IN3070/4070 – Logic – Autumn 2020

Lecture 13: Intuitionistic Logic

Martin Giese

12th November 2020

Department of Informatics

University of Oslo

Today's Plan

- Motivation
- Syntax and Semantics
- Satisfiability & Validity
- Sequent Calculus
- Summary

Outline

- Motivation
- Syntax and Semantics
- Satisfiability & Validity
- Sequent Calculus
- Summary

Intuitionistic Logic – Overview

- has applications in, e.g., program synthesis and verification
- formalizing computation, "proofs as programs" (NuPRL, Coq)

Syntax and Semantics

- same syntax as classical logic, but different semantics
- standard connectives and quantifiers ($\neg$, $\wedge$, $\vee$, $\rightarrow$, $\forall$, $\exists$), predicates, functions, variables

Examples

- $p \vee \neg p$ (law of excluded middle) is not valid in intuitionistic logic
- $(\neg\forall x\ \neg p(x)) \rightarrow \exists x\ p(x)$ is not valid in intuitionistic logic

Proof search calculi

- natural deduction, sequent, tableau and connection calculi

A Non-Constructive Proof

Theorem 1.1 ($x^y = z$).

There exist a solution of $x^y = z$ such that x and y are irrational numbers and z is a rational number.

Proof.

We know that $\sqrt{2}$ is irrational. We distinguish two cases: ${\sqrt{2}}^{\sqrt{2}}$ is either rational or irrational.

a. *If ${\sqrt{2}}^{\sqrt{2}}$ is rational, then $x = \sqrt{2}$ and $y = \sqrt{2}$ are irrational and $z = {\sqrt{2}}^{\sqrt{2}}$ is rational.*
b. *If ${\sqrt{2}}^{\sqrt{2}}$ is irrational, then $x = {\sqrt{2}}^{\sqrt{2}}$ and $y = \sqrt{2}$ are irrational and $z = ({\sqrt{2}}^{\sqrt{2}})^{\sqrt{2}} = {\sqrt{2}}^{(\sqrt{2}\cdot\sqrt{2})} = {\sqrt{2}}^{2} = 2$ is rational.*

□

Theorem (classically) proven, but we don't know which case holds.

Intuitionism

- is it reasonable to claim the existence of a number n with some property without being able to produce n? (e.g. prove $\exists x\, p(x)$ by showing that its negation $\forall x\, \neg p(x)$ leads to a contradiction)
- is it reasonable to accept the validity of $A \vee B$ without knowing whether A or B is valid? – is it reasonable to claim the existence of function f without providing a way to calculate f?

The mathematician L.E.J. Brouwer

- rejected much of early twentieth century mathematics (dominated by, e.g., Frege and Hilbert)
- in his paper "The untrustworthiness of the principles of logic" he challenged the belief that the rules of classical logic are valid
- rejected the validity of the "law of excluded middle" $A \vee \neg A$ and non-constructive existence proofs

Intuitionistic Logic

- in Brouwer's opinion a proof of A or B must consist of either a proof of A or a proof of B; a proof of $\exists x\, p(x)$ must consist of a construction of an element c and a proof of $p(c)$

Intuitionistic (or constructive) logic

- first formal system/logic that attempts to capture Brouwer's logic was given 1930 by his student Arend Heyting
- later Saul Kripke's "possible worlds" semantics gave a "state of knowledge" interpretation of Heyting's formalism

Constructive definition of computability

- write a "logical" specification of a program; if there is a proof for the specification, the program that satisfies the specification can be extracted from the proof ("proof as programs")
- for example the proof of $\forall x\, \exists y\, p(x, y)$ contains the construction of an algorithm for computing a value of y from one for x

Outline

Semantics – Classical Logic

Let $\mathcal{F}^n$ be a set of function symbols with arity n for every $n \in \mathbb{N}_0$, and $\mathcal{P}^n$ be a set of predicate symbols with arity n for every $n \in \mathbb{N}_0$.

Definition 2.1 (Classical Interpretation).

A classical interpretation (or structure) is a tuple $\mathcal{I}_C = (D, \iota)$ where

- *D is a non-empty set, the domain*
- *ι ("iota") is a function, the interpretation, which assigns every*
  - *constant $a \in \mathcal{F}^0$ an element $a^\iota \in D$*
  - *function symbol $f \in \mathcal{F}^n$ with $n > 0$ a function $f^\iota : D^n \to D$*
  - *propositional variable $p \in \mathcal{P}^0$ a truth value $p^\iota \in \{T, F\}$*
  - *predicate symbol $p \in \mathcal{P}^n$ with $n > 0$ a relation $p^\iota \subseteq D^n$*

Kripke Semantics

- is a formal semantics created in the late 1950s and early 1960s by Saul Kripke and André Joyal; was first used for modal logics, later adapted to intuitionistic logic and other non-classical logics

Definition 2.2 (Kripke Frame).

A (Kripke) frame $F = (W, R)$ consists of a

- *a non-empty set of worlds W*
- *a binary accessibility relation $R \subseteq W \times W$ on the worlds in W*

Definition 2.3 (Intuitionistic Frame).

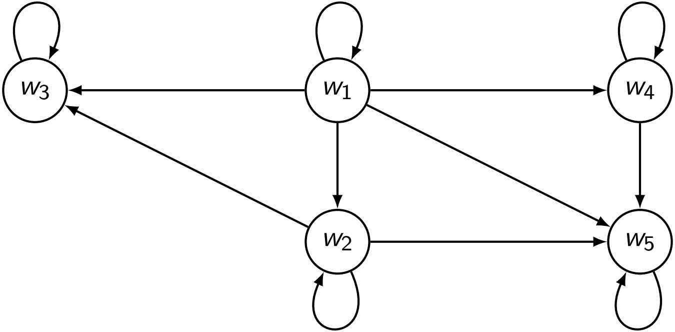
An intuitionistic frame $F_J = (W, R)$ is a Kripke frame (W, R) with a reflexive and transitive accessibility relation R.

($R \subseteq W \times W$ is reflexive iff $(w_1, w_1) \in R$ for all $w_1 \in W$; R is transitive iff for all $w_1, w_2, w_3 \in W$: if $(w_1, w_2) \in R$ and $(w_2, w_3) \in R$ then $(w_1, w_3) \in R$)

Intuitionistic Frame – Example

Example: $F'_J = (W', R')$ with $W' = \{w_1, w_2, w_3, w_4, w_5\}$ and

$$R' = \{(w_1, w_1), (w_2, w_2), (w_3, w_3), (w_4, w_4), (w_5, w_5),$$
$$(w_1, w_2), (w_2, w_3), (w_1, w_4), (w_4, w_5), (w_2, w_5)$$
$$(w_1, w_3), (w_1, w_5)\}$$



Intuitionistic Interpretation

Definition 2.4 (Intuitionistic Interpretation).

An intuitionistic interpretation (J-structure) $\mathcal{I}_J := (F_J, \{\mathcal{I}_C(w)\}_{w \in W})$ *consists of*

- *an intuitionistic frame* $F_J = (W, R)$
- *a set of class. interpretations* $\{\mathcal{I}_C(w)\}_{w \in W}$ *with* $\mathcal{I}_C(w) := (D^w, \iota^w)$ *assigning a domain* D^w *and an interpretation* ι^w *to every* $w \in W$

Furthermore, the following holds:

1. *cumulative domains, i.e. for all* $w, v \in W$ *with* $(w, v) \in R$: $D^w \subseteq D^v$
2. *interpretations only "increase", i.e. for all* $w, v \in W$ *with* $(w, v) \in R$:
   a. $a^{\iota^w} = a^{\iota^v}$ *for every constant* a
   b. $f^{\iota^w} \subseteq f^{\iota^v}$ *for every function* f
   c. $p^{\iota^w} = T$ *implies* $p^{\iota^v} = T$ *for every* $p \in \mathcal{P}^0$
   d. $p^{\iota^w} \subseteq p^{\iota^v}$ *for every predicate* $p \in \mathcal{P}^n$ *with* $n > 0$

   ($g \subseteq h$ *holds for* g *and* h *iff* $g(x) = h(x)$ *for all* x *of the domain of* g*)*

Intuitionistic Truth Value

Definition 2.5 (Intuitionistic Truth Value).

Let $\mathcal{I}_J = ((W, R), \{(D^w, \iota^w)\}_{w \in W})$ be a J-structure. The intuitionistic truth value $v_{\mathcal{I}_J}(w, G)$ of a formula G in the world w under the structure $\mathcal{I}_J$ is T (true) if "w forces G under $\mathcal{I}_J$", denoted $w \Vdash G$, and F (false), otherwise. $v_{\mathcal{I}_J}(w, t)$ is the (classic) evaluation of the term t in world w.

The forcing relation $w \Vdash G$ is defined as follows:

- $w \Vdash p$ for $p \in \mathcal{P}^0$ iff $p^{\iota^w} = T$
- $w \Vdash p(t_1, ..., t_n)$ for $p \in \mathcal{P}^n$, $n>0$, iff $(v_{\mathcal{I}_J}(w, t_1), ..., v_{\mathcal{I}_J}(w, t_n)) \in P^{\iota^w}$
- $w \Vdash \neg A$ iff $v \nVdash A$ for all $v \in W$ with $(w, v) \in R$
- $w \Vdash A \wedge B$ iff $w \Vdash A$ and $w \Vdash B$
- $w \Vdash A \vee B$ iff $w \Vdash A$ or $w \Vdash B$
- $w \Vdash A \rightarrow B$ iff $v \Vdash A$ implies $v \Vdash B$ for all $v \in W$ with $(w, v) \in R$
- $w \Vdash \exists x A$ iff $w \Vdash A[x \backslash d]$ for some $d \in D^w$
- $w \Vdash \forall x A$ iff $v \Vdash A[x \backslash d]$ for all $d \in D^v$ for all $v \in W$ with $(w, v) \in R$

Outline

Satisfiability and Validity

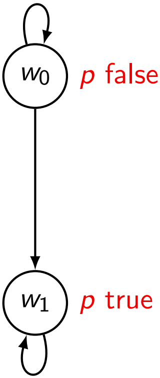
In intuitionistic logic a formula G is valid, if it evaluates to *true* in all worlds and for all intuitionistic interpretations.

Definition 3.1 (Satisfiable,Model,Unsatisfiable,Valid,Invalid).

Let G be a closed (first-order) formula.

- *Let $\mathcal{I}_J$ be an intuitionistic interpretation. $\mathcal{I}_J$ is an intuitionistic model for G, denoted $\mathcal{I}_J \models G$, iff $v_{\mathcal{I}_J}(w, G){=}T$ for all $w{\in}W$.*
- *G is intuitionistically satisfiable iff $\mathcal{I}_J \models G$ for some intuitionistic interpretation $\mathcal{I}_J$.*
- *F is intuitionistically unsatisfiable iff G is not intuit. satisfiable.*
- *G is intuitionistically valid, denoted $\models G$, iff $\mathcal{I}_J \models G$ for all intuitionistic interpretations $\mathcal{I}_J$.*
- *G is intuitionistically invalid/falsifiable iff G is not intuit. valid.*

Satisfiability and Validity – Examples



- $F_1 \equiv p \vee \neg p$

  $w_0 \Vdash \neg p$ iff $v \Vdash p$ does not hold
  for any $v \in W$ with $(w_0, v) \in R$

  but $(w_0, w_1) \in R$ and $w_1 \Vdash p$ holds

  hence, neither $w_0 \Vdash p$ nor $w_0 \Vdash \neg p$

  $\leadsto$ F_1 is not *true* in w_0 $\leadsto$ F_1 not valid

- $F_2 \equiv p \rightarrow p$

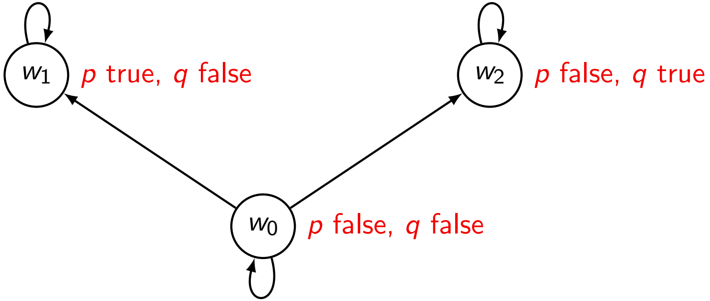
  $w_0 \Vdash p \rightarrow p$ iff $v \Vdash p$ implies $v \Vdash p$
  for all $v \in W$ with $(w_0, v) \in R$

  $\leadsto$ F_2 is *true* in w_0 (and w_1)

Satisfiability and Validity – More Examples

Example: $(p \rightarrow q) \vee (q \rightarrow p)$ is not intuitionistically valid

See [Nerode & Shore 1997] (page 269).

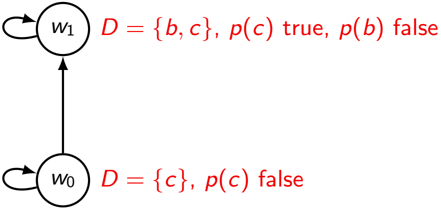


$w_1 \Vdash p,\ w_1 \nVdash q \Longrightarrow w_0 \nVdash p \rightarrow q$
$w_2 \Vdash q,\ w_1 \nVdash p \Longrightarrow w_0 \nVdash q \rightarrow p$
$w_0 \nVdash (p \rightarrow q) \vee (q \rightarrow p)$

Satisfiability and Validity – More Examples

Example: $\neg\forall x\, p(x) \rightarrow \exists x\, \neg p(x)$ is not intuitionistically valid

See [Nerode & Shore 1997] (page 269).



$w_1 \not\Vdash p(b) \Longrightarrow w_1 \not\Vdash \forall x\, p(x)$ and $w_0 \not\Vdash \forall x\, p(x) \Longrightarrow w_0 \Vdash \neg\forall x\, p(x)$
$w_1 \Vdash p(c) \Longrightarrow w_0 \not\Vdash \neg p(c) \Longrightarrow w_0 \not\Vdash \exists x\, \neg p(x)$
Together: $w_0 \not\Vdash \neg\forall x\, p(x) \rightarrow \exists x\, \neg p(x)$

Satisfiability and Validity – More Examples

Example: $\neg(p \land \neg p)$ is intuitionistically valid

Let u be an arbitrary world.
We have to show that $v \nVdash p \land \neg p$ for all v with $(u, v) \in R$.

Assume that $v \Vdash p \land \neg p$ for the sake of contradiction.
I.e. $v \Vdash p$ and $v \Vdash \neg p$.

Then $w \nVdash p$ for all w with $(v, w) \in R$.
Due to reflexivity, $(v, v) \in R$, so $v \nVdash p$.

Contradiction!

Theorems on Intuitionistic Logic

Theorem 3.1 (Intuitionistic Disjunction/Existential Unifier).

- *If $A \vee B$ is intuitionistically valid, then either A or B is intuitionistically valid.*
- *If $\exists x\, p(x)$ is intuitionistically valid, then so is $p(c)$ for some constant c.*

Theorem 3.2 (Intuitionistic and Classical Validity).

If a formula F is valid in intuitionistic logic, then F is also valid in classical logic.

Theorem 3.3 ("Monotonicity").

For every formula F and for all worlds w, v, if $w \Vdash F$ and $R(w, v)$, then $v \Vdash F$.

Outline

- Motivation
- Syntax and Semantics
- Satisfiability & Validity
- **Sequent Calculus**
- Summary

Gentzen's Original Sequent Calculus for Intuitionistic Logic

Gentzen's orignal sequent calculus LJ for first-order intuitionistic logic [Gentzen 1935] is obtained from the classical one by restricting the succedent (right side) of all sequents to at most one formula.

- rules for disjunction of the classical calculus LK:

$$\frac{A, \Gamma \;\Rightarrow\; \Delta \qquad B, \Gamma \;\Rightarrow\; \Delta}{A \vee B, \Gamma \;\Rightarrow\; \Delta}\ \vee\text{-left}$$

$$\frac{\Gamma \;\Rightarrow\; \Delta, A, B}{\Gamma \;\Rightarrow\; \Delta, A \vee B}\ \vee\text{-right}$$

- corresponding rules in Genten's original intuitionistic calculus LJ:

$$\frac{A, \Gamma \;\Rightarrow\; C \qquad B, \Gamma \;\Rightarrow\; C}{A \vee B, \Gamma \;\Rightarrow\; C}\ \vee\text{-left}$$

$$\frac{\Gamma \;\Rightarrow\; A}{\Gamma \;\Rightarrow\; A \vee B}\ \vee\text{-right} \qquad \frac{\Gamma \;\Rightarrow\; B}{\Gamma \;\Rightarrow\; A \vee B}\ \vee\text{-right}$$

LJ — Rules for Conjunction and Disjunction

- rules for $\wedge$ (conjunction)

$$\frac{\Gamma, A, B \;\Rightarrow\; D}{\Gamma, A \wedge B \;\Rightarrow\; D}\ \wedge\text{-left} \qquad \frac{\Gamma \;\Rightarrow\; A \qquad \Gamma \;\Rightarrow\; B}{\Gamma \;\Rightarrow\; A \wedge B}\ \wedge\text{-right}$$

- rules for $\vee$ (disjunction)

$$\frac{\Gamma, A \;\Rightarrow\; D \qquad \Gamma, B \;\Rightarrow\; D}{\Gamma, A \vee B \;\Rightarrow\; D}\ \vee\text{-left}$$

$$\frac{\Gamma \;\Rightarrow\; A}{\Gamma \;\Rightarrow\; A \vee B}\ \vee\text{-right}_1 \qquad \frac{\Gamma \;\Rightarrow\; B}{\Gamma \;\Rightarrow\; A \vee B}\ \vee\text{-right}_2$$

LJ — Rules for Implication and Negation, Axiom

- rules for $\rightarrow$ (implication)

$$\frac{\Gamma, A \rightarrow B \;\Rightarrow\; A \qquad \Gamma, B \;\Rightarrow\; D}{\Gamma, A \rightarrow B \;\Rightarrow\; D} \rightarrow\text{-left} \qquad \frac{\Gamma, A \;\Rightarrow\; B}{\Gamma \;\Rightarrow\; A \rightarrow B} \rightarrow\text{-right}$$

- rules for $\neg$ (negation)

$$\frac{\Gamma, \neg A \;\Rightarrow\; A}{\Gamma, \neg A \;\Rightarrow\; D} \neg\text{-left} \qquad \frac{\Gamma, A \;\Rightarrow\;}{\Gamma \;\Rightarrow\; \neg A} \neg\text{-right}$$

- the axiom

$$\frac{}{\Gamma, A \;\Rightarrow\; A} \text{axiom}$$

LK — Rules for Universal and Existential Quantifier

- rules for $\forall$ (universal quantifier)

$$\frac{\Gamma, A[x\backslash t], \forall x\, A \;\Rightarrow\; D}{\Gamma, \forall x\, A \;\Rightarrow\; D}\ \forall\text{-left} \qquad \frac{\Gamma \;\Rightarrow\; A[x\backslash a]}{\Gamma \;\Rightarrow\; \forall x\, A}\ \forall\text{-right}$$

  - t is an arbitrary closed term
  - Eigenvariable condition for the rule $\forall$-right: a must not occur in the conclusion, i.e. in Γ or A
  - the formula $\forall x\, A$ is preserved in the premise of the rule $\forall$-left

- rules for $\exists$ (existential quantifier)

$$\frac{\Gamma, A[x\backslash a] \;\Rightarrow\; D}{\Gamma, \exists x\, A \;\Rightarrow\; D}\ \exists\text{-left} \qquad \frac{\Gamma \;\Rightarrow\; A[x\backslash t]}{\Gamma \;\Rightarrow\; \exists x\, A}\ \exists\text{-right}$$

  - t is an arbitrary closed term
  - Eigenvariable condition for the rule $\exists$-left: a must not occur in the conclusion, i.e. in Γ, D, or A
  - the formula $\exists x\, A$ is not preserved in the premise of the rule $\exists$-right

Intuitionistic Sequent Calculus – Examples

- Example 1: $q \to (p \vee q)$

$$\dfrac{\dfrac{q \;\Rightarrow\; p}{q \;\Rightarrow\; p \vee q}\ \vee\text{-right}_1}{\Rightarrow\; q \to (p \vee q)}\ \to\text{-right}$$

$$\dfrac{\dfrac{\dfrac{}{q \;\Rightarrow\; q}\ \text{ax}}{q \;\Rightarrow\; p \vee q}\ \vee\text{-right}_2}{\Rightarrow\; q \to (p \vee q)}\ \to\text{-right}$$

- Example 2: $p \vee \neg p$

$$\dfrac{\Rightarrow\; p}{\Rightarrow\; p \vee \neg p}\ \vee\text{-right}_1$$

$$\dfrac{\dfrac{p \;\Rightarrow}{\Rightarrow\; \neg p}\ \neg\text{-left}}{\Rightarrow\; p \vee \neg p}\ \vee\text{-right}_2$$

Intuitionistic Sequent Calculus – Examples

- Example 3: $\neg\neg(p \lor \neg p)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p, \neg(p \lor \neg p) \Rightarrow p}\ \text{ax}}{p, \neg(p \lor \neg p) \Rightarrow p \lor \neg p}\ \lor\text{-right}_1}{p, \neg(p \lor \neg p) \Rightarrow}\ \neg\text{-left}}{\neg(p \lor \neg p) \Rightarrow \neg p}\ \neg\text{-right}}{\neg(p \lor \neg p) \Rightarrow p \lor \neg p}\ \lor\text{-right}_2}{\neg(p \lor \neg p) \Rightarrow}\ \neg\text{-left}}{\Rightarrow \neg\neg(p \lor \neg p)}\ \neg\text{-right}
$$

Intuitionistic Sequent Calculus – More Examples

Example: $(p \to q) \lor (q \to p)$ is not intuitionistically valid

$$\frac{\Rightarrow \; p \to q}{\Rightarrow \; (p \to q) \lor (q \to p)} \lor\text{-right}_1 \qquad \frac{\Rightarrow \; q \to p}{\Rightarrow \; (p \to q) \lor (q \to p)} \lor\text{-right}_2$$

Example: $\neg \forall x\, p(x) \to \exists x\, \neg p(x)$ is not intuitionistically valid

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{p(c), \neg \forall x\, p(x) \;\Rightarrow\; p(a)}{p(c), \neg \forall x\, p(x) \;\Rightarrow\; \forall x\, p(x)} \forall\text{-right}}{p(c), \neg \forall x\, p(x) \;\Rightarrow\;} \neg\text{-left}}{\neg \forall x\, p(x) \;\Rightarrow\; \neg p(c)} \neg\text{-right}}{\neg \forall x\, p(x) \;\Rightarrow\; \exists x\, \neg p(x)} \exists\text{-right}}{\Rightarrow\; \neg \forall x\, p(x) \to \exists x\, \neg p(x)} \to\text{-right}$$

Gödel's Translation from Intuitionistic to Modal Logic

Definition 4.1 (Gödel's Translation).

Gödel's translation T_G for embedding propositional intuitionistic logic into the modal logic S4 is defined as follows.

1. $T_G(p) = \Box p$ *iff* p *is an atomic formula*
2. $T_G(A \wedge B) = T_G(A) \wedge T_G(B)$
3. $T_G(A \vee B) = T_G(A) \vee T_G(B)$
4. $T_G(A \rightarrow B) = \Box(T_G(A) \rightarrow T_G(B))$
5. $T_G(\neg A) = \Box(\neg T_G(A))$

Theorem 4.1 (Gödel's Translation).

A formula F is valid in propositional intuitionistic logic iff the formula $T_G(F)$ is valid in the modal logic S4.

Outline

Summary

- in intuitionistic logic the law of excluded middle is not valid; non-constructive existence proofs are also not allowed
- intuit. logic has applications in program synthesis and verification
- the Kripke semantics of intuitionistic logic uses a set of worlds and an accessibility relation between these worlds
- in each world the classical semantics holds, but the semantics of $\neg$, $\rightarrow$ and $\forall$ is defined with respect to the set of worlds
- validity in propositional intuitionistic logic is decidable, but *PSPACE*-complete [Statman 1979] (*PSPACE*: polynomial space)