Grading Guidelines
IN3070/IN4070
Autumn 2021

January 4, 2022

The maximum number of marks for the whole exam was 100.

The minimum number of marks required for each grade was as follows: For E, min. 40. For D, min. 50. For C, min. 60. For B, min. 77. For A, min. 89.

These boundaries are based on an alignment of the delivered work with the grade definitions given here:

`https://www.uio.no/studier/eksamen/karakterer/`

The final grade for each candidate is based on an evaluation of the delivered work as a whole, and may therefore in some cases deviate from the boundaries given above.

Question 1 – Sequent Calculus

Prove the **validity** of the following formulae using the given calculus.

Upload a file with your proofs

A) $(\neg p \to (q \vee r)) \to ((\neg p \to q) \vee (\neg p \to r))$ using propositional LK [6 marks]

Answer:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$p \Rightarrow p,\, q,\, r$}
\RightLabel{$\neg$-r}
\UnaryInfC{$\Rightarrow \neg p,\, p,\, q,\, r$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$q \Rightarrow p,\, q,\, r$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$r \Rightarrow p,\, q,\, r$}
\RightLabel{$\vee$-l}
\BinaryInfC{$q \vee r \Rightarrow p,\, q,\, r$}
\RightLabel{$\to$-l}
\BinaryInfC{$\neg p \to (q \vee r) \Rightarrow p,\, q,\, r$}
\RightLabel{$\neg$-l}
\UnaryInfC{$\neg p,\, \neg p \to (q \vee r) \Rightarrow q,\, r$}
\RightLabel{$\to$-r}
\UnaryInfC{$\neg p,\, \neg p \to (q \vee r) \Rightarrow q,\, \neg p \to r$}
\RightLabel{$\to$-r}
\UnaryInfC{$\neg p \to (q \vee r) \Rightarrow \neg p \to q,\, \neg p \to r$}
\RightLabel{$\vee$-r}
\UnaryInfC{$\neg p \to (q \vee r) \Rightarrow (\neg p \to q) \vee (\neg p \to r)$}
\RightLabel{$\to$-r}
\UnaryInfC{$\Rightarrow \ (\neg p \to (q \vee r)) \to ((\neg p \to q) \vee (\neg p \to r))$}
\end{prooftree}
$$

Grading: The order of rule applications can differ. It is also possible to close a branch with $\neg p$ on both sides, saving two rule applications. But there have to be three branches. 2 marks for each. Total 1 credit subtracted for missing labels.

B) $(\forall x\,(p(x) \to q)) \to ((\exists x\, p(x)) \to q)$ using first-order LK [7 marks]

Answer:

\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$p(c),\, \forall\cdots \Rightarrow p(c),\, q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$q,\, p(c),\, \forall\cdots \Rightarrow q$}
\RightLabel{$\to$-l}
\BinaryInfC{$p(c) \to q,\, p(c),\, \forall\cdots \Rightarrow q$}
\RightLabel{$\forall$-l}
\UnaryInfC{$\forall x\,(p(x) \to q),\, p(c) \Rightarrow q$}
\RightLabel{$\exists$-l}
\UnaryInfC{$\forall x\,(p(x) \to q),\, \exists x\, p(x) \Rightarrow q$}
\RightLabel{$\to$-r}
\UnaryInfC{$\forall x\,(p(x) \to q) \Rightarrow (\exists x\, p(x)) \to q$}
\RightLabel{$\to$-r}
\UnaryInfC{$\Rightarrow\ (\forall x\,(p(x) \to q)) \to ((\exists x\, p(x)) \to q)$}
\end{prooftree}

Grading:

- Propositonal structure (2 branches): 1 mark
- Correct quantifier rules: 3 marks each
- Abbreviations (name for $\forall$ formula, ellipsis...) is OK.

C) $\forall x((q(x) \wedge (\exists y p(x,y))) \to (\exists y(q(x) \wedge p(x,y))))$ using first-order LK [7 marks]

Answer:

\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$q(c),\, p(c,d) \Rightarrow q(c),\, \exists\cdots$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$q(c),\, p(c,d) \Rightarrow p(c,d),\, \exists\cdots$}
\RightLabel{$\wedge$-r}
\BinaryInfC{$q(c),\, p(c,d) \Rightarrow q(c) \wedge p(c,d),\, \exists\cdots$}
\RightLabel{$\exists$-r}
\UnaryInfC{$q(c),\, p(c,d) \Rightarrow \exists y\,(q(c) \wedge p(c,y))$}
\RightLabel{$\exists$-l}
\UnaryInfC{$q(c),\, \exists y\, p(c,y) \Rightarrow \exists y\,(q(c) \wedge p(c,y))$}
\RightLabel{$\wedge$-l}
\UnaryInfC{$q(c) \wedge \exists y\, p(c,y) \Rightarrow \exists y\,(q(c) \wedge p(c,y))$}
\RightLabel{$\to$-r}
\UnaryInfC{$\Rightarrow\ (q(c) \wedge (\exists y\, p(c,y))) \to (\exists y\,(q(c) \wedge p(c,y)))$}
\RightLabel{$\forall$-right}
\UnaryInfC{$\Rightarrow\ \forall x\,((q(x) \wedge (\exists y\, p(x,y))) \to (\exists y\,(q(x) \wedge p(x,y))))$}
\end{prooftree}

Grading:

- Propositonal structure (2 branches): 1 mark
- Quantifier rules: 2 marks each

D) $(\Box(p \to q)) \to (\Diamond p \to \Diamond q)$ using the modal sequent calculus [5 marks]

Answer:

\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$2 : p,\, \ldots \Rightarrow 2 : p, \ldots$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$2 : q,\, \ldots \Rightarrow 2 : q,\, \ldots$}
\RightLabel{$\to$-l}
\BinaryInfC{$2 : p \to q,\, 1 : \Box\cdots,\, 2 : p,\, 1R2 \Rightarrow 2 : q, 1 : \Diamond q$}
\RightLabel{$\Diamond$-r}
\UnaryInfC{$2 : p \to q,\, 1 : \Box\cdots,\, 2 : p,\, 1R2 \Rightarrow 1 : \Diamond q$}
\RightLabel{$\Box$-l}
\UnaryInfC{$1 : \Box(p \to q),\, 2 : p, 1R2 \Rightarrow 1 : \Diamond q$}
\RightLabel{$\Diamond$-l}
\UnaryInfC{$1 : \Box(p \to q),\, 1 : \Diamond p \Rightarrow 1 : \Diamond q$}
\RightLabel{$\to$-r}
\UnaryInfC{$1 : \Box(p \to q) \Rightarrow 1 : \Diamond p \to \Diamond q$}
\RightLabel{$\to$-r}
\UnaryInfC{$\Rightarrow\ 1 : (\Box(p \to q)) \to (\Diamond p \to \Diamond q)$}
\end{prooftree}

Grading:

- $\Box$-r and $\Diamond$-l rules: 1 marks each
- $\Box$-l and $\Diamond$-r rules: 2 marks each

E) $\Box\Box(p \lor q) \to \Diamond\Box p \lor \Box\Diamond q$ using the modal sequent calculus [8 marks]

Answer:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{3:p,\,\ldots \Rightarrow 3:p,\,\ldots}\,\text{ax} \qquad \dfrac{}{3:q,\,\ldots \Rightarrow 3:q,\,\ldots}\,\text{ax}}{3:p\lor q,\, 2:\Box\cdots,\, 1:\Box\cdots,\, 1R2,\, 2R3 \Rightarrow 3:p,\, 1:\Diamond\cdots,\, 3:q,\, 2:\Diamond q}\,{\to}\text{-l}}{3:p\lor q,\, 2:\Box\cdots,\, 1:\Box\cdots,\, 1R2,\, 2R3 \Rightarrow 3:p,\, 1:\Diamond\cdots,\, 2:\Diamond q}\,\Diamond\text{-r}}{2:\Box(p\lor q),\, 1:\Box\cdots,\, 1R2,\, 2R3 \Rightarrow 3:p,\, 1:\Diamond\cdots,\, 2:\Diamond q}\,\Box\text{-l}}{2:\Box(p\lor q),\, 1:\Box\cdots,\, 1R2 \Rightarrow 2:\Box p,\, 1:\Diamond\cdots,\, 2:\Diamond q}\,\Box\text{-r}}{2:\Box(p\lor q),\, 1:\Box\cdots,\, 1R2 \Rightarrow 1:\Diamond\Box p,\, 2:\Diamond q}\,\Diamond\text{-r}}{1:\Box\Box(p\lor q),\, 1R2 \Rightarrow 1:\Diamond\Box p,\, 2:\Diamond q}\,\Box\text{-l}}{1:\Box\Box(p\lor q) \Rightarrow 1:\Diamond\Box p,\, 1:\Box\Diamond q}\,\Box\text{-r}}{1:\Box\Box(p\lor q) \Rightarrow 1:\Diamond\Box p \lor \Box\Diamond q}\,\lor\text{-r}}{\Rightarrow\; 1:\Box\Box(p\lor q) \to \Diamond\Box p \lor \Box\Diamond q}\,{\to}\text{-r}$$

Grading:

- Propositonal structure (2 branches): 1 mark
- Modal structure (1R2, 2R3): 1 mark
- $\Box$ and $\Diamond$ rules: 1 marks each

Question 2 – Resolution

Prove that the following formula is valid, using the resolution calculus

$$(\forall x \exists y\, p(x,y)) \land (\forall x \forall y\, (p(x,y) \to p(y,x))) \to (\forall y \exists x\, p(x,y))$$

Remember that resolution is a refutation calculus, i.e. you can derive that a set of clauses is unsatisfiable.

Arriving at a correct set of clauses: 10 credits

Correct resolution proof: 5 credits

Answer:

- Negated formula:

$$\neg((\forall x \exists y\, p(x,y)) \land (\forall x \forall y\, (p(x,y) \to p(y,x))) \to (\forall y \exists x\, p(x,y)))$$

- Pushing negation inwards:

$$(\forall x \exists y\, p(x,y)) \land (\forall x \forall y\, (\neg p(x,y) \lor p(y,x))) \land (\exists y \forall x\, \neg p(x,y))$$

- Renaming variables:

$$(\forall x \exists u\, p(x,u)) \wedge (\forall x \forall y\, (\neg p(x,y) \vee p(y,x))) \wedge (\exists v \forall x\, \neg p(x,v))$$

- Prenex Normal Form: (we push out the existential variables first, if possible, to avoid too many univseral quantifiers before them)

$$\exists v \forall x \exists u \forall y (p(x,u) \wedge (\neg p(x,y) \vee p(y,x)) \wedge \neg p(x,v))$$

- Skolemisation:

$$\forall x \forall y (p(x,f(x)) \wedge (\neg p(x,y) \vee p(y,x)) \wedge \neg p(x,c))$$

- Clauses with renamed variables:

$$\{\{p(x,f(x))\}, \{\neg p(x',y'), p(y',x')\}, \{\neg p(x'',c)\}\}$$

Resolution proof,:

1. $p(x,f(x))$
2. $\neg p(x',y'), p(y',x')$
3. $\neg p(x'',c)$
4. $p(f(x),x)$ — resolvent of 1 and 2, $\sigma = \{x' \backslash x, y' \backslash f(x)\}$
5. $\Box$ — resolvent of 3 and 4, $\sigma = \{x'' \backslash f(c), x \backslash c\}$

Grading:

- Depending on the order in which quantifiers are moved to the prefix, the skolem terms will have different arity. This doesn't matter for the resolution steps and has no influence on the grades.
- 2 marks for remembering to negate
- 2 marks for correct CNF
- 2 marks per correct skolemisation (2 variables)
- 2 marks for correct clauses
- 1 mark for variable renaming before or in resolution
- 2 marks for each correct resolution step (max 2 steps...)
- -1 for not keeping all variables before $\exists$ in the prefix as Skolem function without explanation, but otherwise sound (i.e. $p(x,f(x))$)
- -1 for a substitution like $\{x \backslash c, x'' \backslash f(\underline{x})\}$ instead of $\{x \backslash c, x'' \backslash f(c)\}$
- Correct resolution from a wrong clause set gives 4 marks.

Question 3 – An inductive proof

Here is an inductively defined function f that takes a propositional logic formula and returns a different one:

- $f(A) := \neg A$ if A is an atomic formula
- $f(\neg A) := A$
- $f(A \wedge B) := f(A) \vee f(B)$
- $f(A \vee B) := f(A) \wedge f(B)$
- $f(A \to B) := A \wedge f(B)$

Intuitively, f negates the formula, but then moves the negation inwards until it either meets another negation (the second case) or an atom (the first case).

Show by structural induction on A that $f(A)$ is logically equivalent to $\neg A$ for all propositional formulae A.

[10 credits]

Answer: Let $\mathcal{I}$ be a propositional interpretation. We prove by structural induction on A that $v_{\mathcal{I}}(f(A)) = v_{\mathcal{I}}(\neg A)$, which is the same as showing that $v_{\mathcal{I}}(f(A)) = T$ if and only if $v_{\mathcal{I}}(A) = F$.

Induction Hypothesis: $v_{\mathcal{I}}(f(A)) = v_{\mathcal{I}}(\neg A)$ for the subformulae of A.

Base case, A is atomic: Then $f(A) = \neg A$ by definition, so clearly $v_{\mathcal{I}}(f(A)) = v_{\mathcal{I}}(\neg A)$.

Induction step, $A = \neg A_1$: Then $f(A) = A_1$ by definition, so $v_{\mathcal{I}}(f(A)) = v_{\mathcal{I}}(A_1) = T$ iff $v_{\mathcal{I}}(A) = F$ (by model semanics). The IH is not needed here.

Induction step, $A = A_1 \wedge A_2$: Then $f(A) = f(A_1) \vee f(A_2)$ by definition.
So $v_{\mathcal{I}}(f(A)) = v_{\mathcal{I}}(f(A_1) \vee f(A_2)) = T$
iff $v_{\mathcal{I}}(f(A_1)) = T$ or $v_{\mathcal{I}}(f(A_2)) = T$ (by model semantics)
iff $v_{\mathcal{I}}(A_1) = F$ or $v_{\mathcal{I}}(A_2) = F$ (by the induction hypothesis)
iff $v_{\mathcal{I}}(A_1 \wedge A_2) = v_{\mathcal{I}}(A) = F$ (by model semantics).

Induction step, $A = A_1 \vee A_2$: Then $f(A) = f(A_1) \wedge f(A_2)$ by definition.
So $v_{\mathcal{I}}(f(A)) = v_{\mathcal{I}}(f(A_1) \wedge f(A_2)) = T$
iff $v_{\mathcal{I}}(f(A_1)) = T$ and $v_{\mathcal{I}}(f(A_2)) = T$ (by model semantics)
iff $v_{\mathcal{I}}(A_1) = F$ and $v_{\mathcal{I}}(A_2) = F$ (by the induction hypothesis)
iff $v_{\mathcal{I}}(A_1 \vee A_2) = v_{\mathcal{I}}(A) = F$ (by model semantics).

Induction step, $A = A_1 \to A_2$: Then $f(A) = A_1 \wedge f(A_2)$ by definition.
So $v_{\mathcal{I}}(f(A)) = v_{\mathcal{I}}(A_1 \wedge f(A_2)) = T$
iff $v_{\mathcal{I}}(A_1) = T$ and $v_{\mathcal{I}}(f(A_2)) = T$ (by model semantics)
iff $v_{\mathcal{I}}(A_1) = T$ and $v_{\mathcal{I}}(A_2) = F$ (by the induction hypothesis)
iff $v_{\mathcal{I}}(A_1 \to A_2) = v_{\mathcal{I}}(A) = F$ (by model semantics).

Grading:

- 2 marks for a correct structural induction argument
  - 1 mark for an explicit induction hypothesis
  - 1 mark for the general structure
- 1 mark each for literal and negation case
- 2 marks each for disjunction, conjunction, implication
  - 1 mark for referring to the IH and the semantics
  - 1 mark for the general reasoning

Question 4 – Multiple Choice Mix

Grading: Automatically graded. A correct answer will give 2 marks, an incorrect one −2 marks, and a missing answer 0 marks. However, the whole question will never be given less than 0 marks.

A) If A is a valid formula of propositional logic, then $\Diamond A$ is valid in modal logic D [2 marks]
Answer: True. If u is a world, then in a D-frame, there is at least one world v reachable from u. A is true in any propositional interpretation, then $v \Vdash A$. Therefore $u \Vdash \Diamond A$.

B) Are these terms unifiable? $f(x, g(x))$ and $f(h(y), g(y))$
Answer: No. Unifying the first subterms gives $\{x \backslash h(y)\}$. After applying this, we still need to unify $f(h(y), g(h(y)))$ and $f(h(y), g(y))$. The next critical pair requires unifying y and $h(y)$ which fails due to the occurs check.

C) Are these terms unifiable? $f(h(x), y, g(x))$ and $f(h(y), z, g(z))$
Answer: Yes, $\{x \backslash y, z \backslash y\}$ is a most general unifier. There are others that are equal up to variable renaming.

D) The order of resolution steps in a resolution proof matters: starting with a particular resolution might mean that no proof can found even though a proof could be found by starting differently.
Answer: False. There is no 'dead end' in resolution proof search. Starting from an unsatisfiable clause det, you can add as many resolvents (or in fact any other clauses) as you want, the set will remain unsatisfiable, and therefore there is a resolution proof.

E) The same statement as the last, but for SLD resolution.
Answer: True. SLD resolution has to start with a goal clause and after that each step has to involve the previously derived clause. This may not lead to a proof, which is why Prolog proof search has to backtrack.

Question 5 – Intuitionistic Logic

Consider the formulas:

$$
\begin{aligned}
A &= (\neg\exists\neg p(x)) \to (\forall x \neg\neg p(x)) && (1)\\
B &= (\neg(p \wedge q)) \to (\neg p \vee \neg q) && (2)
\end{aligned}
$$

A) Show that A is valid in intuitionistic logic using the LJ sequent calculus [8 credits]
Answer:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\exists x\,\neg p(x), \neg p(c) \Rightarrow \neg p(c)}\,\text{ax}}{\neg\exists x\,\neg p(x), \neg p(c) \Rightarrow \exists x\,\neg p(x)}\,\exists\text{-r}}{\neg\exists x\,\neg p(x), \neg p(c) \Rightarrow}\,\neg\text{-l}}{\neg\exists x\,\neg p(x) \Rightarrow \neg\neg p(c)}\,\neg\text{-r}}{\neg\exists x\,\neg p(x) \Rightarrow \forall x \neg\neg p(x)}\,\forall\text{-r}}{\Rightarrow\ (\neg\exists x\,\neg p(x)) \to (\forall x \neg\neg p(x))}\,\to\text{-r}
$$

Grading:

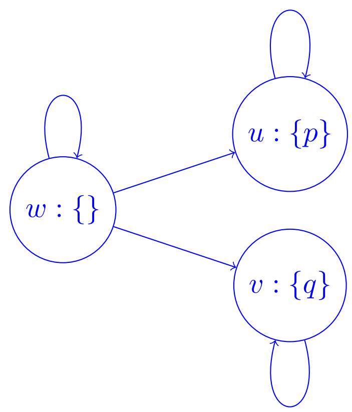
- 8 credits for a correct proof
- minus 2 for forgetting to keep a copy in $\neg$-left
- minus 3 for more than one formula in a succedent

B) Show that B is not valid in intuitionistic logic using the model semantics. I.e. construct an intuitionistic structure with a world w such that $w \Vdash \neg(p \wedge q)$ but $w \nVdash \neg p \vee \neg q$. [10 credits]
Answer: We need a structure in which

1. $p \wedge q$ is not forced in any successor of w, i.e. there is no successor in which both p and q are forced.
2. w forces neither $\neg p$ nor $\neg q$, i.e. there is a successor world u in which p is forced, and a successor world v in which q is forced.

Combining these, we see that the worlds u and v cannot be the same. Otherwise p and q would both be forced.

This leads to a structure in which w has two distinct successors u and v, where $u \Vdash p$ and $v \Vdash q$.



Grading:

- 2 marks for arguing about the forcing relation
- 2 marks for treating the model semantics of $\wedge$ and $\vee$ correctly
- 2 marks for treating the model semantics of $\neg$ correctly
- 2 marks for correctly introducing the successor worlds
- 2 marks for correctly arguing that p and q can't be forced in the same world.
- -2 marks for a wrong model, like e.g. one that is not monotonic.
- Trying really hard to show that *if* $w \Vdash \neg(p \wedge q)$ *then* $w \nVdash \neg p \vee \neg q$ (which will fail), instead of giving an intuitionistic structure: 2 marks out of 10

Question 6 – Soundness for Modal Sequents

In the lecture, we proved the soundness of propositional and first order LK by showing that all LK rules preserve «falsifiability upwards.»

It is possible to prove the soundness of our labeled sequent calculus for the modal logic K in a similar way. For this, we have to define what it means for an interpretation to falsify a labeled sequent.

As a reminder, the labeled sequents the calculus works with can contain

- labeled formulae $u : A$ where A is a formula and u is a label taken from some set of labels $\mathcal{L}$, and
- accessibility formulae uRv for labels $u, v \in \mathcal{L}$. These only occur in the antecedent Γ.

Definition 1: Given a set of labels $\mathcal{L}$, and a Kripke frame (W, R), a *label assignment* is a function $\alpha : \mathcal{L} \to W$ that selects some world $\alpha(u)$ for every label $u \in \mathcal{L}$.
Definition 2: A modal interpretation $\mathcal{I}_M := ((W, R), \{\mathcal{I}(w)\}_{w \in W})$ falsifies a labeled sequent $\Gamma \Rightarrow \Delta$ if there is a label assignment α such that:

- $\langle \alpha(u), \alpha(v) \rangle \in R$ for every accessibility formula $uRv \in \Gamma$,
- $\alpha(u) \Vdash A$ in $\mathcal{I}_M$ for every labeled formula $u : A \in \Gamma$,
- $\alpha(u) \nVdash A$ in $\mathcal{I}_M$ for every labeled formula $u : A \in \Delta$,

The condition of preserving falsifiability upwards is then the same as before: if the conclusion of a rule is falsifiable, then at least one of the premises is falsifiable.

For instance, to see that $\wedge$-left,

$$\frac{u : A, u : B, \Gamma \Rightarrow \Delta}{u : A \wedge B, \Gamma \Rightarrow \Delta}$$

preserves falsifiability upwards also in the labeled calculus, let $\mathcal{I}_M := ((W, R), \{\mathcal{I}(w)\}_{w \in W})$ be a modal interpretation that falsifies the conclusion with label assignment α. Def. 2 tells us that $\alpha(u) \Vdash A \wedge B$. The model semantics for modal logic then tells us that $\alpha(u) \Vdash A$ and $\alpha(u) \Vdash B$. Since the formulae in Γ and Δ are unchanged, all formulas in the new antecedent are forced at the worlds indicated by α, so $\mathcal{I}_M$ also falsifies the premise, using the same label assignment α.

A) Show in a similar way that the rule $\Box$-left:

$$\frac{v : A, u : \Box A, uRv, \Gamma \Rightarrow \Delta}{u : \Box A, uRv, \Gamma \Rightarrow \Delta}$$

preserves falsifiability upwards [6 marks]

Hint: you can use the same falsifying interpretation and label assignment for the conclusion and the premise.

Answer: Let $\mathcal{I}_M := ((W, R), \{\mathcal{I}(w)\}_{w \in W})$ be a modal interpretation that falsifies the conclusion with label assignment α. Def. 2 tells us that (1) $\alpha(u) \Vdash \Box A$, and (2) $\langle \alpha(u), \alpha(v) \rangle \in R$. From the model semantics for $\Box$, and since $\alpha(v)$ is an R-successor of $\alpha(u)$, we can conclude that $\alpha(v) \Vdash A$. Since the formulae in Γ and Δ are unchanged, all formulas in the antecedent of the premiss are forced at the worlds indicated by their labels and α, so $\mathcal{I}_M$ also falsifies the premise, using the same label assignment α.

Grading:

- 1 mark for an arbitrary but fixed $\mathcal{I}_M$ and α that falsifes the conclusion
- 1 mark for what that means for the formulae in the conclusion

- 2 marks for using the model semantics of $\Box$ correctly
- 2 marks for correctly transfering from conclusion to premise he same world.
- -1 mark for not keeping labels and worlds properly apart.

B) Show in a similar way that the rule $\Diamond$-left:

$$\frac{v : A, uRv, \Gamma \Rightarrow \Delta}{u : \Diamond A, \Gamma \Rightarrow \Delta} \quad \text{where } v \text{ is a fresh label}$$

preserves falsifiability upwards [8 marks]

Hint: you can use the same falsifying interpretation for the conclusion and the premise. You will need to modify the label assignment to deal with the new label.

Answer: Let $\mathcal{I}_M{:=}((W, R), \{\mathcal{I}(w)\}_{w \in W})$ be a modal interpretation that falsifies the conclusion with label assignment α. Def. 2 tells us that $\alpha(u) \Vdash \Diamond A$. From the model semantics for $\Diamond$, it follows that there is a world $w \in W$ such that $w \Vdash A$ and $\langle \alpha(u), w \rangle \in R$.

We now define a new label assignment α' such that $\alpha'(u) = \alpha(u)$ for all previous labels u. In addition $\alpha'(v) = w$, i.e. the label assignment for the new label v introduced by the rule is the world w that is reachable from $\alpha(u)$ and that forces A.

With this new label assigment α', it follows that $\alpha'(v) \Vdash A$ and $\langle \alpha'(u), \alpha'(v) \rangle \in R$. In the labeled formulae in Γ and Δ, v does not occur, so the change from α to α' makes no difference.

Therefore, $\mathcal{I}_M$ also falsifies the premise, but with the new label assignment α'.

Grading:

- 1 mark for an arbitrary but fixed $\mathcal{I}_M$ and α that falsifes the conclusion
- 1 mark for what that means for the formulae in the conclusion
- 2 marks for using the model semantics of $\Diamond$ correctly
- 2 marks for the correct construction of α
- 1 mark for correctly transfering from conclusion to premise
- 1 mark for arguing that the change from α to α' does not affect Γ and Δ.
- -1 mark for not keeping labels and worlds properly apart.