q for "The light is red."
- The light is green now: p
- Whenever the light is green, it eventually turns red: $\Box(p \to \Diamond q)$
- Whenever the light is red, it eventually turns green: $\Box(q \to \Diamond p)$
- at any point in time, the light will eventually turn from red to green: $\Box\Diamond(q \land \Diamond p)$
- for temporal logic, we want:

$$\{p, \Box(p \to \Diamond q), \Box(q \to \Diamond p)\} \models \Box\Diamond(q \land \Diamond p)$$

Kripke Semantics

Definition 2.2 (Kripke Frame).

A (Kripke) frame $F = (W, R)$ *consists of*

- *a non-empty set of worlds* W
- *a binary accessibility relation* $R \subseteq W \times W$ *on the worlds in* W

Definition 2.3 (Reminder: Propositional Interpretation).

A propositional interpretation is a function $\mathcal{I} : \mathcal{P} \to \{T, F\}$ *that assigns a truth value to every propositional variable.*

Definition 2.4 (Modal Interpretation).

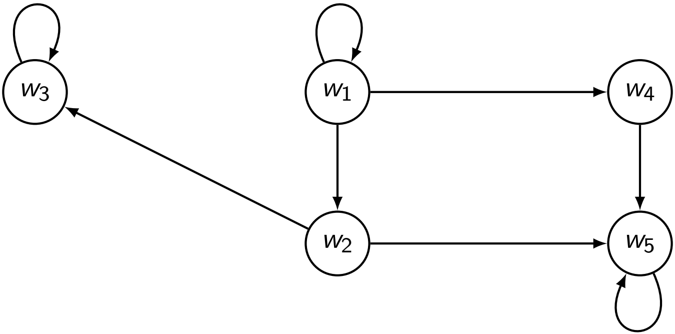
A modal interpretation (Kripke model) $\mathcal{I}_M := (F, \{\mathcal{I}(w)\}_{w \in W})$ *consists of*

- *a Kripke frame* $F = (W, R)$
- *one propositional interpretation* $\mathcal{I}(w)$ *for each* $w \in W$

Kripke Frame – Example

Example: $F = (W, R)$ with $W = \{w_1, w_2, w_3, w_4, w_5\}$ and

$$R = \{(w_1, w_1), (w_3, w_3), (w_5, w_5), (w_1, w_2), (w_2, w_3), (w_1, w_4), (w_4, w_5), (w_2, w_5)\}$$

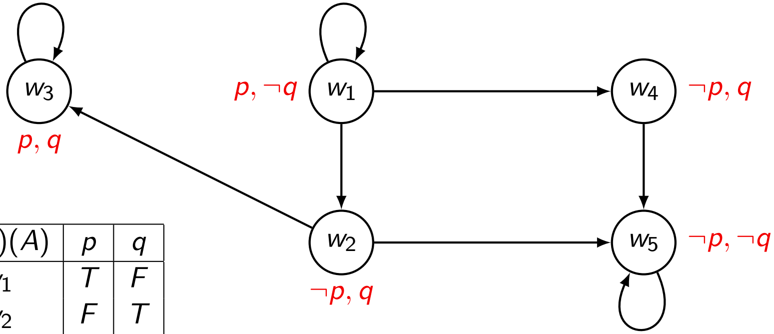


Worlds, Accessibility

- The meaning of "worlds" and "accessibility" depends on the modality
- Knowledge Logic:
  - Worlds: possible states of the world, e.g. it rains in w_0 but not in w_1
  - $w_1 R w_2$: in world w_1, it is consistent with the actor's knowledge that we are in w_2
- Temporal logic:
  - Worlds: states at different points in time
  - $w_1 R w_2$: w_1 is earlier than w_2.

Kripke Model – Example

Example: $F = (W, R)$ as before



$\mathcal{I}(w)(A)$	p	q
w_1	T	F
w_2	F	T
w_3	T	T
w_4	F	T
w_5	F	F

Modal Truth Value

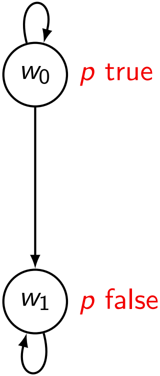
Definition 2.5 (Modal Truth Value).

Let $\mathcal{I}_M = ((W, R), \{\mathcal{I}(w)\}_{w \in W})$ be a Kripke model. The modal truth value $v_{\mathcal{I}_M}(w, A)$ of a formula A in the world w in the model $\mathcal{I}_M$ is T (true) if "w forces A under $\mathcal{I}_M$", denoted $w \Vdash A$, and F (false), otherwise.

The forcing relation $w \Vdash A$ is defined inductively as follows:

- $w \Vdash p$ for $p \in \mathcal{P}$ iff $\mathcal{I}(w)(p) = T$
- $w \Vdash \neg A$ iff not $w \Vdash A$
- $w \Vdash A \wedge B$ iff $w \Vdash A$ and $w \Vdash B$
- $w \Vdash A \vee B$ iff $w \Vdash A$ or $w \Vdash B$
- $w \Vdash A \rightarrow B$ iff not $w \Vdash A$ or $w \Vdash B$
- $w \Vdash \Diamond A$ iff $v \Vdash A$ for some $v \in W$ with $(w, v) \in R$
- $w \Vdash \Box A$ iff $v \Vdash A$ for all $v \in W$ with $(w, v) \in R$

Modal truth value – Examples



- $F_1 \equiv p \vee \Box\neg p$

  $w_0 \Vdash \Box\neg p$ iff
  $v \Vdash \neg p$ for all $v \in W$ with $(w_0, v) \in R$

  but $(w_0, w_1) \in R$ and $w_1 \Vdash p$ holds

  hence, neither $w_0 \Vdash p$ nor $w_0 \Vdash \Box\neg p$

  $\leadsto$ F_1 is not *true* in w_0

- $F_2 \equiv p \vee \Diamond\neg p$

  $w_0 \Vdash \Diamond\neg p$ iff
  $v \Vdash \neg p$ for some $v \in W$ with $(w_0, v) \in R$

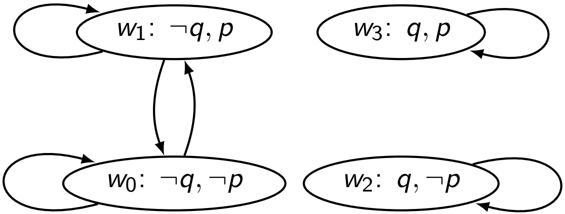
  $\leadsto$ F_2 is *true* in w_0 (and w_1)

Modal Truth Value – Intuition

- Knowledge Logic:
  - $w \Vdash \Box A$: A holds in all worlds accessible from w
  - A holds in all worlds consistent with what we know (have observed) in w
  - I.e. we know A
  - $w \Vdash \Diamond A$: we consider A to be possible
- Temporal Logic:
  - $w \Vdash \Box A$: A holds in all worlds accessible from w
  - A holds at all future points in time
  - $w \Vdash \Diamond A$: A holds at some future point in time

Checking the weather

- p: it rains
- q: we have looked out of the window to check the weather



- If we look out of the window, we know whether it rains:

$$\mathcal{I} \models q \to \big((p \to \Box p) \land (\neg p \to \Box \neg p)\big)$$

Outline

- Motivation
- Modal Logic
- **Satisfiability & Validity**
- Different Modal Logics
- A Modal Sequent Calculus

Satisfiability and Validity

In modal logic a formula F is valid, if it evaluates to *true* in all worlds of all Kripke models.

Definition 3.1 (Satisfiable,Model,Unsatisfiable,Valid,Invalid).

Let A be a formula. and $\mathcal{I}_M$ be a Kripke model.

- *$\mathcal{I}_M$ is a* model in modal logic *for A, denoted $\mathcal{I}_M \models A$, iff $v_{\mathcal{I}_M}(w, A) = T$ for all $w \in W$.*
- *A is satisfiable in modal logic iff $\mathcal{I}_M \models A$ for some Kripke model $\mathcal{I}_M$.*
- *A is unsatisfiable in modal logic iff A is not satisfiable.*
- *A is valid, denoted $\models A$, iff $\mathcal{I}_M \models A$ for all modal interpretations $\mathcal{I}_M$.*
- *A is invalid/falsifiable in modal logic iff A is not valid.*

Example: K

Proposition 3.1.

$$K := \Box(p \to q) \to (\Box p \to \Box q)$$

is a valid formula of modal logic.

Proof.

Let $\mathcal{I}_M = ((W, R), \{\mathcal{I}(w)\}_{w \in W})$ and $w \in W$. To show that $w \Vdash K$, we assume that $w \Vdash \Box(p \to q)$ (†) and $w \Vdash \Box p$ (‡) and have to show $w \Vdash \Box q$.
To show $w \Vdash \Box q$, we show that $v \Vdash q$ for an arbitrary $v \in W$ with wRv. Due to (†), $v \Vdash p \to q$ (*). And due to (‡), $v \Vdash p$ (**). It follows that $v \Vdash q$. □

Logical Consequence: Local and Global

There are two ways of defining logical consequence in modal logic.

Definition 3.2 (Global Logical Consequence).

Let U be a set of formulae and A be a formula. A is a global consequence of U, denote $U \models^G A$, iff for every modal interpretation $\mathcal{I}_M$ the following holds: if $w \Vdash F$ for all $F \in U$ and all worlds $w \in W$ then $w \Vdash A$ for all worlds $w \in W$.

Definition 3.3 (Local Logical Consequence).

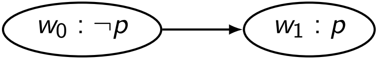
Let U be a set of formulae and A be a formula. A is a local consequence of U, denote $U \models^L A$, iff for every modal interpretation $\mathcal{I}_M$ the following holds: for all world $w \in W$ if $w \Vdash F$ for all $F \in U$ then $w \Vdash A$.

- the deduction theorem does not hold for the global consequence
- if $U \models^L A$ then $U \models^G A$; the opposite direction does not hold

Outline

I Know it's True

- Intuitions about knowledge: to know something means it's true
- That's not the case for *belief* for instance.
- Not for *obligation under law* either.
- For knowledge, $\Box A \to A$ should be valid for all A.
- Not the case in every Kripke model, e.g. not in w_0 here:



- it turns out:

A frame (W, R) is reflexive
iff
$\mathcal{I}_M \models \Box p \to p$ for all Kripke models $\mathcal{I}_M = ((W, R), \{\mathcal{I}(w)\}_{w \in W})$

- Reminder: (W, R) is reflexive if wRw for all $w \in W$.

Reflexivity and $\Box p \to p$

$\Longrightarrow$ Let (W, R) be reflexive, $\mathcal{I}_M = ((W, R), \{\mathcal{I}(w)\}_{w \in W})$, and $w \in W$. If $w \Vdash \Box p$, then since wRw, also , $w \Vdash p$, and so $w \Vdash \Box p \to p$ for all $w \in W$.

$\Longleftarrow$ Let (W, R) be a frame such that $w \Vdash \Box p \to p$ for all $\mathcal{I}_M = ((W, R), \{\mathcal{I}(w)\}_{w \in W})$, and $w \in W$. Assume (W, R) is *not* reflexive. So there is a $u \in W$ with $(u, u) \notin R$. Consider the Kripke model with these propositional interpretations:

$$\mathcal{I}(w)(p) = \begin{cases} \text{T} & \text{if } uRw \\ \text{F} & \text{otherwise} \end{cases}$$

So p is true in all worlds reachable from u. $u \Vdash \Box p$. So since $u \Vdash \Box p \to p$, also $u \Vdash p$, which means that uRu. Contradiction!

Modal Logics K and T

- The modal logic we have defined so far is called K.
- Modal logic T has the same syntax and truth values.
- But for satisfiability, validity, etc. we consider only reflexive frames.
- So $\Box p \to p$ is *not* valid in K.
- But $\Box p \to p$ *is* valid in T.

More Modal Logics

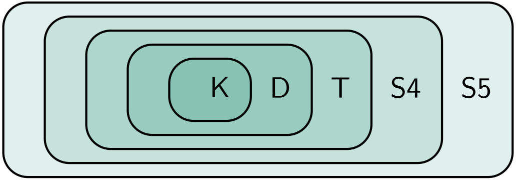
modal logic	condition on R	axioms
K	(no condition)	–
K4	transitive	$\Box A \to \Box\Box A$
D	serial	$\Box A \to \Diamond A$
D4	serial, transitive	$\Box A \to \Diamond A$, $\Box A \to \Box\Box A$
T	reflexive	$\Box A \to A$
S4	reflexive, transitive	$\Box A \to A$, $\Box A \to \Box\Box A$
S5	equivalence (reflexive, euclidean)	$\Box A \to A$, $\Diamond A \to \Box\Diamond A$

(A relation $R \subseteq W \times W$ is *serial* iff for all $w_1 \in W$ there is some $w_2 \in W$ with $(w_1, w_2) \in R$; a relation $R \subseteq W \times W$ is *euclidean* iff for all $w_1, w_2, w_3 \in W$ the following holds: if $(w_1, w_2) \in R$ and $(w_1, w_3) \in R$ then $(w_2, w_3) \in R$.)

Lemma: if a relation is reflexive and euclidean, it is also symmetric and transitive, i.e. an equivalence relation.

Validity Relation for Different Modal Logics

The validity relationship between different modal logics and domain conditions is depicted in the following figure:



E.g. a formula that is valid in D is also valid in T, S4, etc.

Outline

A Sequent Calculus for K

- Let $\mathcal{L}$ be a set of labels
- A labeled formula is a pair $u : A$ where $u \in \mathcal{L}$ and A a formula.
- An accessibility formula has the shape uRv for two labels $u, v \in \mathcal{L}$.
- Use labeled sequents, containing labeled formulae and accessibility formulae
- Propositional rules for labeled formulas: just copy labels, e.g.

$$\frac{\Gamma \Rightarrow u : A, \Delta \qquad \Gamma \Rightarrow u : B, \Delta}{\Gamma \Rightarrow u : A \wedge B, \Delta} \wedge\text{-right}$$

- The $\Diamond$-left rule creates a new label:

$$\frac{\Gamma, uRv, v : A \Rightarrow \Delta}{\Gamma, u : \Diamond A \Rightarrow \Delta} \Diamond\text{-left} \qquad \text{for a fresh label } v$$

- The $\Box$-left rule transfers info to other labels:

$$\frac{\Gamma, uRv, v : A, u : \Box A \Rightarrow \Delta}{\Gamma, uRv, u : \Box A \Rightarrow \Delta} \Box\text{-left}$$

- Axioms require same labels: $u : A, \Gamma \Rightarrow u : A, \Gamma$

Rules for the Succedent

- The $\Box$-right rule creates a new label:

$$\frac{\Gamma, uRv \;\Rightarrow\; v : A,\ \Delta}{\Gamma \;\Rightarrow\; u : \Box A, \Delta}\ \Box\text{-right} \qquad \text{for a fresh label } v$$

- The $\Diamond$-right rule transfers info to other labels:

$$\frac{\Gamma, uRv \;\Rightarrow\; v : A,\ u : \Diamond A,\ \Delta}{\Gamma, uRv \;\Rightarrow\; u : \Diamond A,\ \Delta}\ \Diamond\text{-right}$$

Proof Example

$$\ldots, 2:p, 2:q, 1R2 \Rightarrow 2:p, \ldots \qquad \ldots, 2:p, 2:q, 1R2 \Rightarrow 2:q, \ldots$$

$$\begin{array}{ll}
\ldots, 2:p, 2:q, 1R2 \Rightarrow 2:p\wedge q, 1:\Diamond(p\wedge q) & \wedge\text{-left} \\
1:\Box p, 2:p, 2:q, 1R2 \Rightarrow 1:\Diamond(p\wedge q) & \Diamond\text{-right} \\
1:\Box p, 2:q, 1R2 \Rightarrow 1:\Diamond(p\wedge q) & \Box\text{-left} \\
1:\Box p, 1:\Diamond q \Rightarrow 1:\Diamond(p\wedge q) & \Diamond\text{-left} \\
1:\Box p\wedge\Diamond q \Rightarrow 1:\Diamond(p\wedge q) & \wedge\text{-left} \\
\Rightarrow 1:(\Box p\wedge\Diamond q)\rightarrow\Diamond(p\wedge q) & \rightarrow\text{-right}
\end{array}$$

Other Modal Logics, Termination

- For other modal logics, add rules about the accessibility formulae.
- E.g. for transitive frames:

$$\frac{\Gamma, uRv, vRw, uRw \;\Rightarrow\; \Delta}{\Gamma, uRv, vRw \;\Rightarrow\; \Delta}\text{trans}$$

- The calculi are sound and complete for the repsective modal logics
  - Proofs somewhat like for "ground" first-order logic
- Termination is not guaranteed for all of them!
  - Nested $\Box$ and $\Diamond$ can lead to ∞ many labeled formulae
- A "blocking condition" is needed to enforce termination
  - More about that next week!