Grading Guidelines
IN3070/IN4070
Autumn 2021

December 2, 2021

The maximum number of marks for the whole exam was 100.

The minimum number of marks required for each grade will be published after the exam was graded.

Question 1 – Sequent Calculus

Prove the **validity** of the following formulae using the given calculus.

Upload a file with your proofs

A) $(\neg p \rightarrow (q \vee r)) \rightarrow ((\neg p \rightarrow q) \vee (\neg p \rightarrow r))$ using propositional LK [6 marks]

B) $(\forall x\, (p(x) \rightarrow q)) \rightarrow ((\exists x\, p(x)) \rightarrow q)$ using first-order LK [7 marks]

C) $\forall x((q(x) \wedge (\exists y p(x, y))) \rightarrow (\exists y(q(x) \wedge p(x, y))))$ using first-order LK [7 marks]

D) $(\Box(p \rightarrow q)) \rightarrow (\Diamond p \rightarrow \Diamond q)$ using the modal sequent calculus [5 marks]

E) $\Box\Box(p \vee q) \rightarrow \Diamond\Box p \vee \Box\Diamond q$ using the modal sequent calculus [8 marks]

Question 2 – Resolution

Prove that the following formula is valid, using the resolution calculus

$$(\forall x \exists y\, p(x, y)) \wedge (\forall x \forall y\, (p(x, y) \rightarrow p(y, x))) \rightarrow (\forall y \exists x\, p(x, y))$$

Remember that resolution is a refutation calculus, i.e. you can derive that a set of clauses is unsatisfiable.

Arriving at a correct set of clauses: 10 credits

Correct resolution proof: 5 credits

Question 3 – An inductive proof

Here is an inductively defined function f that takes a propositional logic formula and returns a different one:

- $f(A) := \neg A$ if A is an atomic formula
- $f(\neg A) := A$
- $f(A \wedge B) := f(A) \vee f(B)$
- $f(A \vee B) := f(A) \wedge f(B)$
- $f(A \to B) := A \wedge f(B)$

Intuitively, f negates the formula, but then moves the negation inwards until it either meets another negation (the second case) or an atom (the first case).

Show by structural induction on A that $f(A)$ is logically equivalent to $\neg A$ for all propositional formulae A.

[10 credits]

Question 4 – Multiple Choice Mix

A) If A is a valid formula of propositional logic, then $\Diamond A$ is valid in modal logic D [2 marks]

B) Are these terms unifiable? $f(x, g(x))$ and $f(h(y), g(y))$

C) Are these terms unifiable? $f(h(x), y, g(x))$ and $f(h(y), z, g(z))$

D) The order of resolution steps in a resolution proof matters: starting with a particular resolution might mean that no proof can found even though a proof could be found by starting differently.

E) The same statement as the last, but for SLD resolution.

Question 5 – Intuitionistic Logic

Consider the formulas:

$$A = (\neg\exists\neg p(x)) \to (\forall x \neg\neg p(x)) \quad (1)$$

$$B = (\neg(p \wedge q)) \to (\neg p \vee \neg q) \quad (2)$$

A) Show that A is valid in intuitionistic logic using the LJ sequent calculus [8 credits]

B) Show that B is not valid in intuitionistic logic using the model semantics. I.e. construct an intuitionistic structure with a world w such that $w \Vdash \neg(p \wedge q)$ but $w \nVdash \neg p \vee \neg q$. [10 credits]

Question 6 – Soundness for Modal Sequents

In the lecture, we proved the soundness of propositional and first order LK by showing that all LK rules preserve «falsifiability upwards.»

It is possible to prove the soundness of our labeled sequent calculus for the modal logic K in a similar way. For this, we have to define what it means for an interpretation to falsify a labeled sequent.

As a reminder, the labeled sequents the calculus works with can contain

- labeled formulae $u : A$ where A is a formula and u is a label taken from some set of labels $\mathcal{L}$, and
- accessibility formulae uRv for labels $u, v \in \mathcal{L}$. These only occur in the antecedent Γ.

Definition 1: Given a set of labels $\mathcal{L}$, and a Kripke frame (W, R), a *label assignment* is a function $\alpha : \mathcal{L} \to W$ that selects some world $\alpha(u)$ for every label $u \in \mathcal{L}$.

Definition 2: A modal interpretation $\mathcal{I}_M{:=}((W, R), \{\mathcal{I}(w)\}_{w \in W})$ falsifies a labeled sequent $\Gamma \Rightarrow \Delta$ if there is a label assignment α such that:

- $\langle \alpha(u), \alpha(v) \rangle \in R$ for every accessibility formula $uRv \in \Gamma$,
- $\alpha(u) \Vdash A$ in $\mathcal{I}_M$ for every labeled formula $u : A \in \Gamma$,
- $\alpha(u) \nVdash A$ in $\mathcal{I}_M$ for every labeled formula $u : A \in \Delta$,

The condition of preserving falsifiability upwards is then the same as before: if the conclusion of a rule is falsifiable, then at least one of the premises is falsifiable.

For instance, to see that $\wedge$-left,

$$\frac{u : A, u : B, \Gamma \Rightarrow \Delta}{u : A \wedge B, \Gamma \Rightarrow \Delta}$$

preserves falsifiability upwards also in the labeled calculus, let $\mathcal{I}_M{:=}((W, R), \{\mathcal{I}(w)\}_{w \in W})$ be a modal interpretation that falsifies the conclusion with label assignment α. Def. 2 tells us that $\alpha(u) \Vdash A \wedge B$. The model semantics for modal logic then tells us that $\alpha(u) \Vdash A$ and $\alpha(u) \Vdash B$. Since the formulae in Γ

and Δ are unchanged, all formulas in the new antecedent are forced at the worlds indicated by α, so $\mathcal{I}_M$ also falsifies the premise, using the same label assignment α.

A) Show in a similar way that the rule $\Box$-left:

$$\frac{v : A, u : \Box A, uRv, \Gamma \Rightarrow \Delta}{u : \Box A, uRv, \Gamma \Rightarrow \Delta}$$

preserves falsifiability upwards [6 marks]

Hint: you can use the same falsifying interpretation and label assignment for the conclusion and the premise.

B) Show in a similar way that the rule $\Diamond$-left:

$$\frac{v : A, uRv, \Gamma \Rightarrow \Delta}{u : \Diamond A, \Gamma \Rightarrow \Delta} \quad \text{where } v \text{ is a fresh label}$$

preserves falsifiability upwards [8 marks]

Hint: you can use the same falsifying interpretation for the conclusion and the premise. You will need to modify the label assignment to deal with the new label.