IN3070/4070 – Logic – Autumn 2020

Lecture 12: Description Logics and Termination

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Today's Plan

- Motivation and Examples
- $\mathcal{ALC}$ Syntax and Semantics
- Calculus for $\mathcal{ALC}$ Terminological Reasoning
- Discussion

Outline

- Motivation and Examples
- $\mathcal{ALC}$ Syntax and Semantics
- Calculus for $\mathcal{ALC}$ Terminological Reasoning
- Discussion

Motivation

Undecidable	first-order reasoning
|	|
Non-Elementary decidable	...
|	|
...	...
|	|
ExpTime	...
|	|
PSpace	...
|	|
NP	propositional reasoning
|	|
P	...

What are Description (and Modal) Logics?

- Fragments of first-order with decidable reasoning
- Typically 'in-between' propositional and first-order
  - Complexity typically between PSpace and 2NExpTime
  - Some lightweight DLs have polynomial reasoning

Motivation

Many applications (e.g., in Knowledge Representation, the Semantic Web) do not require full power of first-order

What can we leave out?

- Key reasoning problems should become decidable
- Sufficient expressive power to model application domain

Description Logics are a family of first-order fragments that meet these requirements for many applications:

- Underlying formalisms of modern ontology languages
- Widely-used in information systems (bio-medical, oil and gas, etc.)
- Core component of the Semantic Web

Motivation

Consider an example from the bio-medical domain:

- A juvenile disease affects only children or teens
- Children and teens are not adults
- A person is a child, a teen, or an adult
- Juvenile arthritis is a kind of arthritis and a juvenile disease
- Every kind of arthritis damages some joint

The types of objects given by unary first-order predicates:

juvenile disease, child, teen, adult, ...

The types of relationships given by binary first-order predicates:

affects, damages, ...

Motivation

The vocabulary of a Description Logic is composed of

- Unary first-order predicates

  Arthritis, Child, ...
- Binary first-order predicates

  Affects, Damages, ...
- first-order constants

  JohnSmith, MaryJones, JRA, ...

We are already restricting the expressive power of first-order logic

- No function symbols
- No predicates of arity greater than 2

Motivation

Now, let's take a look at the first-order formulas for our example:

$$\forall x.(JuvDis(x) \to \forall y.(Affects(x,y) \to Child(y) \lor Teen(y)))$$
$$\forall x.(Child(x) \lor Teen(x) \to \neg Adult(x))$$
$$\forall x.(Person(x) \to Child(x) \lor Teen(x) \lor Adult(x))$$
$$\forall x.(JuvArthritis(x) \to Arthritis(x) \land JuvDis(x))$$
$$\forall x.(Arthritis(x) \to \exists y.(Damages(x,y) \land Joint(y))$$

We can find several regularities in these formulas:

- There is an outermost universal quantifier on a single variable x
- They can be split into two parts by the implication symbol

  Each part is a formula with one free variable
- Atomic formulas involving a binary predicate occur only quantified in a syntactically restricted way

Motivation

Consider as an example one of our formulas:

$$\forall x.(\mathit{Child}(x) \lor \mathit{Teen}(x) \to \neg \mathit{Adult}(x))$$

Let's look at all its sub-formulas at each side of the implication

$\mathit{Child}(x)$	Set of all children
$\mathit{Teen}(x)$	Set of all teens
$\mathit{Child}(x) \lor \mathit{Teen}(x)$	Set of all people that are either children or teens
$\mathit{Adult}(x)$	Set of all adults
$\neg \mathit{Adult}(x)$	Set of all objects that are not adult people

Important observations concerning formulas with one free variable:

- Some are atomic (e.g., $\mathit{Child}(x)$)

  do not contain other formulas as subformulas
- Others are complex (e.g., $\mathit{Child}(x) \lor \mathit{Teen}(x)$)
- Variables are redundant!

Basic Definitions

Idea: Define operators for constructing complex formulas with one free variable out of simple building blocks

Atomic concept: Represents an atomic formula with one free variable

$$\mathit{Child} \quad \rightsquigarrow \quad \mathit{Child}(x)$$

Complex concepts (part 1):

- Concept Union ($\sqcup$): applies to two concepts

$$\mathit{Child} \sqcup \mathit{Teen} \quad \rightsquigarrow \quad \mathit{Child}(x) \lor \mathit{Teen}(x)$$

- Concept Intersection ($\sqcap$): applies to two concepts

$$\mathit{Arthritis} \sqcap \mathit{JuvDis} \quad \rightsquigarrow \quad \mathit{Arthritis}(x) \land \mathit{JuvDis}(x)$$

- Concept Negation ($\neg$): applies to one concept

$$\neg \mathit{Adult} \quad \rightsquigarrow \quad \neg \mathit{Adult}(x)$$

Motivation

Consider examples with binary predicates:

$$\forall x.(\mathit{Arthritis}(x) \to \exists y.(\mathit{Damages}(x,y) \land \mathit{Joint}(y))$$
$$\forall x.(\mathit{JuvDis}(x) \to \forall y.(\mathit{Affects}(x,y) \to \mathit{Child}(y) \lor \mathit{Teen}(y)))$$

- We have a concept and a binary predicate (called role) mentioning concept's free variable
- The role and the concept are connected via conjunction (existential quantification) or implication (universal quantification)

Basic Definitions

Atomic role: Represents an atom with two free variables

$$\mathit{Affects} \quad \rightsquigarrow \quad \mathit{Affects}(x,y)$$

Complex concepts (part 2): apply to an atomic role and a concept

- Existential Restriction:

$$\exists \mathit{Damages}.\mathit{Joint} \quad \rightsquigarrow \quad \exists y.(\mathit{Damages}(x,y) \land \mathit{Joint}(y))$$

- Universal Restriction:

$$\forall \mathit{Affects}.(\mathit{Child} \sqcup \mathit{Teen}) \quad \rightsquigarrow \quad \forall y.(\mathit{Affects}(x,y) \to \mathit{Child}(y) \lor \mathit{Teen}(y))$$

Outline

- Motivation and Examples
- $\mathcal{ALC}$ Syntax and Semantics
- Calculus for $\mathcal{ALC}$ Terminological Reasoning
- Discussion

$\mathcal{ALC}$ Concepts

$\mathcal{ALC}$ is the basic description logic (*Attributive Language with Complements*)

$\mathcal{ALC}$ concepts inductively defined from atomic concepts and roles:

- Every atomic concept is a concept
- $\top$ and $\bot$ are concepts
- If C is a concept, then $\neg C$ is a concept
- If C and D are concepts, then so are $C \sqcap D$ and $C \sqcup D$
- If C a concept and R a role, then $\forall R.C$ and $\exists R.C$ are concepts

Concepts describe sets of objects with certain common features:

$Woman \sqcap \exists hasChild.(\exists hasChild.Person)$	Women with a grandchild
$Disease \sqcap \forall Affects.Child$	Diseases affecting only children
$Person \sqcap \neg\exists owns.DetHouse$	People not owning a detached house
$Man \sqcap \exists hasChild.\top \sqcap \forall hasChild.Man$	Fathers having only sons

Very useful idea for Knowledge Representation!

General Concept Inclusion Axioms

Recall our example formulas:

$$\forall x.(JuvDis(x) \to \forall y.(Affects(x,y) \to Child(y) \vee Teen(y)))$$
$$\forall x.(Child(x) \vee Teen(x) \to \neg Adult(x))$$
$$\forall x.(Person(x) \to Child(x) \vee Teen(x) \vee Adult(x))$$
$$\forall x.(JuvArthritis(x) \to Arthritis(x) \wedge JuvDis(x))$$
$$\forall x.(Arthritis(x) \to \exists y.(Damages(x,y) \wedge Joint(y))$$

They are of the following form, with $\alpha_C(x)$ and $\alpha_D(x)$ corresponding to $\mathcal{ALC}$ concepts C and D

$$\forall x.(\alpha_C(x) \to \alpha_D(x))$$

Such closed formulas (sentences) are $\mathcal{ALC}$ General Concept Inclusions (GCIs)

$$C \sqsubseteq D$$

Where C and D are $\mathcal{ALC}$-concepts

General Concept Inclusion Axioms

$$\forall x.(JuvDis(x) \to \forall y.(Affects(x,y) \to Child(y) \vee Teen(y))) \rightsquigarrow JuvDis \sqsubseteq \forall Affects.(Child \sqcup Teen)$$
$$\forall x.(Child(x) \vee Teen(x) \to \neg Adult(x)) \rightsquigarrow Child \sqcup Teen \sqsubseteq \neg Adult$$
$$\forall x.(Person(x) \to Child(x) \vee Teen(x) \vee Adult(x)) \rightsquigarrow Person \sqsubseteq Child \sqcup Teen \sqcup Adult$$
$$\forall x.(JuvArth(x) \to Arth(x) \wedge JuvDis(x)) \rightsquigarrow JuvArth \sqsubseteq Arth \sqcap JuvDis$$
$$\forall x.(Arth(x) \to \exists y.(Damages(x,y) \wedge Joint(y)) \rightsquigarrow Arth \sqsubseteq \exists Damages.Joint$$

Why call $C \sqsubseteq D$ a concept inclusion axiom?

- Intuitively, every object belonging to C should belong also to D
- States that C is more specific than D

Terminological Statements

GCIs allow us to represent a surprising variety of terminological statements

- Sub-type statements

$$\forall x.(JuvArth(x) \to Arth(x)) \quad \rightsquigarrow \quad JuvArth \sqsubseteq Arth$$

- Full definitions:

$$\forall x.(JuvArth(x) \leftrightarrow Arth(x) \land JuvDis(x)) \quad \rightsquigarrow \quad \begin{array}{l} JuvArth \sqsubseteq Arth \sqcap JuvDis \\ Arth \sqcap JuvDis \sqsubseteq JuvArth \end{array}$$

- Disjointness statements:

$$\forall x.(Child(x) \to \neg Adult(x)) \quad \rightsquigarrow \quad Child \sqsubseteq \neg Adult$$

- Covering statements:

$$\forall x.(Person(x) \to Adult(x) \lor Child(x)) \quad \rightsquigarrow \quad Person \sqsubseteq Adult \sqcup Child$$

- Type restrictions:

$$\forall x.(\forall y.(Affects(x,y) \to Arth(x) \land Person(y))) \quad \rightsquigarrow \quad \begin{array}{l} \exists Affects.\top \sqsubseteq Arth \\ \top \sqsubseteq \forall Affects.Person \end{array}$$

Data Assertions

In description logics, we can also represent data:

$Child(JohnSmith)$	John Smith is a child
$JuvenileArthritis(JRA)$	JRA is a juvenile arthritis
$Affects(JRA, MaryJones)$	Mary Jones is affected by JRA

Usually data assertions correspond to first-order ground (variable-free) atoms

In $\mathcal{ALC}$, we have two types of data assertions, for a,b constants:

$$C(a) \quad \rightsquigarrow \quad C \text{ is an } \mathcal{ALC} \text{ concept}$$
$$R(a,b) \quad \rightsquigarrow \quad R \text{ is an atomic role}$$

Examples of acceptable data assertions in $\mathcal{ALC}$:

$$\exists hasChild.Teacher(John) \quad \rightsquigarrow \exists y.(hasChild(John,y) \land Teacher(y))$$
$$HistorySt \sqcup ClassicsSt(John) \quad \rightsquigarrow HistorySt(John) \lor ClassicsSt(John)$$

DL Knowledge Base: TBox + ABox

An $\mathcal{ALC}$ knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ is composed of

- a TBox $\mathcal{T}$ (Terminological component):

  Finite set of GCIs

- an ABox $\mathcal{A}$ (Assertional component):

  Finite set of assertions

TBox:

$$JuvArthritis \sqsubseteq Arthritis \sqcap JuvDisease$$
$$Arthritis \sqcap JuvDisease \sqsubseteq JuvArthritis$$
$$Arthritis \sqsubseteq \exists Damages.Joint$$
$$JuvDisease \sqsubseteq \forall Affects.(Child \sqcup Teen)$$
$$Child \sqcup Teen \sqsubseteq \neg Adult$$

ABox:

$$Child(JohnSmith)$$
$$JuvArthritis(JRA)$$
$$Affects(JRA, MaryJones)$$
$$Child \sqcup Teen(MaryJones)$$

Semantics via First-Order Translation

Semantics of $\mathcal{ALC}$ can be defined via translation into first-order logic:

- Concepts translated as formulas with one free variable

$$\begin{array}{rl} \pi_x(A) = A(x) & \pi_y(A) = A(y) \\ \pi_x(\neg C) = \neg \pi_x(C) & \pi_y(\neg C) = \neg \pi_y(C) \\ \pi_x(C \sqcap D) = \pi_x(C) \land \pi_x(D) & \pi_y(C \sqcap D) = \pi_y(C) \land \pi_y(D) \\ \pi_x(C \sqcup D) = \pi_x(C) \lor \pi_x(D) & \pi_y(C \sqcup D) = \pi_y(C) \lor \pi_y(D) \\ \pi_x(\exists R.C) = \exists y.(R(x,y) \land \pi_y(C)) & \pi_y(\exists R.C) = \exists x.(R(y,x) \land \pi_x(C)) \\ \pi_x(\forall R.C) = \forall y.(R(x,y) \to \pi_y(C)) & \pi_y(\forall R.C) = \forall x.(R(y,x) \to \pi_x(C)) \end{array}$$

- GCIs and assertions translated as closed formulas

$$\begin{array}{rcl} \pi(C \sqsubseteq D) & = & \forall x.(\pi_x(C) \to \pi_x(D)) \\ \pi(R(a,b)) & = & R(a,b) \\ \pi(C(a)) & = & \pi_{x/a}(C) \end{array}$$

- TBoxes, ABoxes and KBs are translated in the obvious way

Semantics via First-Order Translation

Note that concept-forming operators are not independent:

$$\begin{aligned} \bot &\rightsquigarrow \neg\top \\ C \sqcup D &\rightsquigarrow \neg(\neg C \sqcap \neg D) \\ \forall R.C &\rightsquigarrow \neg(\exists R.\neg C) \end{aligned}$$

These equivalences can be proved using first-order semantics:

$$\begin{aligned} \pi_x(\neg\exists R.\neg C) &= \neg\exists y.(R(x,y) \wedge \neg\pi_y(C)) \\ &\equiv \forall y.(\neg(R(x,y) \wedge \neg\pi_y(C))) \\ &\equiv \forall y.(\neg R(x,y) \vee \pi_y(C)) \\ &\equiv \forall y.(R(x,y) \rightarrow \pi_y(C)) \\ &= \pi_x(\forall R.C) \end{aligned}$$

We can define syntax of $\mathcal{ALC}$ using only conjunction and negation operators and the existential role operator

Direct (Model-Theoretic) Semantics

Direct semantics: An alternative (and convenient) way of specifying semantics

DL interpretation $\mathcal{I} = \langle \mathrm{D}, \cdot^{\mathcal{I}} \rangle$ is a first-order interpretation over the DL vocabulary:

- each constant a interpreted as an object $a^{\mathcal{I}} \in \mathrm{D}$
- each atomic concept A interpreted as a set $A^{\mathcal{I}} \subseteq \mathrm{D}$
- each atomic role R interpreted as a binary relation $R^{\mathcal{I}} \subseteq \mathrm{D} \times \mathrm{D}$

We specify a mechanism for interpreting concepts:

$$\begin{aligned} \top^{\mathcal{I}} &= \mathrm{D} \\ \bot^{\mathcal{I}} &= \emptyset \\ (\neg C)^{\mathcal{I}} &= \mathrm{D} \setminus C^{\mathcal{I}} \\ (C \sqcap D)^{\mathcal{I}} &= C^{\mathcal{I}} \cap D^{\mathcal{I}} \\ (C \sqcup D)^{\mathcal{I}} &= C^{\mathcal{I}} \cup D^{\mathcal{I}} \\ (\exists R.C)^{\mathcal{I}} &= \{u \in \mathrm{D} \mid \exists w \in \mathrm{D} \text{ s.t. } \langle u, w\rangle \in R^{\mathcal{I}} \text{ and } w \in C^{\mathcal{I}}\} \\ (\forall R.C)^{\mathcal{I}} &= \{u \in \mathrm{D} \mid \forall w \in \mathrm{D},\ \langle u, w\rangle \in R^{\mathcal{I}} \text{ implies } w \in C^{\mathcal{I}}\} \end{aligned}$$

Direct (Model-Theoretic) Semantics

Consider the interpretation $\mathcal{I} = \langle \mathrm{D}, \cdot^{\mathcal{I}} \rangle$

$$\begin{aligned} \mathrm{D} &= \{u, v, w\} \\ \mathit{JuvDis}^{\mathcal{I}} &= \{u\} \\ \mathit{Child}^{\mathcal{I}} &= \{w\} \\ \mathit{Teen}^{\mathcal{I}} &= \emptyset \\ \mathit{Affects}^{\mathcal{I}} &= \{\langle u, w\rangle\} \end{aligned}$$

We can then interpret any concept as a subset of D:

$$\begin{aligned} (\mathit{JuvDis} \sqcap \mathit{Child})^{\mathcal{I}} &= \emptyset \\ (\mathit{Child} \sqcup \mathit{Teen})^{\mathcal{I}} &= \{w\} \\ (\exists \mathit{Affects}.(\mathit{Child} \sqcup \mathit{Teen}))^{\mathcal{I}} &= \{u\} \\ (\neg \mathit{Child})^{\mathcal{I}} &= \{u, v\} \\ (\forall \mathit{Affects}.\mathit{Teen})^{\mathcal{I}} &= \{v, w\} \end{aligned}$$

Direct (Model-Theoretic) Semantics

We can now determine whether $\mathcal{I}$ is a model of ...

- A General Concept Inclusion Axiom $C \sqsubseteq D$:

$$\mathcal{I} \models (C \sqsubseteq D) \quad \text{iff} \quad C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$$

- An assertion $C(a)$:

$$\mathcal{I} \models C(a) \quad \text{iff} \quad a^{\mathcal{I}} \in C^{\mathcal{I}}$$

- An assertion $R(a, b)$:

$$\mathcal{I} \models R(a, b) \quad \text{iff} \quad \langle a^{\mathcal{I}}, b^{\mathcal{I}}\rangle \in R^{\mathcal{I}}$$

- A TBox $\mathcal{T}$, ABox $\mathcal{A}$, and knowledge base:

$$\begin{aligned} \mathcal{I} \models \mathcal{T} &\quad \text{iff} \quad \mathcal{I} \models \alpha \text{ for each } \alpha \in \mathcal{T} \\ \mathcal{I} \models \mathcal{A} &\quad \text{iff} \quad \mathcal{I} \models \alpha \text{ for each } \alpha \in \mathcal{A} \\ \mathcal{I} \models \mathcal{K} &\quad \text{iff} \quad \mathcal{I} \models \mathcal{T} \text{ and } \mathcal{I} \models \mathcal{A} \end{aligned}$$

Direct (Model-Theoretic) Semantics

Consider our previous example interpretation:

$$D = \{u, v, w\} \quad \mathit{Affects}^{\mathcal{I}} = \{\langle u, w\rangle\}$$
$$\mathit{JuvDis}^{\mathcal{I}} = \{u\} \quad \mathit{Child}^{\mathcal{I}} = \{w\} \quad \mathit{Teen}^{\mathcal{I}} = \emptyset$$

$\mathcal{I}$ is a model of the following axioms:

$$\mathit{JuvDis} \sqsubseteq \exists \mathit{Affects}.\mathit{Child} \rightsquigarrow \{u\} \subseteq \{u\}$$
$$\mathit{Child} \sqsubseteq \neg \mathit{Teen} \rightsquigarrow \{w\} \subseteq \{u, v, w\}$$
$$\mathit{JuvDisease} \sqsubseteq \forall \mathit{Affects}.\mathit{Child} \rightsquigarrow \{u\} \subseteq \{u, v, w\}$$

However $\mathcal{I}$ is not a model of the following axioms:

$$\mathit{JuvDis} \sqsubseteq \exists \mathit{Affects}.(\mathit{Child} \sqcap \mathit{Teen}) \rightsquigarrow \{u\} \not\subseteq \emptyset$$
$$\neg \mathit{Teen} \sqsubseteq \mathit{Child} \rightsquigarrow \{u, v, w\} \not\subseteq \{w\}$$
$$\exists \mathit{Affects}.\top \sqsubseteq \mathit{Teen} \rightsquigarrow \{u\} \not\subseteq \emptyset$$

Observations

- The 'square' syntax of DLs looks odd at first, but it is less verbose than that of first-order logic (and may even be more intuitive for engineers not biased towards theoretical CS)
- $\mathcal{ALC}$ (and other DLs) underlies OWL ontology language with its own 'serialisation' syntax using RDF triples
- Modal Logic K formulas are essentially $\mathcal{ALC}$ concepts with a single role R:

$$\Diamond A \rightsquigarrow \exists R.A$$
$$\Box A \rightsquigarrow \forall R.A$$

- Other Modal Logics also have corresponding DLs

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Ontology Design

Scenario: Ontology design

- We are building a conceptual model (a TBox) for our domain
- At this design stage we have not included the data (no ABox)

Our TBox should be

- Error-free:

  No unintended logical consequences

- Sufficiently detailed:

  Contain all relevant knowledge for our application

Ontology Design

$$JuvArthritis \sqsubseteq Arthritis \sqcap JuvDisease$$
$$JuvDisease \sqsubseteq Disease$$
$$Arthritis \sqsubseteq \exists Damages.Joint \sqcap \forall Damages.Joint$$
$$JuvDisease \sqsubseteq \forall Affects.(Child \sqcup Teen)$$
$$Child \sqcup Teen \sqsubseteq \neg Adult$$
$$Arthritis \sqsubseteq \exists Affects.Adult$$
$$Disease \sqcap \exists Damages.Joint \sqsubseteq JointDisease$$

This TBox contains modeling errors:

- Juvenile arthritis is a kind of juvenile disease
  - Juvenile disease affects only children or teens, which are not adults
    - A juvenile arthritis cannot affect any adult
- Juvenile arthritis is a kind of arthitis
  - Each arthritis affects some adult
    - Each juvenile arthritis affects some adult

Concept Satisfiability

What is the impact of the error?

- All models $\mathcal{I}$ of $\mathcal{T}$ must be such that $JuvArthritis^{\mathcal{I}} = \emptyset$
  - A juvenile arthritis cannot exist!
    - We cannot add data concerning juvenile arthritis

Such errors can be detected by solving the following problem:

> Concept satisfiability w.r.t. a TBox:
> An instance is a pair $\langle C, \mathcal{T} \rangle$ with C a concept and $\mathcal{T}$ a TBox.
> The answer is *true* iff a model $\mathcal{I} \models \mathcal{T}$ exists such that $C^{\mathcal{I}} \neq \emptyset$.

In a first-order setting, C is satisfiable w.r.t. $\mathcal{T}$ if and only if

$$\pi(\mathcal{T}) \wedge \exists x.(\pi_x(C)) \quad \text{is satisfiable}$$

Concept Subsumption

Parts of our arthritis TBox, however, do conform to our intuitions

$$JuvArthritis \sqsubseteq Arthritis \sqcap JuvDisease$$
$$JuvDisease \sqsubseteq Disease$$
$$Arthritis \sqsubseteq \exists Damages.Joint \sqcap \forall Damages.Joint$$
$$JuvDisease \sqsubseteq \forall Affects.(Child \sqcup Teen)$$
$$Child \sqcup Teen \sqsubseteq \neg Adult$$
$$Arthritis \sqsubseteq \exists Affects.Adult$$
$$Disease \sqcap \exists Damages.Joint \sqsubseteq JointDisease$$

- Juvenile arthritis is a kind of juvenile disease
  - Juvenile disease is a kind of disease
    - Juvenile arthritis is a kind of disease
- Juvenile arthritis is a kind of arthitis
  - Each arthritis damages some joint
    - Each juvenile arthritis affects some joint
      - Juvenile arthritis is a joint disease.

Concept Subsumption

We have discovered new interesting information

- All models $\mathcal{I}$ of $\mathcal{T}$ must be such that $JuvArthritis^{\mathcal{I}} \subseteq JointDisease^{\mathcal{I}}$
  - Juvenile arthritis is a sub-type of joint disease
    - All instances of juvenile arthitis are also joint diseases

Such implicit information detectable by solving the following problem:

> Concept subsumption w.r.t. a TBox:
> An instance is a triple $\langle C, D, \mathcal{T} \rangle$ with C, D concepts, $\mathcal{T}$ a TBox.
> The answer is *true* iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for each $\mathcal{I} \models \mathcal{T}$ (written $\mathcal{T} \models C \sqsubseteq D$).

In the first-order setting, C is subsumed by D w.r.t. $\mathcal{T}$ if and only if

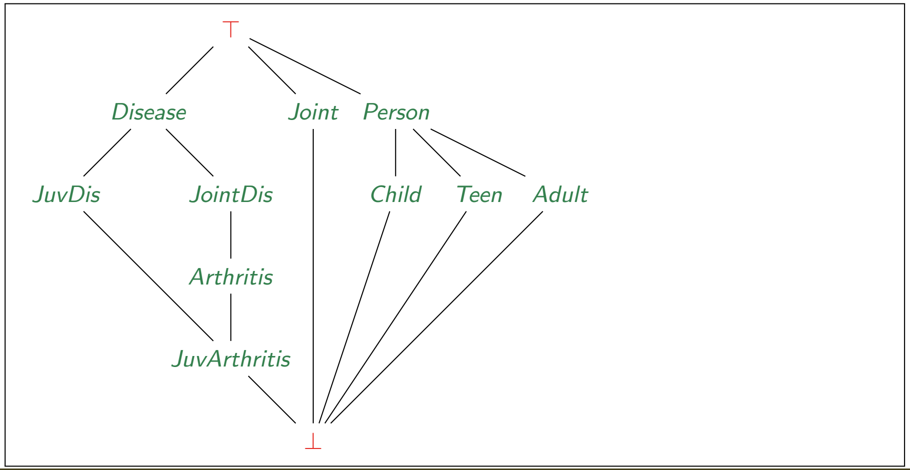
$$\pi(\mathcal{T}) \quad \models \quad \forall x.(\pi_x(C) \rightarrow \pi_x(D))$$

In the Modal Logic setting, subsumption is local logical consequence

TBox Classification

Problem of finding all subsumptions between atomic concepts in $\mathcal{T}$

Allows us to organise atomic concepts in a subsumption hierarchy



Calculus for $\mathcal{ALC}$ Terminological Reasoning

Reductions and Special Cases

In $\mathcal{ALC}$, concept subsumption reducible to concept satisfiability:

$$\mathcal{T} \models C \sqsubseteq D \quad \text{iff} \quad (C \sqcap \neg D) \text{ is unsatisfiable w.r.t. } \mathcal{T}$$

In $\mathcal{ALC}$, concept satisfiability is reducible to subsumption:

$$C \text{ satisfiable w.r.t. } \mathcal{T} \quad \text{iff} \quad \mathcal{T} \not\models (C \sqsubseteq \bot)$$

Interesting particular cases:

- $C \sqsubseteq \bot$ with $\mathcal{T} = \emptyset$: Can a concept be instantiated at all?
- $\top \sqsubseteq \bot$: Does $\mathcal{T}$ have a model?

Validity, etc. can be defined and reduced in a similar way

We focus on algorithms for $\mathcal{ALC}$ concept subsumption w.r.t. TBox

Calculus for $\mathcal{ALC}$ Terminological Reasoning

Sequent Calculus for $\mathcal{ALC}$ Subsumption (Empty TBox)

We start with concept subsumption w.r.t. empty TBox ($\mathcal{T} = \emptyset$)
We can reuse the calculus for consequence for K (generalised to several roles)

- A labelled formula is a pair $u : A$ where u is a label and A a concept, an accessibility formula is uRv for two labels u, v and R a role
- Propositional rules for labelled formulas ('square' version): e.g.

$$\frac{\Gamma \Rightarrow u : A, \Delta \qquad \Gamma \Rightarrow u : B, \Delta}{\Gamma \Rightarrow u : A \sqcap B, \Delta} \wedge\text{-right}$$

- The $\exists R$-left rule, for each role R, creates a new label:

$$\frac{\Gamma, uRv, v : A \Rightarrow \Delta}{\Gamma, u : \exists R.A \Rightarrow \Delta} \exists R\text{-left} \qquad \text{for a fresh label } v$$

- The $\forall R$-left rule, for each role R, transfers info to other labels:

$$\frac{\Gamma, uRv, v : A, u : \forall R.A \Rightarrow \Delta}{\Gamma, uRv, u : \forall R.A \Rightarrow \Delta} \forall R\text{-left}$$

- Axioms for $\top$ and $\bot$ (or get rid of them using $A \sqcup \neg A$ for $\top$, etc.):

$$\frac{}{\Gamma, u : \bot \Rightarrow \Delta} \text{axiom} \qquad \frac{}{\Gamma \Rightarrow u : \top, \Delta} \text{axiom}$$

- The $\exists R$- and $\forall R$-right rules, other axioms: the same as for K

Calculus for $\mathcal{ALC}$ Terminological Reasoning

Sequent Calculus for $\mathcal{ALC}$ Subsumption (Empty TBox)

- The calculi are sound and complete
- Termination is guaranteed
  - Proof by structural induction: along each branch, the formulas become simpler and simpler
  - May take quite long time (exponential, in fact PSpace-complete)
- A non-closed branch can be used for extracting counter-model
  - the domain is the set of labels, labelled formulas $u : A$ define concept interpretations, accessibility formulas uRv define role interpretations
  - this counter-model is always finite and tree-shaped
- What about the general case with non-empty TBox?

Sequent Calculus for $\mathcal{ALC}$ Subsumption (with TBox)

A TBox contains GCIs of the form $C \sqsubseteq D$

Each GCI equivalent to $\top \sqsubseteq \neg C \sqcup D$

We can 'compile' the whole TBox

$$\mathcal{T} = \{C_i \sqsubseteq D_i \mid 1 \leq i \leq n\}$$

into a single, equivalent GCI:

$$\top \sqsubseteq \bigsqcap_{1 \leq i \leq n} \neg C_i \sqcup D_i$$

Let's call $C_{\mathcal{T}}$ the concept on the right-hand side of this GCI

Sequent Calculus for $\mathcal{ALC}$ Subsumption (with TBox)

- Check concept subsumption $C \sqsubseteq D$ w.r.t. $\mathcal{T}$
- Intuitively, $C_{\mathcal{T}}$ should hold in all labels, so add $C_{\mathcal{T}}$ to Γ when creating new v
- The $\exists R$-left rule w.r.t. $\mathcal{T}$, for each role R:

$$\frac{\Gamma, uRv, v : A, v : C_{\mathcal{T}} \ \Rightarrow\ \Delta}{\Gamma, u : \exists R.A \ \Rightarrow\ \Delta}\ \exists R\text{-left} \qquad \text{for a fresh label } v$$

- The $\Box$-right rule w.r.t. $\mathcal{T}$, for each role R:

$$\frac{\Gamma, uRv, v : C_{\mathcal{T}} \ \Rightarrow\ v : A, \Delta}{\Gamma \ \Rightarrow\ u : \forall R.A, \Delta}\ \forall R\text{-right} \qquad \text{for a fresh label } v$$

- Start with $1 : C_{\mathcal{T}} \ \Rightarrow\ 1 : \neg C \sqcup D$
- The rest as in the $\mathcal{T} = \emptyset$ case
- Soundness and completeness as before, but termination is not guaranteed: no decrease in the formula size along branches

Sequent Calculus for $\mathcal{ALC}$ Subsumption (with TBox)

Example: $A \sqsubseteq \bot$ w.r.t. $\mathcal{T} = \{A \sqsubseteq \exists R.A\}$

Essentially, (un)satisfiability of concept A w.r.t. $\mathcal{T}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{1:A \Rightarrow 1:A,\ 1:\bot}}{1:\neg A,\ 1:A \Rightarrow 1:\bot} \qquad \dfrac{\dfrac{\cdots \qquad \dfrac{\cdots}{1:\exists R.A,\ 1:A,\ 1R2,\ 2:\exists R.A,\ 2:A \Rightarrow 1:\bot}}{1:\exists R.A,\ 1:A,\ 1R2,\ 2:\neg A \sqcup \exists R.A,\ 2:A \Rightarrow 1:\bot}}{1:\exists R.A,\ 1:A \Rightarrow 1:\bot}}{1:\neg A \sqcup \exists R.A,\ 1:A \Rightarrow 1:\bot}\ \sqcup\text{-left}}{1:\neg A \sqcup \exists R.A \Rightarrow 1:\neg A,\ 1:\bot}\ \neg\text{-right}}{1:\neg A \sqcup \exists R.A \Rightarrow 1:\neg A \sqcup \bot}\ \sqcup\text{-right}$$

Sequent Calculus for $\mathcal{ALC}$ Subsumption (with TBox)

Solution: Regain termination with cycle detection

Definition 3.1.

Label v' is reachable from label v in $\Gamma \Rightarrow \Delta$ if there are $v_0 R_1 v_1, \ldots, v_{n-1} R_n v_n$ in Γ with $v' = v_0$ and $v = v_n$.

A label v' is directly blocked by a label v (in Γ and Δ) if

- *v' is reachable from v*
- *$v : C \in \Gamma$ if and only if $v' : C \in \Gamma$, and $v : C \in \Delta$ if and only if $v' : C \in \Delta$ for every concept C.*

A label v' is blocked if either

- *it is directly blocked by some v or*
- *there exists a directly blocked v such that v' is reachable from v.*

Restrict application of $\exists R$-left and $\forall R$-right rules to labels that are not blocked

Intuitively, a branch where everything is blocked is a finite representation of an infinite branch

Sequent Calculus for $\mathcal{ALC}$ Subsumption (with TBox)

Example: $A \sqsubseteq \bot$ w.r.t. $\mathcal{T} = \{A \sqsubseteq \exists R.A\}$

Essentially, (un)satisfiability of concept A w.r.t. $\mathcal{T}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{1:A \Rightarrow 1:A,\ 1:\bot}}{1:\neg A,\ 1:A \Rightarrow 1:\bot} \qquad \dfrac{\dfrac{\cdots \quad \dfrac{}{1:\exists R.A,\ 1:A,\ 1R2,\ 2:\exists R.A,\ 2:A \Rightarrow 1:\bot}}{1:\exists R.A,\ 1:A,\ 1R2,\ 2:\neg A \sqcup \exists R.A,\ 2:A \Rightarrow 1:\bot}}{1:\exists R.A,\ 1:A \Rightarrow 1:\bot}}{1:\neg A \sqcup \exists R.A,\ 1:A \Rightarrow 1:\bot}\ \sqcup\text{-left}}{\dfrac{1:\neg A \sqcup \exists R.A \Rightarrow 1:\neg A,\ 1:\bot}{1:\neg A \sqcup \exists R.A \Rightarrow 1:\neg A \sqcup \bot}\ \sqcup\text{-right}}\ \neg\text{-right}$$

Label 2 is directly blocked by label 1

Label 2 is blocked

$\exists R$-left does not apply to 2

Other rules can apply, and even can 'unblock' $\exists R$-left for 2!

Sequent Calculus for $\mathcal{ALC}$ Subsumption (with TBox)

Theorem 3.1.

Calculus for $\mathcal{ALC}$ subsumption with blocking is sound, complete and terminating.

Proof idea.

- Soundness as before
- Completeness since every block can be 'infinitely unrolled' to a counter-model
- Termination is guaranteed since there are finite number of (sets of) labelled formulae □

Corollary: reasoning in $\mathcal{ALC}$ is decidable (in fact ExpTime-complete)

Observation: The 'unrolled' counter-model is tree-shaped (but may be infinite)

A general reason for decidability

Comment: adding ABox (assertions as $A(a)$, $R(a, b)$) does not change anything conceptually

The Picture

Undecidable	first-order
|	|
Non-Elementary decidable	. . .
|	|
. . .	. . .
|	|
ExpTime	$\mathcal{ALC}$ w.r.t. TBox
|	|
PSpace	$\mathcal{ALC}$ w.r.t. $\emptyset$ (and K)
|	|
NP	propositional
|	|
P	. . .

Outline

Other Description Logics

- This was $\mathcal{ALC}$, the *Attributive Language with Complements*.
- The $\mathcal{C}$ actually denotes an extension of a more restrictive language $\mathcal{AL}$.
- In a similar way, we have the following possible extensions of our logic:
  - $\mathcal{H}$: Role hierarchies;
  - $\mathcal{R}$: Complex role hierarchies;
  - $\mathcal{N}$: Cardinality restrictions;
  - $\mathcal{Q}$: Qualified cardinality restrictions;
  - $\mathcal{O}$: Closed classes;
  - $\mathcal{I}$: Inverse roles;
  - ...
- We name the languages by adding the letters of the features to $\mathcal{ALC}$. So e.g. $\mathcal{ALCN}$ is $\mathcal{ALC}$ extended with cardinality restrictions and $\mathcal{ALCHI}$ is $\mathcal{ALC}$ extended with role hierarchies and inverse roles.
- It is common to shorten $\mathcal{ALC}$ (extended with transitive roles) to just $\mathcal{S}$ for more advanced languages, so e.g. $\mathcal{SHOIN}$ is $\mathcal{ALC} + \mathcal{H} + \mathcal{O} + \mathcal{I} + \mathcal{N}$.

Description Logic Applications

- Description logics are decidable
- Can be used to describe large vocabularies (>100 000 concepts)
- E.g., in medicine, engineering, ...
- Reasoning helps to find mistakes when authoring
- Can be used in domain modelling, data integration, etc.
- Interested? Take IN3060/IN4060 – Semantic Technologies next semester!