IN3070
IN4070

Autumn 2023

Exercises for the Course

Logic

Week 3

Exercise 1.8

(Sequent Calculus LK)

Prove the following formulae in the sequent Calculus LK.

1. $\neg(p \lor q) \to (\neg p \land \neg q)$
2. $p \lor \neg p$
3. $(p \to (q \to r)) \to (p \land q \to r)$

Exercise 1.9

(Invalid and Satisfiable in LK)

Use the sequent calculus LK to show that the first formula is invalid and the second formula is satisfiable.

1. $(p \to (q \to r)) \to (p \lor q \to r)$
2. $\neg(p \lor q) \to (\neg p \land q)$

Exercise 2.0

(Some quick test questions)

1. Can you use LK to show that a formula is absurd/contradictory/unsatisfiable?
2. If you remove the Left $\neg$ rule from LK, will it still be sound?
3. Make a proof calculus that is neither sound nor complete.

Exercise 2.1

(Sequents)

Let A be the formula $(p \to (q \to r)) \to (p \lor q \to r)$.

Explain how we can use LK to show that A is valid *if we restrict to interpretations for which r is true*. That is to say, for any interpretation $\mathcal{I}$, if $\mathcal{I}(r) = \mathrm{T}$ then A, $v_{\mathcal{I}}(A) = \mathrm{T}$.

Exercise 2.2

(Invalid and Satisfiable in LK)

(To options here: a) If you feel that you have understood the completeness theorem and its proof, just do the exercise. b) If not, use the exercise as a warm up to the completeness proof by doing the following. For 1., first form the sequent with empty antecedent and the formula as consequent, then construct a maximal LK derivation over the sequent. Pick a branch which ends in a non-axiom, say with sequent $\Gamma \Longrightarrow \Delta$, where Γ and Δ contains only atomic formulas. Define an interpretation by setting all propositional variables in Γ as True and all other propositional variables as False. Check that this interpretation falsifies the formula $(p \to (q \to r)) \to (p \lor q \to r)$. For 2. you can use the same method, but with a twist.)

Use the sequent calculus LK to show that the first formula is invalid and the second formula is satisfiable.

1. $(p \to (q \to r)) \to (p \vee q \to r)$
2. $\neg(p \vee q) \to (\neg p \wedge q)$

Exercise 2.3 **(Induction on formulas)**

If you are not used to proving properties by induction on (the complexity of) formulas, do the following exercises. Such a proof proceeds by:

- *Base case.* Show property holds for atomic formulas.
- *Induction hypothesis.* Property holds for formulas A, B.
- *Induction step.* Show property holds for $\neg A$, $A \vee B$, $A \wedge B$, and $A \to B$.

1. Say that a *positive* formula is a formula in which negation ($\neg$) does not occur (anywhere). Let $\mathcal{I}$ be the interpretation which assigns all atoms the truth value T. Show that for all positive formulas A, $v_{\mathcal{I}}(A) = T$. (No case $\neg A$ here, since that would produce a non-positive formula).
2. Let p and q be two atoms. For a formula A, let $A[q/p]$ be the formula obtained by replacing every occurrence of p in A by q. Let $\mathcal{I}$ be an interpretation such that $\mathcal{I}(p) = \mathcal{I}(q)$. Show that for all A, $v_{\mathcal{I}}(A) = v_{\mathcal{I}}(A[q/p])$

Exercise 2.4 **(Missing cases)**

The lecture left several cases in the lemma for the soundness proof, the lemma for the completeness proof, and the alternative completeness proof as exercises. Fill those out. (Do at least the ones in the soundness proof, and the rest if you still feel that you would benefit from it.)

Exercise 2.5 **(Lemma Generation)**

Consider an alternative sequent rule for conjunctions on the right:

$$\frac{\Gamma \Longrightarrow A, \Delta \qquad \Gamma, A \Longrightarrow B, \Delta}{\Gamma \Longrightarrow A \wedge B, \Delta} \wedge\text{-lg}$$

The intuition is: to prove $A \wedge B$, we first prove A. And then we can use A as a 'lemma' to help us when we prove B. Therefore, this rule is sometimes referred to as 'lemma generation.' Many theorem provers for propositional logic use this or similar techniques to find shorter proofs.

1. Prove that the rule $\wedge$-lg is sound, like we did for the other LK-rules.
2. If we remove $\wedge$-right from LK, but add $\wedge$-lg instead, does the completeness proof still work?

3. Can you write down ‘lemma generation’ variants of the $\vee$-left and $\rightarrow$-left rules too?

4. The rule $\wedge$-lg2 adds B as a lemma when proving A. And $\wedge$-lg3 does it both ways. Will these ‘work,’ i.e. will they give a sound and complete calculus?

$$\frac{\Gamma, B \Longrightarrow A, \Delta \qquad \Gamma \Longrightarrow B, \Delta}{\Gamma \Longrightarrow A \wedge B, \Delta}\ \wedge\text{-lg2}$$

$$\frac{\Gamma, B \Longrightarrow A, \Delta \qquad \Gamma, A \Longrightarrow B, \Delta}{\Gamma \Longrightarrow A \wedge B, \Delta}\ \wedge\text{-lg3}$$

Exercise 2.6 **(Regularity)**

The regularity restriction of the LK calculus says that a branching rule (i.e. $\vee$-left, $\rightarrow$-left, $\wedge$-right) should not need to be applied if it adds a formula to the antecedent or succedent of one of the new branches that was already there before the rule application.

E.g., the following application of $\vee$-left:

$$\frac{p, \boxed{p} \Longrightarrow r \qquad p, q \Longrightarrow r}{p, p \vee q \Longrightarrow r}\ \vee\text{-left}$$

adds the literal p to the succedent of the left branch, even though there was already a p there. The regularity restriction says that such rule applications should not be used. Intuitively, this seems right, because whatever derivation we later use to close the left branch could have been used without this $\vee$-left application.

A *regular LK proof* is an LK proof where all rule applications are regular.

Show that the regularity restriction does not destroy completeness. I.e., every valid sequent has a regular LK proof.

Hints:

- *There are two ways of doing this. First, by slightly modifying our completeness proof for LK. Second, by showing how an irregular proof can be transformed into a regular one. The first way is probably easier.*

- *It should be enough to do this for one of the branching rules, e.g. $\wedge$-right, the reasoning is analogous for the others.*

Exercise 2.7* **(Extra induction exercise)**

In LK, an axiom is any sequent where the same formula occurs in both the antecedent and succedent/consequent. Intuitively, whenever that happens we should be able to continue the proof and get axioms where the formula occurring on both sides is a propositional variable. If this is correct, we could have defined an axiom to be a sequent where there is a propositional variable occurring in both the antecedent and succedent/consequent, and we would have ended up with exactly the same provable sequents.

Prove that the intuition is correct. That is, prove by induction on A that any sequent of the form $\Gamma, A \Rightarrow A, \Delta$ is LK-provable with a proof where all the axioms have a propositional variable occurring in both the antecedent and succedent.