Grading Guidelines
IN3070/IN4070
Autumn 2023

The maximum number of marks for the whole exam was 100.

The minimum number of marks required for each grade will be published after the exam was graded.

Question 1 – Sequent Calculi LK and LJ

Prove the validity of the following formulae using the given calculus. Note that the first two formulas are to be proven in the classical logic LK, while the last one is to be proven in the intuitionistic logic LJ.

In this task you can submit a hand-written answer. Use the sketching paper handed to you in the exam room for this. See instructions in the link below the task bar.

A) $\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$ using propositional LK [5 marks]

Answer:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p, q \;\Rightarrow\; p}\,\text{ax} \qquad \dfrac{}{p, q \;\Rightarrow\; q}\,\text{ax}}{p, q \;\Rightarrow\; p \wedge q}\,\wedge\text{-r}}{p, q, \neg(p \wedge q) \;\Rightarrow\;}\,\neg\text{-l}}{p, \neg(p \wedge q) \;\Rightarrow\; \neg q}\,\neg\text{-r}}{\neg(p \wedge q) \;\Rightarrow\; \neg p, \neg q}\,\neg\text{-r}}{\neg(p \wedge q) \;\Rightarrow\; \neg p \vee \neg q}\,\vee\text{-r}}{\Rightarrow\; \neg(p \wedge q) \rightarrow (\neg p \vee \neg q)}\,\rightarrow\text{-r}$$

B) $\forall x(p \vee r(x)) \rightarrow (p \vee \forall x r(x))$, using first-order LK [5 marks]

Answer:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p, \forall x(p \vee r(x)) \;\Rightarrow\; p, r(c)}\,\text{ax} \qquad \dfrac{}{r(c), \forall x(p \vee r(x)) \;\Rightarrow\; p, r(c)}\,\text{ax}}{p \vee r(c), \forall x(p \vee r(x)) \;\Rightarrow\; p, r(c)}\,\vee\text{-l}}{\forall x(p \vee r(x)) \;\Rightarrow\; p, r(c)}\,\forall\text{-l},\ [x \backslash c]}{\forall x(p \vee r(x)) \;\Rightarrow\; p, \forall x r(x)}\,\forall\text{-r}}{\forall x(p \vee r(x)) \;\Rightarrow\; p \vee \forall x r(x)}\,\vee\text{-r}}{\Rightarrow\; \forall x(p \vee r(x)) \rightarrow (p \vee \forall x r(x))}\,\rightarrow\text{-r}$$

C) $(p \vee \neg p) \to (\neg(p \wedge q) \to (\neg p \vee \neg q))$, using propositional LJ [10 marks]

Answer:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p, q, \neg(p \wedge q) \;\Rightarrow\; p}\,\text{ax} \qquad \dfrac{}{p, q, \neg(p \wedge q) \;\Rightarrow\; q}\,\text{ax}}{p, q, \neg(p \wedge q) \;\Rightarrow\; p \wedge q}\,\wedge\text{-r}}{p, q, \neg(p \wedge q) \;\Rightarrow\;}\,\neg\text{-l}}{p, \neg(p \wedge q) \;\Rightarrow\; \neg q}\,\neg\text{-r}}{p, \neg(p \wedge q) \;\Rightarrow\; \neg p \vee \neg q}\,\vee\text{-r}_q \qquad \dfrac{\dfrac{}{\neg p, \neg(p \wedge q) \;\Rightarrow\; \neg p}\,\text{ax}}{\neg p, \neg(p \wedge q) \;\Rightarrow\; \neg p \vee \neg q}\,\vee\text{-r}_2}{p \vee \neg p, \neg(p \wedge q) \;\Rightarrow\; \neg p \vee \neg q}\,\vee\text{-l}$$

$$\dfrac{\dfrac{p \vee \neg p, \neg(p \wedge q) \;\Rightarrow\; \neg p \vee \neg q}{p \vee \neg p \;\Rightarrow\; \neg(p \wedge q) \to (\neg p \vee \neg q)}\,\to\text{-r}}{\Rightarrow\; (p \vee \neg p) \to (\neg(p \wedge q) \to (\neg p \vee \neg q))}\,\to\text{-r}$$

Question 2 – Classical first-order semantics

A) Show that the formula $\forall x(p(x) \vee r(x)) \to \forall x p(x) \vee \forall x r(x)$ is not valid by constructing a falsifying interpretation. [8 marks]

Answer: A falsifying interpretation for an implication must make the formula on the left true, but the one on the right false.

In this case, to make the formula on the left true, we need an interpretation where each domain element is included in the interpretation of either p or r. But to make the formula on the right false, neither interpretation should include the whole domain.

One possible interpretation is $\mathcal{I} = (D, \iota)$ where $D = \{a, b\}$, $p^\iota = \{a\}$, and $r^\iota = \{b\}$.

B) Show that the formula $\forall x(p \vee r(x)) \to p \vee \forall x r(x)$ is valid by reasoning semantically. (That is, show that it is true in all interpretations by reasoning about interpretations, do *not* use the calculus and the soundness theorem.) [8 marks]

In this task you can submit a hand-written answer. Use the sketching paper handed to you in the exam room for this. See instructions in the link below the task bar.

Answer: Let $\mathcal{I} = (D, \iota)$ be an arbitrary interpretation. We need to show that $\mathcal{I} \models \forall x(p \vee r(x)) \to p \vee \forall x r(x)$.

For, this, we need to show that *if* $\mathcal{I} \models \forall x(p \vee r(x))$, *then* $\mathcal{I} \models p \vee \forall x r(x)$, so assume that $\mathcal{I} \models \forall x(p \vee r(x))$ (*).

It is convenient to distinguish between two cases, namely whether $\mathcal{I} \models p$ or $\mathcal{I} \not\models p$.

Case 1, $\mathcal{I} \models p$. Then $\mathcal{I} \models p \vee A$ for any formula A and in particular $\mathcal{I} \models p \vee \forall x r(x)$.

Case 2, $\mathcal{I} \not\models p$. We will show $\mathcal{I} \models \forall x r(x)$ from which it follows that $\mathcal{I} \models p \vee \forall x r(x)$. Let $d \in D$ be an arbitrary domain element. Because of

(*) and the definition of the semantics of the universal quantifier, we know that $\mathcal{I}, \{x \leftarrow d\} \models p \lor r(x)$. But $\mathcal{I}, \{x \leftarrow d\} \not\models p$, so by the semantics of disjunction, $\mathcal{I}, \{x \leftarrow d\} \models r(x)$. Since this holds for all $d \in D$, we have that $\mathcal{I} \models \forall x r(x)$, and so $\mathcal{I} \models p \lor \forall x r(x)$.

Question 3 – Modal Sequent Calculus

In the sequent calculus for modal logics, the axiom that closes a branch requires the same label on the formulae in the antecedent and the succedent.

$$u : A, \Gamma \Rightarrow u : A, \Delta$$

If this were not required, i.e. if an axiom had the shape

$$u : A, \Gamma \Rightarrow v : A, \Delta$$

allowing different labels in the antecedent and succedent, the calculus would be unsound. Show this by giving a formula that is not valid in modal logic K but that has a closed derivation in the calculus with the wrong axiom.

[10 marks]

Answer: The branches of a derivation are partial constructions of Kripke models, just like the branches in first order sequent calculi are partial constructions of first-order models. The wrong axiom would allow to infer that a formula is true in one world from the fact that it is true in another. If u is the start label for instance, and some formula holds there, we could use this to prove the same formula in another label v. This label v would have to come from a $\Diamond$-l or $\Box$-r formula, because those introduce new labels.

We can use the formula $p \to \Box p$. This is clearly not valid because one can construct a Kripke model where p is true in one world w, but not in all the others reachable from w. But here is a derivation that can be closed using the wrong axiom:

$$\dfrac{\dfrac{\dfrac{}{1 : p, 1R2 \ \Rightarrow\ 2 : p}\text{wrong!}}{1 : p \ \Rightarrow\ 1 : \Box p}\Box\text{-r}}{\Rightarrow\ 1 : p \to \Box p}\to\text{-r}$$

Another possibility is the formula $\Diamond p \to p$:

$$\dfrac{\dfrac{\dfrac{}{2 : p, 1R2 \ \Rightarrow\ 1 : p}\text{wrong!}}{1 : \Diamond p \ \Rightarrow\ 1 : p}\Diamond\text{-l}}{\Rightarrow\ 1 : \Diamond p \to p}\to\text{-r}$$

But there are many more examples.

Question 4 – An alternative beta rule

Consider replacing the ∨-left rule of the propositional sequent calculus LK by the following rule:

$$\frac{A, \Gamma \vdash B, \Delta \qquad B, \Gamma \vdash A, \Delta}{A \vee B, \Gamma \vdash \Delta}$$

A) Is the resulting calculus still sound? Explain why, or give a formula that can be proven although it is not valid. [6 marks]

Answer: This rules amounts to saying that if $A \vee B$ is true, then either A is true and B is false, or A is false and B is true. This neglects the case where they are both true, which gives reason to believe that the calculus becomes unsound.

A simple way of getting an invalid formula that can be proven is to note that if A and B are the same formula, then both branches are the axiom. E.g. we could prove $\neg(p \vee p)$ as follows:

$$\dfrac{\dfrac{\dfrac{}{p \;\Rightarrow\; p}\,\text{ax} \qquad \dfrac{}{p \;\Rightarrow\; p}\,\text{ax}}{p \vee p \;\Rightarrow}\,\text{wrong } \vee\text{-l}}{\Rightarrow\; \neg(p \vee p)}\,\neg\text{-r}$$

B) Is the resulting calculus still complete? Explain why, or give a formula that cannot be proven all though it is valid. [6 marks]

Answer: Yes, the calculus is still complete. One way of seeing this is that a fair application of this rules on a branch ensures that whenever $A \vee B$ occurs in an antecedent, also A or B occur. And this is all that is needed for the completeness proof.

Another valid argument is that the calculus with the usual ∨-l rule is complete. So there is a 'usual' proof for every valid formula. The alternative rule adds some more formulae to the branches, but they do not get in the way of appluing the rules of the 'usual' proof

C) If $A \vee B$ is true, then either A is true and B is not, or B is true and A is not, or A and B are both true. We could try to capture this using the following rule with three premisses:

$$\frac{A, \Gamma \vdash B, \Delta \qquad B, \Gamma \vdash A, \Delta \qquad A, B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}$$

Would this replacement for the usual ∨-left rule leave the calculus sound? Complete? [5 marks]

Answer: Yes, the resulting calculus would be sound an complete. For the soundness, note that a falsifying interpretation for the conclusion would make either A or B or both false, and thus it would falsify at least one, and possibly several of the premisses.

Again, a fair application of this rule on a branch ensures that whenever $A \vee B$ occurs in an antecedent, also A or B (or both) occur. And this is all that is needed for the completeness proof.

D) Given the discussions about branching in the lecture about DPLL, would it be a good idea to implement this for automated proof search? [3 marks]

Answer: This is probably not a good idea, since branching is the reason for the exponential complexity in proof search. Making three branches when two would have been sufficient means that even more of the work would be duplicated between branches.

In this task you can submit a hand-written answer. Use the sketching paper handed to you in the exam room for this. See instructions in the link below the task bar.

1 Question 5 – Hintikka Sets

In this question, we will work with **propositional** formulas in **negation normal form**, i.e. the set of formulas F is inductively defined as the smallest set such that

- $p \in F$ for any atomic formula p
- $\neg p \in F$ for any atomic formula p
- $A \vee B \in F$ if $A, B \in F$
- $A \wedge B \in F$ if $A, B \in F$

A set of formulae $H \subseteq F$ is called a **Hintikka set** if it satisfies the following conditions:

- There is no atomic formula p with both $p \in H$ and $\neg p \in H$
- For every $A \vee B \in H$, either $A \in H$ or $B \in H$ (or both)
- For every $A \wedge B \in H$, $A \in H$ and $B \in H$

Hintikka sets, named after the Finnish philosopher and logician Jaakko Hintikka, can be used in the completeness proof of one-sided sequent calculi: the formulae in the antecedents of a saturated open branch form a Hintikka set.

Show that every Hintikka set is satisfiable, i.e. that there is a propositional interpretation that makes all formulae in the set true.

Hints:

- you can define the interpretation from the literals in H, just like in the completeness proof shown in the course.

- to show that *all* formulae in H are satisfied, use structural induction on formulas.
- remember to properly explain what is the base of the induction, what are the induction steps, when you use the induction hypothesis, etc.

[16 marks]

Answer: As the hints suggest, we should first define an interpretation that makes all literals in H true, and then prove by structural induction over NNF formulae that all formulae in H are true. Both parts are exactly as in the completeness proof for propositional LK.

Given a Hintikka set H, let $\mathcal{I}_H$ be the interpretation that makes all atomic formulae $p \in H$ true, and all other atomic formulae false.

We now show that $\mathcal{I}_H \models F$ all formulae $F \in H$, bu structural induction on the formulae.

Base case 1: for an atomic formula p, $\mathcal{I}_H \models p$ by definition of $\mathcal{I}_H$.

Base case 2: for a negated atomic formula $\neg p \in H$, we know that $p \notin H$, because of the first condition on Hintikka sets. Therefore $\mathcal{I}_H \not\models p$ by definition of $\mathcal{I}_H$, and so $\mathcal{I}_H \models \neg p$

Induction hypothesis, if $A \in H$, then $\mathcal{I}_H \models A$, and the same for B.

Induction step 1: for a disjunction $A \vee B \in H$, we know that $A \in H$ or $B \in H$ by definition of a Hintikka set. By the induction hypothesis, we get that $\mathcal{I}_H \models A$ or $\mathcal{I}_H \models B$, and therefore $\mathcal{I}_H \models A \vee B$.

Induction step 2: for a conjunction $A \wedge B \in H$, we know that $A \in H$ and $B \in H$ by definition of a Hintikka set. By the induction hypothesis, we get that $\mathcal{I}_H \models A$ and $\mathcal{I}_H \models B$, and therefore $\mathcal{I}_H \models A \wedge B$.

2 Question 6 – Resolution

Prove that the following formula is valid, using the resolution calculus

$$(\forall x(p(x) \rightarrow p(f(x)))) \rightarrow (\forall x(p(x) \rightarrow p(f(f(x)))))$$

Remember that resolution is a refutation calculus, i.e. you can derive that a set of clauses is unsatisfiable. Also remember that variables should be made disjoint before applying resolution.

- Arriving at a correct set of clauses: 5 credits
- Correct resolution proof: 5 credits

Answer: Negation: $\neg((\forall x(p(x) \rightarrow p(f(x)))) \rightarrow (\forall x(p(x) \rightarrow p(f(f(x))))))$

NNF: $(\forall x(\neg p(x) \vee p(f(x)))) \wedge \exists x(p(x) \wedge \neg p(f(f(x))))$

Prenex (pull out $\exists$ first): $\exists y \forall x((\neg p(x) \vee p(f(x))) \wedge p(y) \wedge \neg p(f(f(y))))$

Skolemisation: $\forall x((\neg p(x) \vee p(f(x))) \wedge p(c) \wedge \neg p(f(f(c))))$

Clauses:
(1) $\neg p(x), p(f(x))$
(2) $p(c)$
(3) $\neg p(f(f(c)))$

Resolution derivation:
(4) $p(f(c))$ from (1) and (2) with $\sigma = \{x \backslash c\}$
(5) $p(f(f((c))))$ from (1) and (4) with $\sigma = \{x \backslash f(c)\}$
(6) $\Box$ from (3) and (6)

An alternative route is to resolve (1) with itself. For this, consider a copy of (1) with renamed variables:
(1') $\neg p(x'), p(f(x'))$

Then derive
(7) $p(x), p(f(f(x)))$ from (1) and (1') with $\sigma = \{x' \backslash f(x)\}$

This can then be resolved with (2) and (3) to arrive at the empty clause.

Question 7 – Description Logics

A) The calculus presented for the description logic ALC has a "blocking condition." The application of ∃R-left and ∀R-right rules is restricted to "labels that are not blocked." You do not need to give the definition of these blocking conditions, but please write in one sentence why they are needed, i.e. what their effect on the calculus is. [4 marks]

Answer: The blocking conditions prevent the application of rules that could lead to a cycle in the proof search, and therefore non-termination of the calculus. They guarantee that the proof search always terminates, giving a decision procedure, i.e. a guaranteed yes/no answer to the question of validity/satisfiability.

B) One way of defining the semantics of description logics, as shown in the lecture, is by a translation to first order logic. Concepts are translated to formulas with one free variable, while ABox and TBox assertions are translated into closed formulas.

Can a similar translation be given from first-order logic to a description logic like ALC? Write why in one sentence. [4 marks]

Answer: No. First order validity is not decidable, but ALC is. A translation from first-order logic to ALC would give us a way of deciding first-order validity, via a terminating calculus for ALC, and we know that that is impossible.