IN3070
IN4070

Autumn 2023

Exercises for the Course

Logic

Week 5

Exercise Warm-up

(Using definitions)

Let A be a formula and x a variable that is not bound in A (so either free or not occurring in A at all). Let u be a fresh variable, i.e. a variable different from x and that does not occur in A, neither free nor bound. Let $A' = A[u/x]$.

Let I be an interpretation and α a variable assignment. Show that $v_I(\forall x A, \alpha) = v_I(\forall u A', \alpha)$.

Exercise 3.3

(Sequent Calculus LK & Eigenvariables)

Try to prove the validity of the following formulae in the sequent calculus LK. If you cannot find a proof in LK then provide a counter-model.

1. The first-order formula F_S from Exercise 3.2 a).
2. $\forall x\, \exists y\, p(x,y) \rightarrow \forall u\, \exists v\, p(u,v)$
3. $\forall x\, \exists y\, p(x,y) \rightarrow \exists v\, \forall u\, p(u,v)$

Exercise 3.4

(Symmetry of LK)

Let A be a formula containing only $\forall$, $\exists$ and the connectives $\neg$, $\vee$, and $\wedge$. The dual formula A' of A is obtained by exchanging $\forall$ and $\exists$, and exchanging $\vee$ and $\wedge$. Prove that $\vdash A$ iff $\vdash \neg A'$.

Hint: try to explain how to transform an LK proof for A into a proof for $\neg A'$, and vice versa

Exercise 3.7

(Induction proofs)

Let I be an interpretation.

a) Let t be a term, possibly with free variables.

Prove by induction on t that $v_I(t, \alpha) = v_I(t, \beta)$ for all variable assignments α and β such that for all free variables x in t we have $\alpha(x) = \beta(x)$ (we say that α and β *agree on* $FV(t)$, where $FV(t)$ is the set of free variables in t.).

b) Let A be a formula, possibly with free variables.

Prove by induction on A that $v_I(A, \alpha) = v_I(A, \beta)$ for all variable assignments α and β that agree on $FV(A)$.

c) Conclude Thm 4.1 on slide 24/41 of Lecture 4; for a closed formula the truth value is the same for all variable assignments.

Exercise 3.9

(Semantics practice)

Give semantic proofs of the equivalences and implications on slide 33/41 in Lecture 4.

(If you want to be thorough, you should also prove that the statements “other direction is not valid” are true by giving a countermodel for the other direction).

Exercise 4.0 **(Soundness and completeness proofs)**

a) Fill in the cases for $\forall$-right and $\exists$-right in Lemma 2.1 (slide 15/40), Lecture 5. (Yes, the slide header has a typo)

b) After Monday's class: In the proof of completeness slides 28–35, Lecture 5, Try to do cases in the induction proof that we did not do in class by yourself. Then compare with how it is done in the slides.

Exercise 4.1 **(Counter-models)**

a) Let p be a 1-ary predicate symbol and s a 1-ary function symbol. Construct a falsifying interpretation for the sequent

$$\forall x\,(p(x) \vee p(s(x))),\ \forall x\,(\neg p(x) \vee \neg p(s(x))) \Longrightarrow$$

by beginning an LK derivation, and constructing an interpretation from an open branch.

Hint: start by using $\forall$-left on one of the two formulae with some dummy constant o, then continue using first one then the other of them, with larger and larger terms, until you see the pattern.

b) Try to find a falsifying interpretation $\mathcal{I} = (D, \iota)$ where the domain has only two elements, $|D| = 2$. This can *not* be read off from an open branch.

c) Is there a falsifying interpretation with only one domain element?

Exercise 4.2 **(Bernays-Schönfinkel)**

The Bernays–Schönfinkel class of formulas, named after Paul Bernays and Moses Schönfinkel, is a fragment of first-order logic formulas where satisfiability is decidable.

It is the class of all formulae of the shape

$$B = \exists x_1 \cdots \exists x_m \forall y_1 \cdots \forall y_n\, A$$

where A

- contains no quantifiers
- contains no function symbols (i.e. all terms are variables or constants)

Give an algorithm that decides the satsfiability of this kind of formula.

Hints:

- *a formula B is satisfiable if its negation $\neg B$ is not valid.*
- *consider the first few steps an LK proof for $\neg B$ must have.*
- *remember that there are no function symbols. How does the Herbrand universe of an open branch look?*
- *do we need to have infinite branches to achieve fairness?*