Exercises for the Course

Logic for Computer Science

Week 2

Induction Exercise (Induction and Recursion)

Extra induction exercise to get used to induction and recursion. Use the definition of the set of formulas on slide 16

Prove by induction that formulas contain the same number of right as left parentheses.

Define by recursion a function from the set of formulas to $\mathbb{N}$ that counts the number of parentheses (left and right) of a formula.

Exercise 1.1 (Formalization)

A warm-up exercise. Not very difficult, but the other questions are more important.

"What is the secret of your long life?" a centenarian was asked. "I strictly follow my diet: If I don't drink beer for dinner, then I always have fish. Any time I have both beer and fish for dinner, then I do without ice cream. If I have ice cream or don't have beer, then I never eat fish." Formalize the answer in propositional logic. Can you simplify the answer?

Exercise 1.2 (Interpretation)

Consider the interpretation $\mathcal{I}$ where $\mathcal{I}(p) = F, \mathcal{I}(q) = T, \mathcal{I}(r) = T$.

Does $\mathcal{I}$ satisfy the following propositional formulae?

1. $(p \rightarrow \neg q) \vee \neg(r \wedge q)$
2. $(\neg p \vee \neg q) \rightarrow (p \vee \neg r)$
3. $\neg(\neg p \rightarrow \neg q) \wedge r$
4. $\neg(\neg p \rightarrow q \wedge \neg r)$

Exercise 1.3 (Logical Equivalence)

Two formulas A and B are *logically equivalent* if they have the same truth values under all interpretations. That is, if both $A \vDash B$ and $B \vDash A$.

Show that the following formulae are logically equivalent.

1. $p \rightarrow (q \wedge \neg q)$ and $\neg p$
2. $(\neg p \vee q) \rightarrow q \wedge (q \rightarrow r) \wedge \neg r$ and $(q \wedge \neg r) \wedge (\neg r \rightarrow \neg q) \vee (p \wedge \neg q)$

Exercise 1.4 (Satisfiability and Validity)

Are the following formulae satisfiable, valid, unsatisfiable, or invalid (falsifiable)?

1. $(\neg p \vee q) \wedge (q \rightarrow (\neg r \wedge \neg p)) \wedge (p \vee r)$

2. $((\neg p \vee q) \to q \wedge (q \to r) \wedge \neg r) \to p$
3. $\neg p \wedge (\neg q \vee r) \wedge (\neg p \to q \wedge \neg r)$

Exercise 1.5 **(Propositional Challenge)**

A challenging exercise that might take a lot of your time. The other questions are more important!

Find a formula A that contains the atoms p, q, and r such that changing any of the interpretations $\mathcal{I}(p)$, $\mathcal{I}(q)$, or $\mathcal{I}(r)$ will also change the truth value $v_{\mathcal{I}}(A)$ of A.

Exercise 1.6 **(Deduction Theorem)**

About reasoning about all interpretations: Suppose you are to prove $A \vDash B$, for some formulas A and B. You must show for any *interpretation that it is the case that if it satisfies A then it satisfies B. How to do this, obviously you cannot survey them all? Well, what you do is to suppose that an interpretation I is given to you; it is arbitrary, you know nothing about it. But of course, either it satisfies A or it doesn't. If I does not satisfy A then you are done. Thus the only interesting case is if I does satisfy A. But then you know something about I, it satisfies A, and from that you must deduce that it also satisfies B. For instance, $p \vDash p \vee q$, for if I satisfies p, then $v_I(p) = T$, and then it follows that $v_I(p \vee q) = T$.*

Warm up: Prove first a simple version of the Deduction Theorem:

$$A \models B \text{ if an only if } \models A \to B.$$

Notice that there are two things to prove here: 1) that $A \models B$ implies $\models A \to B$; and 2) that $\models A \to B$ implies $A \models B$. So for (1) you start by assuming that $A \models B$ and then you must show that for a given, arbitrary I it must be the case that $I \models A \to B$.

Now prove the Deduction Theorem: Let $U = \{A_1, ..., A_n\}$ be a finite, non-empty set of formulae. We write $\bigwedge_i A_i$ for the conjunction $A_1 \wedge \cdots \wedge A_n$ so $v_{\mathcal{I}}(\bigwedge_i A_i) = T$ iff $v_{\mathcal{I}}(A_i) = T$ for all $A_i \in U$.

Prove that

$$U \models B \text{ if and only if } \models (\textstyle\bigwedge_i A_i) \to B.$$

Exercise 1.7 **(Logical Consequence)**

In the statement of the Deduction Theorem above it was assumed that U was empty. That assumption is not necessary, but there are a couple of things that need to be clarified in order to remove it. For now, just think about the following. Then check the Solutions to see if you agree.

We write $\emptyset$ for the empty set, the set with no members.

What does it mean if

$\emptyset \models B$? What can you say about B if this is the case?

The following two exercises are optional this week, they will occur in the exercise set for next week instead.

Exercise 1.8* **(Sequent Calculus LK)**

Prove the following formulae in the sequent Calculus LK.

1. $\neg(p \vee q) \to (\neg p \wedge \neg q)$
2. $p \vee \neg p$
3. $(p \to (q \to r)) \to (p \wedge q \to r)$

Exercise 1.9* **(Invalid and Satisfiable in LK)**

Use the sequent calculus LK to show that the first formula is invalid and the second formula is satisfiable.

1. $(p \to (q \to r)) \to (p \vee q \to r)$
2. $\neg(p \vee q) \to (\neg p \wedge q)$