Exercises for the Course

Logic

Modal Logic

Exercise 10.05

(Kripke Semantics Warm Up)

a) In modal logic $\mathbf{K}$, that is, without any assumptions on the accessibility relation, show that the following formulas are satisfiable but not valid (so two things to show per formula).

1. $\Box p$
2. $\Diamond p$
3. $p \to \Diamond p$
4. $\Box p \to \Diamond p$

b) In modal logic $\mathbf{K}$, that is, without any assumptions on the accessibility relation, show that the following formulas are valid.

1. $\Box p \leftrightarrow \neg\Diamond\neg p$
2. $\Box(p \wedge q) \to (\Box p \wedge \Box q)$
3. $\Box(p \wedge q) \leftarrow (\Box p \wedge \Box q)$

Exercise 10.1

(Modal Sequent Calculus)

Using the sequent calculus from the lecture, prove

1. $\Box(p \to q) \to (\Box p \to \Box q)$
2. $\Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$
3. $(\Diamond p \vee \Diamond q) \to \Diamond(p \vee q)$
4. $((p \to \Box p) \wedge (\Diamond\neg p)) \to \neg p$

Exercise 10.2

(Model Semantics)

a) Let A be a *valid* formula of propositional logic, i.e. it is true in all propositional interpretations.

Show that $\Box A$ is valid in modal logic $\mathbf{K}$.

Show that $\Diamond A$ is *not* valid in modal logic $\mathbf{K}$.

b) Prove (using the model semantics) that the formula $\Box p \to \Box\Box p$ is satisfied at all worlds of any Kripke structure if the accessibility relation R is *transitive*.

c) Fix a frame $F = (W, R)$. Show that if all Kripke models that have F as their underlying frame satisfy $\Box p \to \Box\Box p$ at all worlds, then the accessibility relation R is transitive.

d) Construct a Kripke model $((W, R)\{\mathcal{I}_w\}_{w \in W})$ which satisfies $\Box p \to \Box\Box p$ at all worlds where R is not transitive.

(e) Let A be a modal propositional formula such that A is not satisfied at any world in any Kripke model.

Show that $\Diamond A$ is not satisfied at any world in any Kripke model.

Show that there is a Kripke model with a world such that $w \Vdash \Box A$.

(f) Show the last claim on slide 23 (note that there are two things to prove):

- if $U \models^L A$ then $U \models^G A$; the opposite direction does not hold

Exercise 10.3 **(A Logic of Knowledge)**

Here is an attempt at formalising the idea of "knowledge."

We assume that our knowledge is determined by what we have *observed* about the world. E.g. the temperature, the colour of a light, etc. Let O be a set of possible observations, W a set of possible states of the world, and $o : W \to O$ an observation function that tells us for each state of the world what we have observed about it.

E.g. in the scenario where we may or may not have looked outside, and it may or may not rain, the set W would contain four worlds, according to the four possibilities. O would contain only three possible observations: NotLooked, LookedAndRain, LookedAndNotRain.

We can now say that two worlds $u, v \in W$ appear the same to us if our observations are the same, $o(u) = o(v)$. E.g. the two worlds where we have not looked outside appear the same because the observation is NotLooked for both of them

Now we can say that we "know" A in a world $w \in W$ if A is true in all worlds that fit our observations i.e., A is true in all worlds v with $o(u) = o(v)$.

a) Define an accessibility relation $R \subseteq W \times W$ such that the modal semantics of $\Box A$ is the same is this notion of knowledge.

b) Which of the properties of frames (reflexive, serial, transitive, etc.) we saw in the lecture does R have? Which of the modal logics listed in the lecture (K, D, T,...) corresponds to this notion of knowledge?