IN3070/4070 – Logic – Autumn 2020

Lecture 11: Modal Logics

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Today's Plan

- Motivation
- Modal Logic
- Satisfiability & Validity
- Different Modal Logics
- A Modal Sequent Calculus

Outline

Motivation: Valid Argumentation

- Remember: "The subject in which nobody knows what one is talking about, nor whether what one is saying is true" [Bertrand Russell]

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  - Reasoning about concepts and relationships

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- Turns out: many of the reasoning patterns from the previous slide can be turned into decidable logics.

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- $\Diamond\Box$ is not a formula.

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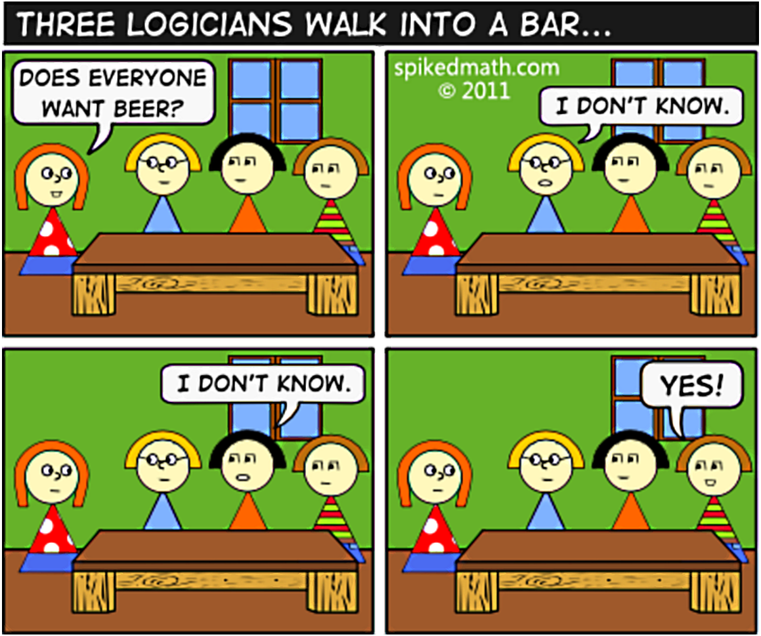
$$\{p \rightarrow \Box p, \Diamond\neg p\} \models \neg p$$


THREE LOGICIANS WALK INTO A BAR...
DOES EVERYONE WANT BEER?
spikedmath.com © 2011
I DON'T KNOW.
I DON'T KNOW.
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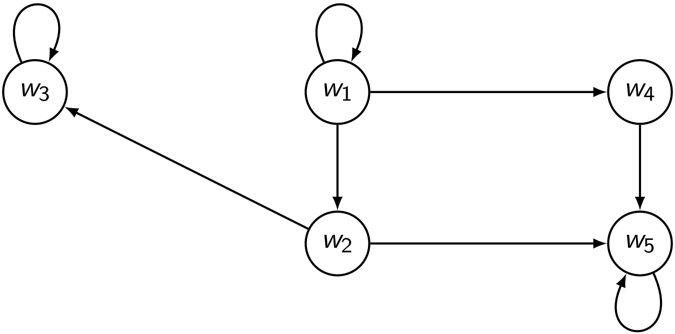
- *a Kripke frame* $F = (W, R)$
- *one propositional interpretation* $\mathcal{I}(w)$ *for each* $w \in W$

Kripke Frame – Example

Example: $F = (W, R)$ with $W = \{w_1, w_2, w_3, w_4, w_5\}$ and

$$R = \{(w_1, w_1), (w_3, w_3), (w_5, w_5), (w_1, w_2), (w_2, w_3), (w_1, w_4), (w_4, w_5), (w_2, w_5)\}$$



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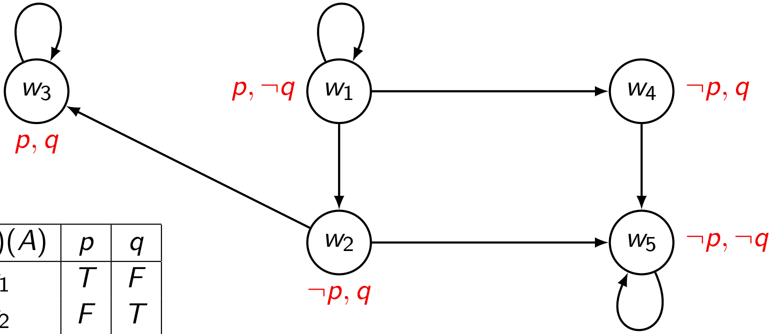
- Temporal logic:
  - Worlds: states at different points in time
  - $w_1 R w_2$: w_1 is earlier than w_2.

Kripke Model – Example

Example: $F = (W, R)$ as before



$\mathcal{I}(w)(A)$	p	q
w_1	T	F
w_2	F	T
w_3	T	T
w_4	F	T
w_5	F	F

Modal Truth Value

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- $w \Vdash A \wedge B$ iff $w \Vdash A$ and $w \Vdash B$
- $w \Vdash A \vee B$ iff $w \Vdash A$ or $w \Vdash B$
- $w \Vdash A \rightarrow B$ iff not $w \Vdash A$ or $w \Vdash B$

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Modal truth value – Examples

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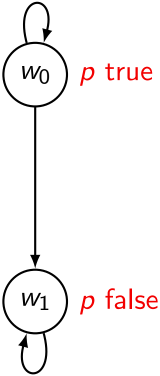

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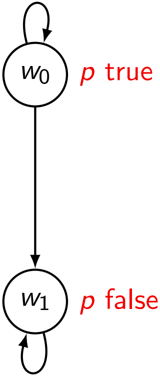

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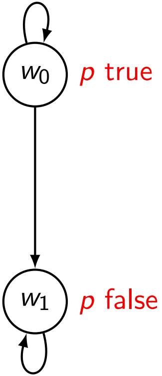

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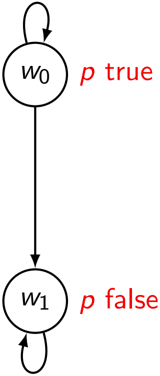

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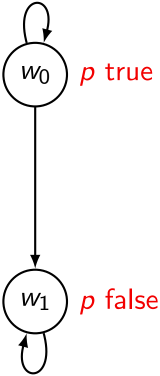

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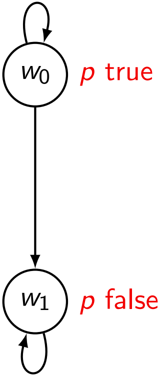

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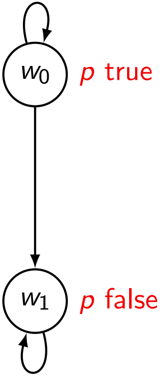

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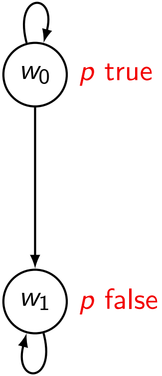

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  - A holds at all future points in time
  - $w \Vdash \Diamond A$: A holds at some future point in time

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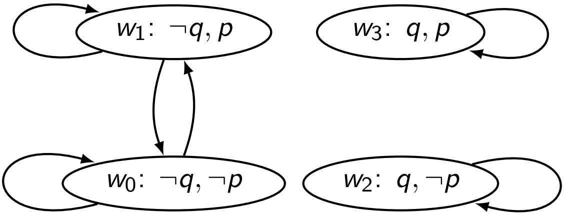

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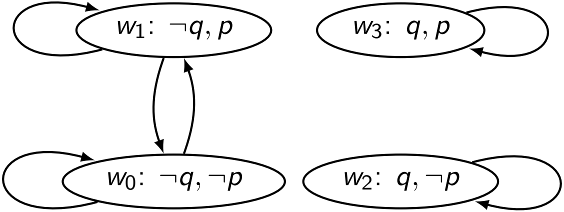

- If we look out of the window, we know whether it rains:

$$\mathcal{I} \models q \to \big((p \to \Box p) \land (\neg p \to \Box \neg p)\big)$$

Outline

- Motivation
- Modal Logic
- Satisfiability & Validity
- Different Modal Logics
- A Modal Sequent Calculus

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- *A is invalid/falsifiable in modal logic iff A is not valid.*

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Proposition 3.1.

$$K := \Box(p \to q) \to (\Box p \to \Box q)$$

is a valid formula of modal logic.

Proof.

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To show $w \Vdash \Box q$, we show that $v \Vdash q$ for an arbitrary $v \in W$ with wRv. Due to (†), $v \Vdash p \to q$ (*). And due to (‡), $v \Vdash p$ (**). It follows that $v \Vdash q$. □

Logical Consequence: Local and Global

There are two ways of defining logical consequence in modal logic.

Definition 3.2 (Global Logical Consequence).

Let U be a set of formulae and A be a formula. A is a global consequence of U, denote $U \models^G A$, iff for every modal interpretation $\mathcal{I}_M$ the following holds: if $w \Vdash F$ for all $F \in U$ and all worlds $w \in W$ then $w \Vdash A$ for all worlds $w \in W$.

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Definition 3.3 (Local Logical Consequence).

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- the deduction theorem does not hold for the global consequence
- if $U \models^L A$ then $U \models^G A$; the opposite direction does not hold

Outline

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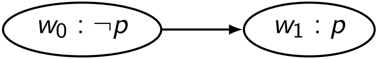
- Not the case in every Kripke model, e.g. not in w_0 here:



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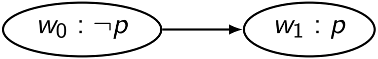

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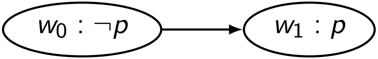

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- Reminder: (W, R) is reflexive if wRw for all $w \in W$.

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modal logic	condition on R	axioms
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K4	transitive	$\Box A \to \Box\Box A$
D	serial	$\Box A \to \Diamond A$
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Lemma: if a relation is reflexive and euclidean, it is also symmetric and transitive, i.e. an equivalence relation.

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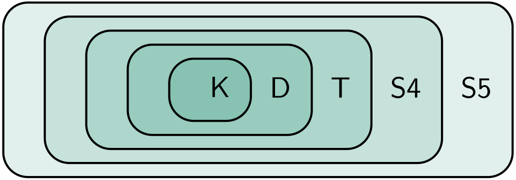

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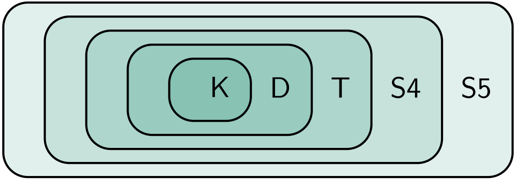

E.g. a formula that is valid in D is also valid in T, S4, etc.

Outline

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- Axioms require same labels: $u : A, \Gamma \Rightarrow u : A, \Gamma$

Rules for the Succedent

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Proof Example

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Proof Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\ldots, 2:p, 2:q, 1R2 \;\Rightarrow\; 2:p, \ldots} \quad \dfrac{}{\ldots, 2:p, {\color{red}2:q}, 1R2 \;\Rightarrow\; {\color{red}2:q}, \ldots}}{\ldots, 2:p, 2:q, 1R2 \;\Rightarrow\; 2:p \land q, 1:\Diamond(p \land q)}\ \land\text{-left}}{1:\Box p, 2:p, 2:q, 1R2 \;\Rightarrow\; 1:\Diamond(p \land q)}\ \Diamond\text{-right}}{1:\Box p, 2:q, 1R2 \;\Rightarrow\; 1:\Diamond(p \land q)}\ \Box\text{-left}}{1:\Box p, 1:\Diamond q \;\Rightarrow\; 1:\Diamond(p \land q)}\ \Diamond\text{-left}}{1:\Box p \land \Diamond q \;\Rightarrow\; 1:\Diamond(p \land q)}\ \land\text{-left}}{\Rightarrow\; 1:(\Box p \land \Diamond q) \to \Diamond(p \land q)}\ \to\text{-right}$$

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  - More about that next week!