IN3070
IN4070

Autumn 2023

Exercises for the Course

Logic

Week 4

Exercise 3.05 **(Recursively defining a function)**

The definition of *free variable* on slide 11 is somewhat informal. Try giving a proper, recursive definition of the function

$$\text{Free} : \mathcal{F} \to \mathcal{P}(\mathcal{V})$$

which sends a formula to the set of variables occurring free in it. ($\mathcal{P}(\mathcal{V}$ is the powerset of $\mathcal{V}$—the set of sets of variables.)

You might want to start with defining the function that sends a term to the variables that occur in it, or you can take that as given if you couldn't be bothered to.

Exercise 3.1 **(Variables & Assignments)**

a) Which variables occur free in the following formulae? Which variables occur bound?

b) Apply the following substitution to both formulae: $\{x \backslash a, y \backslash f(a, b), z \backslash g(x, c)\}$.

c) Is the substitution capture-free for these formulae?

1. $\forall x\, (p(y) \wedge \forall y\, (p(x, z) \to q(y)))$
2. $\forall x\, \exists x\, \forall y\, (p(x) \vee \exists y\, q(y)) \to r(x)$

Exercise 3.2 **(Formalization & Interpretations)**

a) "If there is a man in town who shaves all the men in town who do not shave themselves, then some man in town shaves himself." Formalize these statements by a single first-order formula F_S.

b) What is the value of the term $(5 + 3) * (8 - 5)$ under the interpretation $\mathcal{I} = (\mathbb{N}, \iota)$ with $+^\iota = *$, $*^\iota = -$, $-^\iota = \div$ (division), $3^\iota = 8$, $5^\iota = 6$, $8^\iota = 36$.

c) Show that the following formulae are satisfiable or invalid (or both) by providing a model and/or a counter-model (i.e. an interpretation that falsifies the formula).

1. $\forall x\, p(f(x), a) \to \exists x\, p(g(x), x)$
2. $\forall y\, q(y, b) \to \exists x\, q(a, x)$

Exercise 3.25 **(Interpretations)**

The language of arithmetic has constants $\{\bar{0}, \bar{1}, \bar{2}, \ldots\}$, two function symbols of arity 2, $\{\bar{+}, \bar{\times}\}$, and a relation symbol of arity 2, $\{\bar{\leq}\}$, (and equality). We use infix notation for convenience, that is we write e.g. $x \bar{\leq} y$ instead of $\bar{\leq}(x, y)$.

Give interpretations to *falsify* the following two statements:

1. $\bar{1}\bar{+}\bar{2} = \bar{3}$
2. $\forall x \exists y (x \bar{\leq} y)$

Exercise 3.3 (If we get to LK) **(Sequent Calculus LK & Eigenvariables)**

Try to prove the validity of the following formulae in the sequent calculus LK. If you cannot find a proof in LK then povide a counter-model.

1. The first-order formula F_S from Exercise 3.2 a).
2. $\forall x\, \exists y\, p(x, y) \rightarrow \forall u\, \exists v\, p(u, v)$
3. $\forall x\, \exists y\, p(x, y) \rightarrow \exists v\, \forall u\, p(u, v)$

Exercise 3.4 (If we get to LK) **(Symmetry of LK)**

Let A be a formula containing only $\forall$, $\exists$ and the connectives $\neg$, $\vee$, and $\wedge$. The dual formula A' of A is obtained by exchanging $\forall$ and $\exists$, and exchanging $\vee$ and $\wedge$. Prove that $\vdash A$ iff $\vdash \neg A'$.

Hint: try to explain how to transform an LK proof for A into a proof for $\neg A'$, and vice versa

Exercise 3.5 **(Warm-up induction proofs on terms)**

a. Let a language contain the constants a, b, c and the ternary function symbol f. Let I be the interpretation $(\mathbb{N}, \iota)$ where $a^\iota = 2$, $b^\iota = 8$, $c^\iota = 12$, and f^ι is the function which takes three numbers and adds them all together.

Prove by induction on terms that for any closed term t, $v_\iota(t)$ is an even number.

b. Try proving the Substitution Lemma for Terms (slide 26) without looking at the proof on the slide.

Exercise 3.6 **(Variable Assignments and Closed Formulas)**

The term value, resp. truth value of a *closed* term t, resp. formula A in an interpretation $\mathcal{I} = (D, \iota)$ is independent of the variable assignment. I.e. if α and β are two variable assignments for $\mathcal{I}$, and t is closed, then

$$v_\mathcal{I}(\alpha, t) = v_\mathcal{I}(\beta, t)$$

and if A is closed then

$$v_\mathcal{I}(\alpha, A) = v_\mathcal{I}(\beta, A)$$

Prove these facts by structural induction on t and A.

Hint: when you try to prove this for $\forall x\, A$ and $\exists x\, A$, you will run into the obstacle that the subformula A is not necessarily closed, so it's not possible to apply the induction hypothesis to it. You have to find and prove a more general statement about variable assignments and the free variables occurring in formulae and terms.