IN3070/4070 – Logic – Autumn 2020

Lecture 13: Intuitionistic Logic

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12th November 2020

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Today's Plan

- Motivation
- Syntax and Semantics
- Satisfiability & Validity
- Sequent Calculus
- Summary

Outline

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Proof search calculi

- natural deduction, sequent, tableau and connection calculi

A Non-Constructive Proof

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There exist a solution of $x^y = z$ such that x and y are irrational numbers and z is a rational number.

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Theorem (classically) proven, but we don't know which case holds.

Intuitionism

- is it reasonable to claim the existence of a number n with some property without being able to produce n? (e.g. prove $\exists x\, p(x)$ by showing that its negation $\forall x\, \neg p(x)$ leads to a contradiction)
- is it reasonable to accept the validity of $A \lor B$ without knowing whether A or B is valid? – is it reasonable to claim the existence of function f without providing a way to calculate f?

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The mathematician L.E.J. Brouwer

- rejected much of early twentieth century mathematics (dominated by, e.g., Frege and Hilbert)
- in his paper "The untrustworthiness of the principles of logic" he challenged the belief that the rules of classical logic are valid
- rejected the validity of the "law of excluded middle" $A \vee \neg A$ and non-constructive existence proofs

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- first formal system/logic that attempts to capture Brouwer's logic was given 1930 by his student Arend Heyting
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Constructive definition of computability

- write a "logical" specification of a program; if there is a proof for the specification, the program that satisfies the specification can be extracted from the proof ("proof as programs")
- for example the proof of $\forall x\, \exists y\, p(x, y)$ contains the construction of an algorithm for computing a value of y from one for x

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Semantics – Classical Logic

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Definition 2.1 (Classical Interpretation).

A classical interpretation (or structure) is a tuple $\mathcal{I}_C = (D, \iota)$ where

- *D is a non-empty set, the domain*
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  - *constant $a \in \mathcal{F}^0$ an element $a^\iota \in D$*
  - *function symbol $f \in \mathcal{F}^n$ with $n>0$ a function $f^\iota : D^n \to D$*
  - *propositional variable $p \in \mathcal{P}^0$ a truth value $p^\iota \in \{T, F\}$*
  - *predicate symbol $p \in \mathcal{P}^n$ with $n>0$ a relation $p^\iota \subseteq D^n$*

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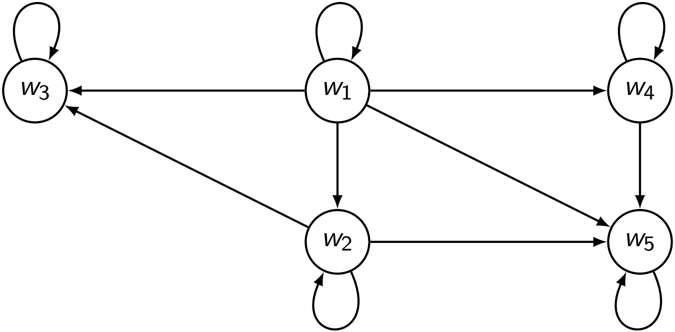
An intuitionistic frame $F_J = (W, R)$ is a Kripke frame (W, R) with a reflexive and transitive accessibility relation R.

($R \subseteq W \times W$ is reflexive iff $(w_1, w_1) \in R$ for all $w_1 \in W$; R is transitive iff for all $w_1, w_2, w_3 \in W$: if $(w_1, w_2) \in R$ and $(w_2, w_3) \in R$ then $(w_1, w_3) \in R$)

Intuitionistic Frame – Example

Example: $F'_J = (W', R')$ with $W' = \{w_1, w_2, w_3, w_4, w_5\}$ and

$$R' = \{(w_1, w_1), (w_2, w_2), (w_3, w_3), (w_4, w_4), (w_5, w_5), (w_1, w_2), (w_2, w_3), (w_1, w_4), (w_4, w_5), (w_2, w_5) (w_1, w_3), (w_1, w_5)\}$$



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   a. $a^{\iota^w} = a^{\iota^v}$ *for every constant* a
   b. $f^{\iota^w} \subseteq f^{\iota^v}$ *for every function* f
   c. $p^{\iota^w} = T$ *implies* $p^{\iota^v} = T$ *for every* $p \in \mathcal{P}^0$
   d. $p^{\iota^w} \subseteq p^{\iota^v}$ *for every predicate* $p \in \mathcal{P}^n$ *with* $n > 0$

   ($g \subseteq h$ *holds for* g *and* h *iff* $g(x) = h(x)$ *for all* x *of the domain of* g*)*

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- $w \Vdash \exists x A$ iff $w \Vdash A[x \backslash d]$ for some $d \in D^w$

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- $w \Vdash A \vee B$ *iff* $w \Vdash A$ *or* $w \Vdash B$
- $w \Vdash A \to B$ *iff* $v \Vdash A$ *implies* $v \Vdash B$ *for all* $v \in W$ *with* $(w, v) \in R$
- $w \Vdash \exists x A$ *iff* $w \Vdash A[x \backslash d]$ *for some* $d \in D^w$
- $w \Vdash \forall x A$ *iff* $v \Vdash A[x \backslash d]$ *for all* $d \in D^v$ *for all* $v \in W$ *with* $(w, v) \in R$

Outline

- Motivation
- Syntax and Semantics
- Satisfiability & Validity
- Sequent Calculus
- Summary

Satisfiability and Validity

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Let G be a closed (first-order) formula.

- *Let $\mathcal{I}_J$ be an intuitionistic interpretation. $\mathcal{I}_J$ is an intuitionistic model for G, denoted $\mathcal{I}_J \models G$, iff $v_{\mathcal{I}_J}(w, G) = T$ for all $w \in W$.*

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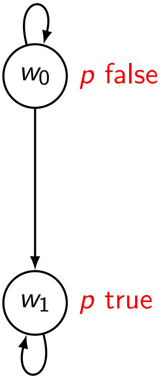
Satisfiability and Validity – Examples

- $F_1 \equiv p \vee \neg p$



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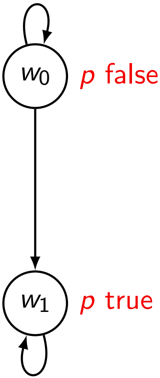

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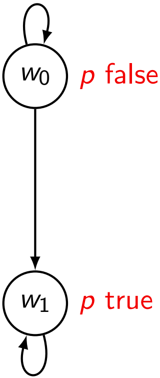

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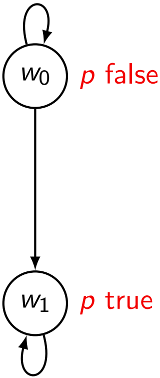

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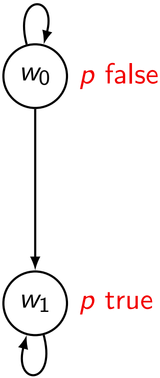

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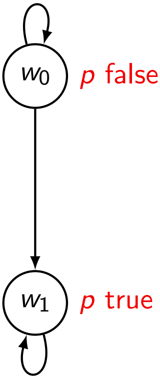

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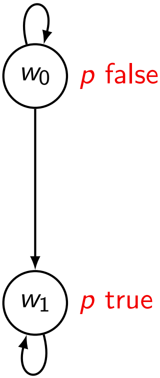

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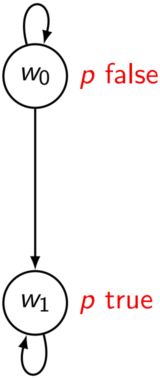

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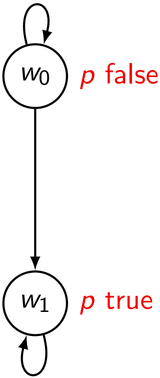

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  $\rightsquigarrow$ F_2 is *true* in w_0 (and w_1)

Satisfiability and Validity – More Examples

Example: $(p \to q) \lor (q \to p)$ is not intuitionistically valid

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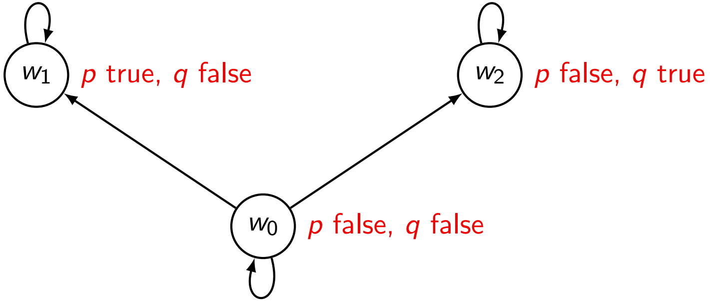
See [Nerode & Shore 1997] (page 269).



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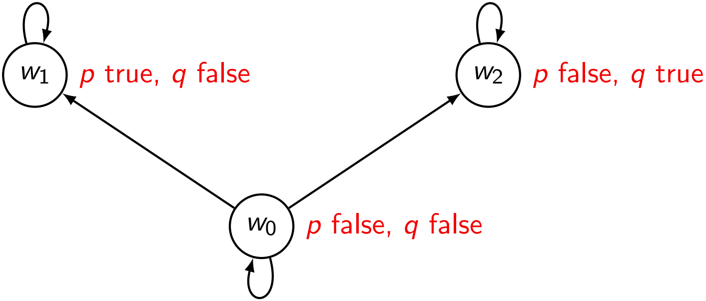

$w_1 \Vdash p,\ w_1 \nVdash q$

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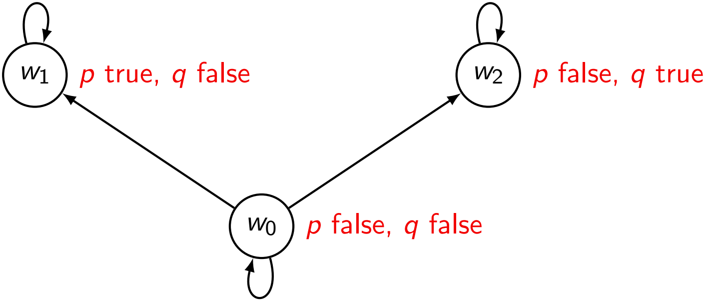

$w_1 \Vdash p,\ w_1 \nVdash q \Longrightarrow w_0 \nVdash p \to q$

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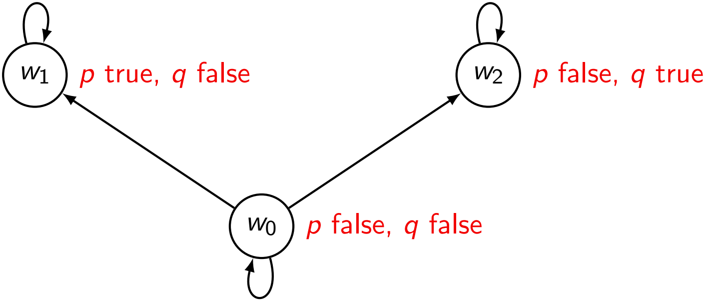

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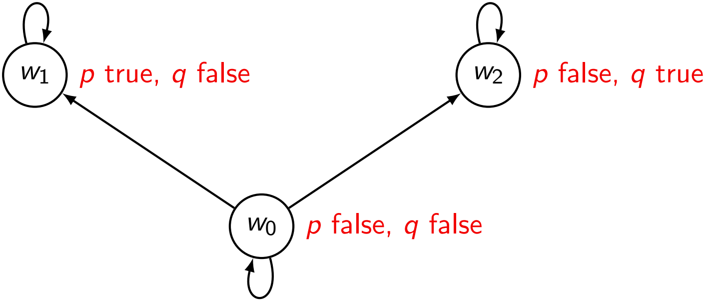

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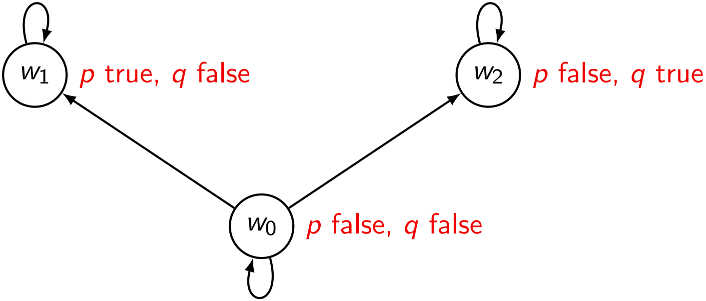

$w_1 \Vdash p,\ w_1 \nVdash q \implies w_0 \nVdash p \to q$
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$w_0 \nVdash (p \to q) \lor (q \to p)$

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Example: $\neg\forall x\, p(x) \rightarrow \exists x\, \neg p(x)$ is not intuitionistically valid

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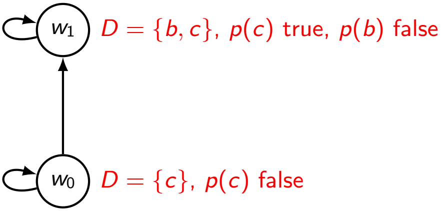

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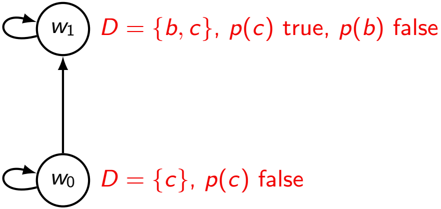

$w_1 \not\Vdash p(b)$

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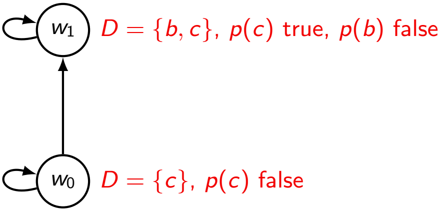

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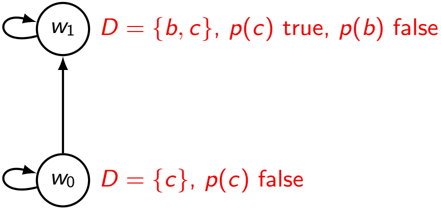

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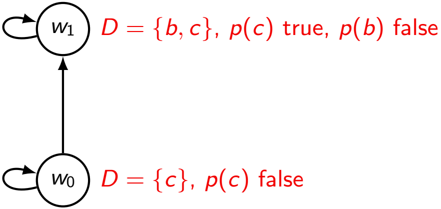

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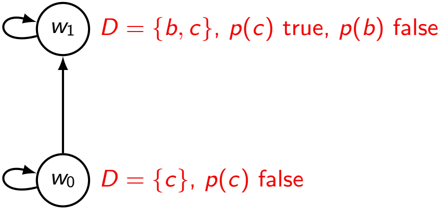

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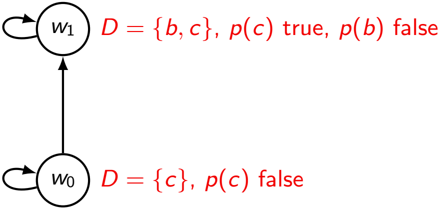

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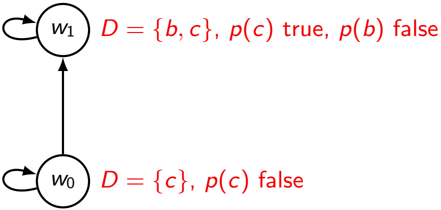

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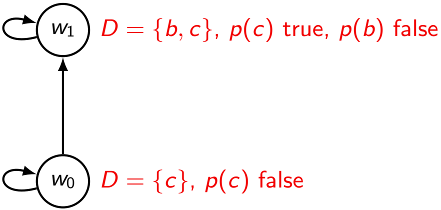

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Together: $w_0 \nVdash \neg\forall x\, p(x) \to \exists x\, \neg p(x)$

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Due to reflexivity, $(v, v) \in R$, so $v \nVdash p$.

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Contradiction!

Theorems on Intuitionistic Logic

Theorem 3.1 (Intuitionistic Disjunction/Existential Unifier).

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Theorem 3.3 ("Monotonicity").

For every formula F and for all worlds w, v, if $w \Vdash F$ and $R(w, v)$, then $v \Vdash F$.

Outline

Gentzen's Original Sequent Calculus for Intuitionistic Logic

Gentzen's orignal sequent calculus LJ for first-order intuitionistic logic [Gentzen 1935] is obtained from the classical one by restricting the succedent (right side) of all sequents to at most one formula.

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- rules for disjunction of the classical calculus LK:

$$\frac{A, \Gamma \;\Rightarrow\; \Delta \qquad B, \Gamma \;\Rightarrow\; \Delta}{A \vee B, \Gamma \;\Rightarrow\; \Delta}\ \vee\text{-left}$$

$$\frac{\Gamma \;\Rightarrow\; \Delta, A, B}{\Gamma \;\Rightarrow\; \Delta, A \vee B}\ \vee\text{-right}$$

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- corresponding rules in Genten's original intuitionistic calculus LJ:

$$\frac{A, \Gamma \;\Rightarrow\; C \qquad B, \Gamma \;\Rightarrow\; C}{A \vee B, \Gamma \;\Rightarrow\; C}\ \vee\text{-left}$$

$$\frac{\Gamma \;\Rightarrow\; A}{\Gamma \;\Rightarrow\; A \vee B}\ \vee\text{-right} \qquad \frac{\Gamma \;\Rightarrow\; B}{\Gamma \;\Rightarrow\; A \vee B}\ \vee\text{-right}$$

LJ — Rules for Conjunction and Disjunction

- rules for $\wedge$ (conjunction)

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$$\frac{\Gamma \;\Rightarrow\; A}{\Gamma \;\Rightarrow\; A \vee B}\ \vee\text{-right}_1$$

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LJ — Rules for Implication and Negation, Axiom

- rules for $\rightarrow$ (implication)

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$$\frac{\Gamma, A \rightarrow B \ \Rightarrow \ A \qquad \Gamma, B \ \Rightarrow \ D}{\Gamma, A \rightarrow B \ \Rightarrow \ D} \rightarrow\text{-left}$$

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$$\frac{\Gamma, \neg A \;\Rightarrow\; A}{\Gamma, \neg A \;\Rightarrow\; D}\neg\text{-left}$$

LJ — Rules for Implication and Negation, Axiom

- rules for $\to$ (implication)

$$\frac{\Gamma, A \to B \;\Rightarrow\; A \qquad \Gamma, B \;\Rightarrow\; D}{\Gamma, A \to B \;\Rightarrow\; D} \to\text{-left} \qquad \frac{\Gamma, A \;\Rightarrow\; B}{\Gamma \;\Rightarrow\; A \to B} \to\text{-right}$$

- rules for $\neg$ (negation)

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$$\dfrac{}{\Gamma, A \;\Rightarrow\; A}\ \text{axiom}$$

LK — Rules for Universal and Existential Quantifier

- rules for ∀ (universal quantifier)

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$$\frac{\Gamma, A[x\backslash t], \forall x\, A \;\Rightarrow\; D}{\Gamma, \forall x\, A \;\Rightarrow\; D}\;\forall\text{-left}$$

LK — Rules for Universal and Existential Quantifier

- rules for $\forall$ (universal quantifier)

$$\frac{\Gamma, A[x\backslash t], \forall x\, A \;\Rightarrow\; D}{\Gamma, \forall x\, A \;\Rightarrow\; D}\,\forall\text{-left} \qquad \frac{\Gamma \;\Rightarrow\; A[x\backslash a]}{\Gamma \;\Rightarrow\; \forall x\, A}\,\forall\text{-right}$$

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  - t is an arbitrary closed term
  - Eigenvariable condition for the rule $\forall$-right: a must not occur in the conclusion, i.e. in Γ or A
  - the formula $\forall x\, A$ is preserved in the premise of the rule $\forall$-left

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- rules for $\exists$ (existential quantifier)

$$\frac{\Gamma, A[x\backslash a] \;\Rightarrow\; D}{\Gamma, \exists x\, A \;\Rightarrow\; D}\ \exists\text{-left} \qquad \frac{\Gamma \;\Rightarrow\; A[x\backslash t]}{\Gamma \;\Rightarrow\; \exists x\, A}\ \exists\text{-right}$$

  - t is an arbitrary closed term
  - Eigenvariable condition for the rule $\exists$-left: a must not occur in the conclusion, i.e. in Γ, D, or A
  - the formula $\exists x\, A$ is not preserved in the premise of the rule $\exists$-right

Intuitionistic Sequent Calculus – Examples

- Example 1: $q \to (p \lor q)$

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$$\Rightarrow \; q \to (p \lor q)$$

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$$\frac{q \;\Rightarrow\; p \lor q}{\;\Rightarrow\; q \to (p \lor q)}\ \to\text{-right}$$

Intuitionistic Sequent Calculus – Examples

- Example 1: $q \to (p \lor q)$

$$\dfrac{\dfrac{q \;\Rightarrow\; p}{q \;\Rightarrow\; p \lor q}\ \lor\text{-right}_1}{\;\Rightarrow\; q \to (p \lor q)}\ \to\text{-right}$$

Intuitionistic Sequent Calculus – Examples

- Example 1: $q \to (p \vee q)$

$$\dfrac{\dfrac{q \;\Rightarrow\; p}{q \;\Rightarrow\; p \vee q}\ \vee\text{-right}_1}{\Rightarrow\; q \to (p \vee q)}\ \to\text{-right}$$

$$\Rightarrow\; q \to (p \vee q)$$

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- Example 2: $p \lor \neg p$

Intuitionistic Sequent Calculus – Examples

- Example 1: $q \to (p \vee q)$

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- Example 2: $p \vee \neg p$

$$\Rightarrow\; p \vee \neg p$$

Intuitionistic Sequent Calculus – Examples

- Example 1: $q \rightarrow (p \vee q)$

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$$\dfrac{\dfrac{\dfrac{}{q \;\Rightarrow\; q}\ \text{ax}}{q \;\Rightarrow\; p \vee q}\ \vee\text{-right}_2}{\Rightarrow\; q \rightarrow (p \vee q)}\ \rightarrow\text{-right}$$

- Example 2: $p \vee \neg p$

$$\dfrac{\Rightarrow\; p}{\Rightarrow\; p \vee \neg p}\ \vee\text{-right}_1$$

Intuitionistic Sequent Calculus – Examples

- Example 1: $q \to (p \vee q)$

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Intuitionistic Sequent Calculus – Examples

- Example 1: $q \to (p \lor q)$

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$$\dfrac{\dfrac{\dfrac{}{q \;\Rightarrow\; q}\ \text{ax}}{q \;\Rightarrow\; p \lor q}\ \lor\text{-right}_2}{\Rightarrow\; q \to (p \lor q)}\ \to\text{-right}$$

- Example 2: $p \lor \neg p$

$$\dfrac{\Rightarrow\; p}{\Rightarrow\; p \lor \neg p}\ \lor\text{-right}_1$$

$$\dfrac{\dfrac{p \;\Rightarrow}{\Rightarrow\; \neg p}\ \neg\text{-left}}{\Rightarrow\; p \lor \neg p}\ \lor\text{-right}_2$$

Intuitionistic Sequent Calculus – Examples

- Example 3: $\neg\neg(p \lor \neg p)$

Intuitionistic Sequent Calculus – Examples

- Example 3: $\neg\neg(p \vee \neg p)$

$$\Rightarrow \neg\neg(p \vee \neg p)$$

Intuitionistic Sequent Calculus – Examples

- Example 3: $\neg\neg(p \vee \neg p)$

$$\dfrac{\neg(p \vee \neg p) \;\Rightarrow}{\Rightarrow\; \neg\neg(p \vee \neg p)}\;\neg\text{-right}$$

Intuitionistic Sequent Calculus – Examples

- Example 3: $\neg\neg(p \vee \neg p)$

$$
\dfrac{\dfrac{\neg(p \vee \neg p) \;\Rightarrow\; p \vee \neg p}{\neg(p \vee \neg p) \;\Rightarrow\;}\ \neg\text{-left}}{\Rightarrow\; \neg\neg(p \vee \neg p)}\ \neg\text{-right}
$$

Intuitionistic Sequent Calculus – Examples

- Example 3: $\neg\neg(p \lor \neg p)$

$$
\dfrac{\dfrac{\dfrac{\neg(p \lor \neg p) \;\Rightarrow\; \neg p}{\neg(p \lor \neg p) \;\Rightarrow\; p \lor \neg p}\,\lor\text{-right}_2}{\neg(p \lor \neg p) \;\Rightarrow}\,\neg\text{-left}}{\Rightarrow\; \neg\neg(p \lor \neg p)}\,\neg\text{-right}
$$

Intuitionistic Sequent Calculus – Examples

- Example 3: $\neg\neg(p \vee \neg p)$

$$\dfrac{\dfrac{\dfrac{\dfrac{p, \neg(p \vee \neg p) \;\Rightarrow\;}{\neg(p \vee \neg p) \;\Rightarrow\; \neg p}\;\neg\text{-right}}{\neg(p \vee \neg p) \;\Rightarrow\; p \vee \neg p}\;\vee\text{-right}_2}{\neg(p \vee \neg p) \;\Rightarrow\;}\;\neg\text{-left}}{\;\Rightarrow\; \neg\neg(p \vee \neg p)}\;\neg\text{-right}$$

Intuitionistic Sequent Calculus – Examples

- Example 3: $\neg\neg(p \vee \neg p)$

$$
\begin{array}{l}
p, \neg(p \vee \neg p) \Rightarrow p \vee \neg p \\
\hline
p, \neg(p \vee \neg p) \Rightarrow \qquad \neg\text{-left} \\
\hline
\neg(p \vee \neg p) \Rightarrow \neg p \qquad \neg\text{-right} \\
\hline
\neg(p \vee \neg p) \Rightarrow p \vee \neg p \qquad \vee\text{-right}_2 \\
\hline
\neg(p \vee \neg p) \Rightarrow \qquad \neg\text{-left} \\
\hline
\Rightarrow \neg\neg(p \vee \neg p) \qquad \neg\text{-right}
\end{array}
$$

Intuitionistic Sequent Calculus – Examples

- Example 3: $\neg\neg(p \vee \neg p)$

$$
\begin{array}{c}
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p, \neg(p \vee \neg p) \Rightarrow p}{p, \neg(p \vee \neg p) \Rightarrow p \vee \neg p}\ \vee\text{-right}_1}{p, \neg(p \vee \neg p) \Rightarrow}\ \neg\text{-left}}{\neg(p \vee \neg p) \Rightarrow \neg p}\ \neg\text{-right}}{\neg(p \vee \neg p) \Rightarrow p \vee \neg p}\ \vee\text{-right}_2}{\neg(p \vee \neg p) \Rightarrow}\ \neg\text{-left}}{\Rightarrow \neg\neg(p \vee \neg p)}\ \neg\text{-right}
\end{array}
$$

Intuitionistic Sequent Calculus – Examples

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$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p, \neg(p \vee \neg p) \Rightarrow p}\ \text{ax}}{p, \neg(p \vee \neg p) \Rightarrow p \vee \neg p}\ \vee\text{-right}_1}{p, \neg(p \vee \neg p) \Rightarrow}\ \neg\text{-left}}{\neg(p \vee \neg p) \Rightarrow \neg p}\ \neg\text{-right}}{\neg(p \vee \neg p) \Rightarrow p \vee \neg p}\ \vee\text{-right}_2}{\neg(p \vee \neg p) \Rightarrow}\ \neg\text{-left}}{\Rightarrow \neg\neg(p \vee \neg p)}\ \neg\text{-right}
$$

Intuitionistic Sequent Calculus – More Examples

Example: $(p \to q) \lor (q \to p)$ is not intuitionistically valid

Intuitionistic Sequent Calculus – More Examples

Example: $(p \to q) \lor (q \to p)$ is not intuitionistically valid

$$\dfrac{\Rightarrow\ p \to q}{\Rightarrow\ (p \to q) \lor (q \to p)}\ \lor\text{-right}_1$$

Intuitionistic Sequent Calculus – More Examples

Example: $(p \to q) \lor (q \to p)$ is not intuitionistically valid

$$\dfrac{\Rightarrow \;\; p \to q}{\Rightarrow \;\; (p \to q) \lor (q \to p)}\ \lor\text{-right}_1 \qquad \dfrac{\Rightarrow \;\; q \to p}{\Rightarrow \;\; (p \to q) \lor (q \to p)}\ \lor\text{-right}_2$$

Intuitionistic Sequent Calculus – More Examples

Example: $(p \to q) \lor (q \to p)$ is not intuitionistically valid

$$\frac{\Rightarrow \;\; p \to q}{\Rightarrow \;\; (p \to q) \lor (q \to p)}\ \lor\text{-right}_1 \qquad \frac{\Rightarrow \;\; q \to p}{\Rightarrow \;\; (p \to q) \lor (q \to p)}\ \lor\text{-right}_2$$

Example: $\neg \forall x\, p(x) \to \exists x\, \neg p(x)$ is not intuitionistically valid

Intuitionistic Sequent Calculus – More Examples

Example: $(p \to q) \lor (q \to p)$ is not intuitionistically valid

$$\dfrac{\Rightarrow \; p \to q}{\Rightarrow \; (p \to q) \lor (q \to p)} \lor\text{-right}_1 \qquad \dfrac{\Rightarrow \; q \to p}{\Rightarrow \; (p \to q) \lor (q \to p)} \lor\text{-right}_2$$

Example: $\neg \forall x \, p(x) \to \exists x \, \neg p(x)$ is not intuitionistically valid

$$\Rightarrow \; \neg \forall x \, p(x) \to \exists x \, \neg p(x)$$

Intuitionistic Sequent Calculus – More Examples

Example: $(p \to q) \vee (q \to p)$ is not intuitionistically valid

$$\dfrac{\Rightarrow\ p \to q}{\Rightarrow\ (p \to q) \vee (q \to p)}\ \vee\text{-right}_1 \qquad \dfrac{\Rightarrow\ q \to p}{\Rightarrow\ (p \to q) \vee (q \to p)}\ \vee\text{-right}_2$$

Example: $\neg \forall x\, p(x) \to \exists x\, \neg p(x)$ is not intuitionistically valid

$$\dfrac{\neg \forall x\, p(x)\ \Rightarrow\ \exists x\, \neg p(x)}{\Rightarrow\ \neg \forall x\, p(x) \to \exists x\, \neg p(x)}\ \to\text{-right}$$

Intuitionistic Sequent Calculus – More Examples

Example: $(p \to q) \lor (q \to p)$ is not intuitionistically valid

$$\frac{\Rightarrow\ p \to q}{\Rightarrow\ (p \to q) \lor (q \to p)}\ \lor\text{-right}_1 \qquad \frac{\Rightarrow\ q \to p}{\Rightarrow\ (p \to q) \lor (q \to p)}\ \lor\text{-right}_2$$

Example: $\neg \forall x\, p(x) \to \exists x\, \neg p(x)$ is not intuitionistically valid

$$\dfrac{\dfrac{\dfrac{p(c), \neg \forall x\, p(x)\ \Rightarrow}{\neg \forall x\, p(x)\ \Rightarrow\ \neg p(c)}\ \neg\text{-right}}{\neg \forall x\, p(x)\ \Rightarrow\ \exists x\, \neg p(x)}\ \exists\text{-right}}{\Rightarrow\ \neg \forall x\, p(x) \to \exists x\, \neg p(x)}\ \to\text{-right}$$

Intuitionistic Sequent Calculus – More Examples

Example: $(p \to q) \lor (q \to p)$ is not intuitionistically valid

$$\frac{\Rightarrow\ p \to q}{\Rightarrow\ (p \to q) \lor (q \to p)}\ \lor\text{-right}_1 \qquad \frac{\Rightarrow\ q \to p}{\Rightarrow\ (p \to q) \lor (q \to p)}\ \lor\text{-right}_2$$

Example: $\neg\forall x\, p(x) \to \exists x\, \neg p(x)$ is not intuitionistically valid

$$\cfrac{\cfrac{\cfrac{\cfrac{p(c), \neg\forall x\, p(x)\ \Rightarrow\ \forall x\, p(x)}{p(c), \neg\forall x\, p(x)\ \Rightarrow}\ \neg\text{-left}}{\neg\forall x\, p(x)\ \Rightarrow\ \neg p(c)}\ \neg\text{-right}}{\neg\forall x\, p(x)\ \Rightarrow\ \exists x\, \neg p(x)}\ \exists\text{-right}}{\Rightarrow\ \neg\forall x\, p(x) \to \exists x\, \neg p(x)}\ \to\text{-right}$$

Intuitionistic Sequent Calculus – More Examples

Example: $(p \to q) \lor (q \to p)$ is not intuitionistically valid

$$\dfrac{\Rightarrow \; p \to q}{\Rightarrow \; (p \to q) \lor (q \to p)}\ \lor\text{-right}_1 \qquad \dfrac{\Rightarrow \; q \to p}{\Rightarrow \; (p \to q) \lor (q \to p)}\ \lor\text{-right}_2$$

Example: $\neg \forall x\, p(x) \to \exists x\, \neg p(x)$ is not intuitionistically valid

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p(c), \neg\forall x\, p(x) \;\Rightarrow\; p(a)}{p(c), \neg\forall x\, p(x) \;\Rightarrow\; \forall x\, p(x)}\ \forall\text{-right}}{p(c), \neg\forall x\, p(x) \;\Rightarrow}\ \neg\text{-left}}{\neg\forall x\, p(x) \;\Rightarrow\; \neg p(c)}\ \neg\text{-right}}{\neg\forall x\, p(x) \;\Rightarrow\; \exists x\, \neg p(x)}\ \exists\text{-right}}{\Rightarrow\; \neg\forall x\, p(x) \to \exists x\, \neg p(x)}\ \to\text{-right}$$

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A formula F is valid in propositional intuitionistic logic iff the formula $T_G(F)$ is valid in the modal logic S4.

Outline

- Motivation
- Syntax and Semantics
- Satisfiability & Validity
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- Summary

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