IN3070
IN4070

Autumn 2023

Exercises for the Course

Logic

Week 8

Exercise 8.1 (Atomic Cut and Unit Propagation)

Using atomic cut and unit propagation, show that the following formulas (sets of clauses) are unsatisfiable, or give a satisfying substitution

1. $(\neg p \vee r \vee \neg s) \wedge (p \vee q \vee r \vee \neg s) \wedge (\neg p \vee \neg t) \wedge \neg r \wedge (p \vee s) \wedge (p \vee r \vee t) \wedge (s \vee t)$
2. $\{\{p, \neg q\}, \{\neg p, q\}, \{q, \neg r\}, \{s\}, \{\neg s, \neg q, \neg r\}, \{s, r\}\}$
3. $\{\{p, q, s, t\}, \{p, s, \neg t\}, \{q, \neg s, t\}, \{p, \neg s, \neg t\}, \{p, \neg q\}, \{\neg r, \neg p\}, \{r\}\}$

Exercise 8.2 (Properties of Simplification)

Prove that the simplification rules are sound.

a) Given a propositional formula A that contains a subformula B, and let $B \equiv B'$ (logical equivalence). Prove A is logically equivalent to the result of replacing B by B' in A. *Hint: use structural induction on A*

b) Given propositional formulae A and B and an interpretation $\mathcal{I}$ with $\mathcal{I} \models B$. Prove that $\mathcal{I} \models A$ if and only if $\mathcal{I} \models A[B]$. *Hint: for the replacement of B by true or false, use structural induction on A similarly to part A. For the simplification steps, use part a).*

c) Prove that the simplification rule

$$\frac{B,\ A[B],\ \Gamma \Longrightarrow \Delta}{B,\ A,\ \Gamma \Longrightarrow \Delta}$$

preserves falsifiability upwards.

Exercise 8.3 (Pure Literal Elimination)

Given a set of clauses S, a literal L is *pure* for S if it occurs only positively or only negatively. I.e. if there are no clauses $C, D \in S$ with $L \in C$ and $\overline{L} \in D$.

In a refutation procedure like DPLL or resolution, if there is a pure literal L, it is OK to remove all clauses that contain L, and this is why:

Let $S' := \{C \in S \mid L \notin C\}$ be the set of clauses in S that do not contain L. Show that if L is pure, then S is satisfiable if and only if S' is satisfiable.

Hint: it is easiest to consider two cases, depending on whether L is an atom like p or a negated atom like $\neg p$. In both cases, if L is pure, it is easy to find an interpretation that satisfies all clauses containing L.