Priority Queues

2nd November 2022

Priority Queues

- Binary heaps
- Leftist heaps
- Binomial heaps
- Fibonacci heaps

Priority queues are important in, among other things, operating systems (process control in multitasking systems), search algorithms (A, A*, D*, etc.), and simulation.

Priority Queues

Priority queues are data structures that hold elements with some kind of priority (*key*) in a queue-like structure, implementing the following operations:

- **`insert()`** – Inserting an element into the queue.
- **`deleteMin()`** – Removing the element with the highest priority.

And maybe also:

- **`buildHeap()`** – Build a queue from a set (>1) of elements.
- **`increaseKey()/DecreaseKey()`** – Change priority.
- **`delete()`** – Removing an element from the queue.
- **`merge()`** – Merge two queues.

Priority Queues

An unsorted linked list can be used. **`insert()`** inserts an element at the head of the list ($O(1)$), and **`deleteMin()`** searches the list for the element with the highest priority and removes it ($O(n)$).

A sorted list can also be used (reversed running times).

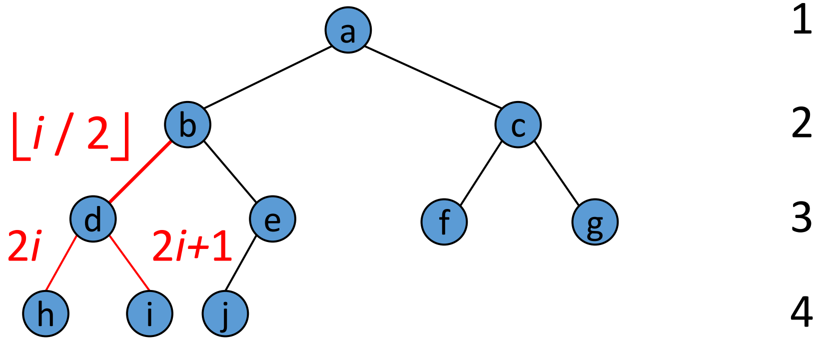
– Not very efficient implementations.

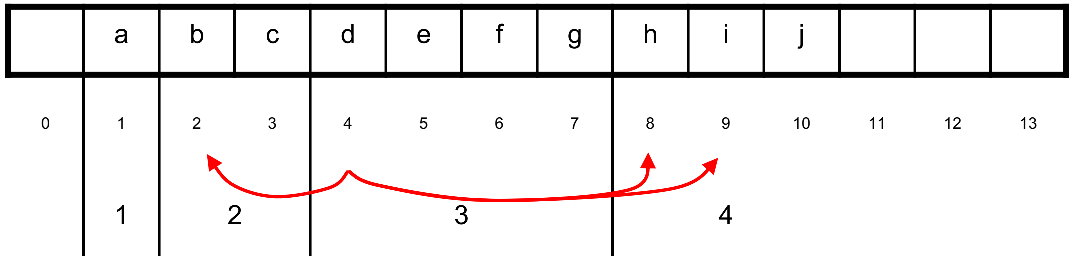
To make an efficient priority queue, it is enough to keeps the elements "almost sorted".

Binary Heaps

A *binary heap* is organized as a complete binary tree. (All levels are full, except possibly the last.)

In a *binary heap* the element in the root must have a key less than or equal to the key of its children, in addition each sub-tree must be a binary heap.





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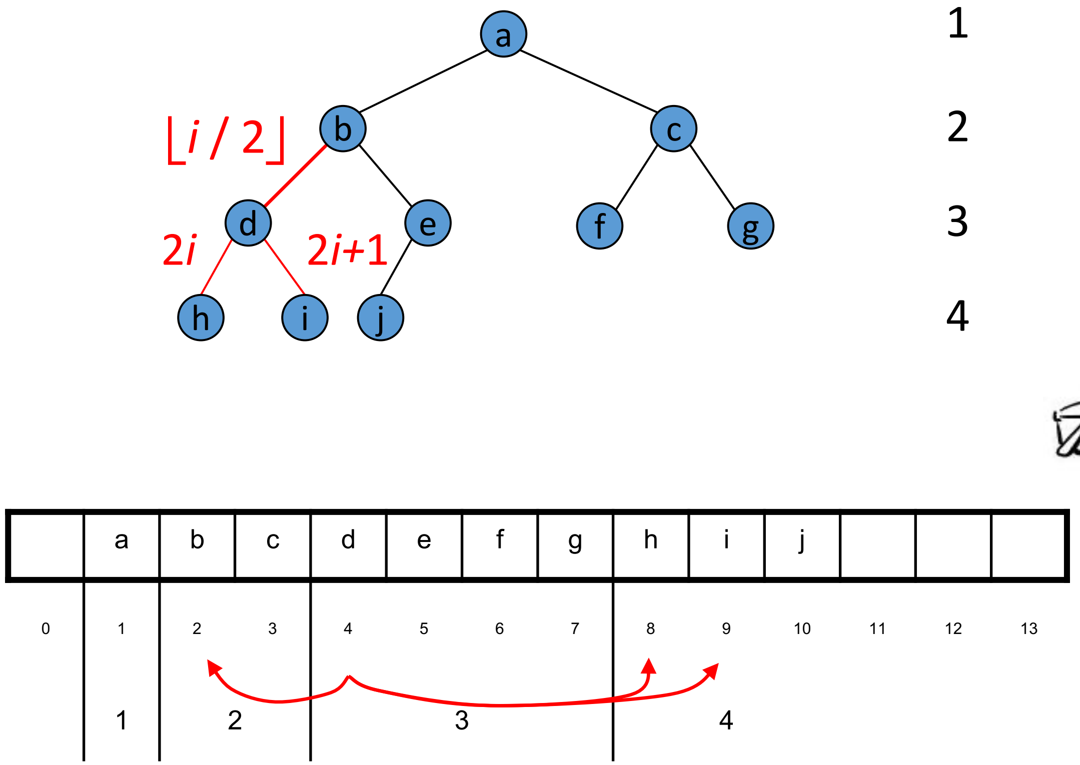
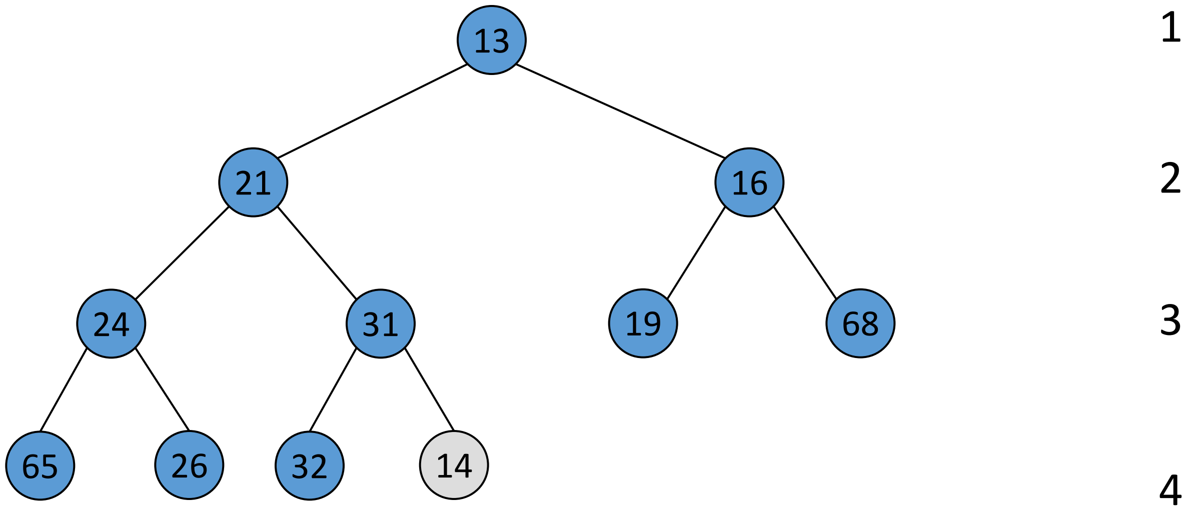


Binary Heaps

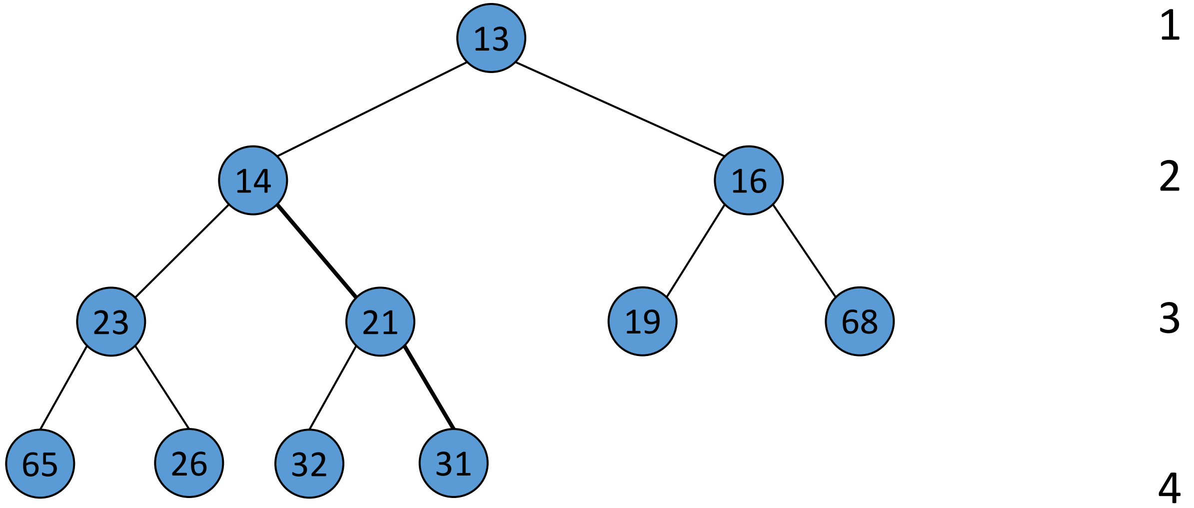
`insert(14)`



	13	21	16	24	31	19	68	65	26	32	14		
0	1	2	3	4	5	6	7	8	9	10	11	12	13
	1	2		3				4					

Binary Heaps

`insert(14)`

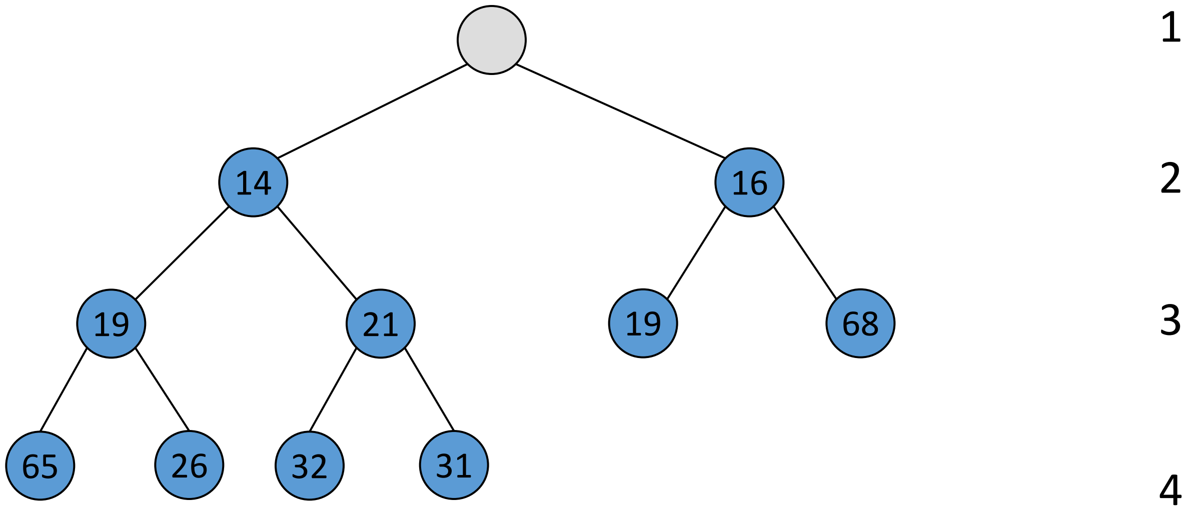


	13	14	16	24	21	19	68	65	26	32	31		
0	1	2	3	4	5	6	7	8	9	10	11	12	13
	1	2		3				4					

”**`percolateUp()`**”

Binary Heaps

`deleteMin()`



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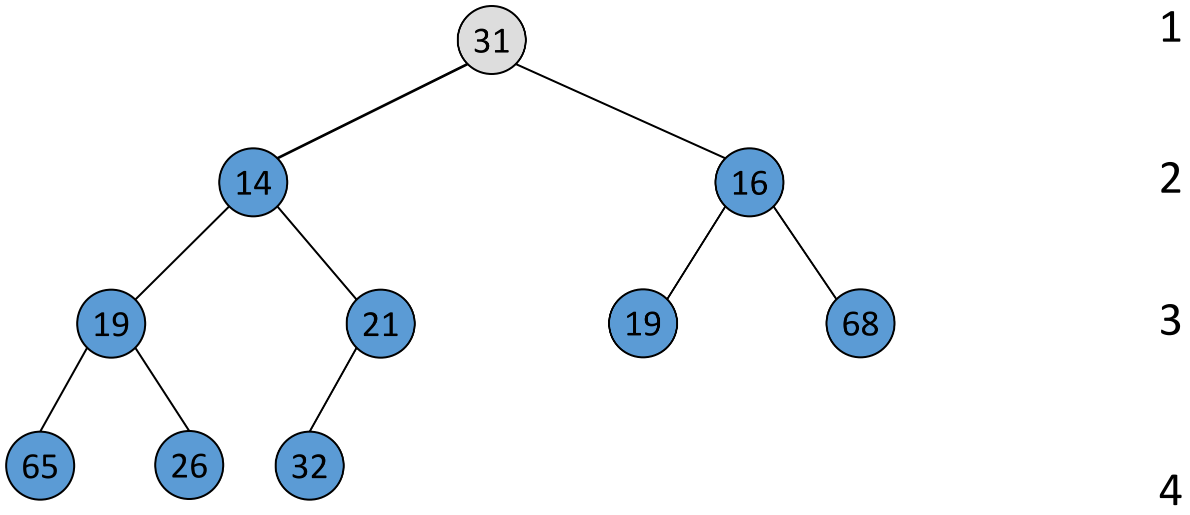
Binary Heaps

`deleteMin()`



	31	14	16	19	21	19	68	65	26	32			
0	1	2	3	4	5	6	7	8	9	10	11	12	13
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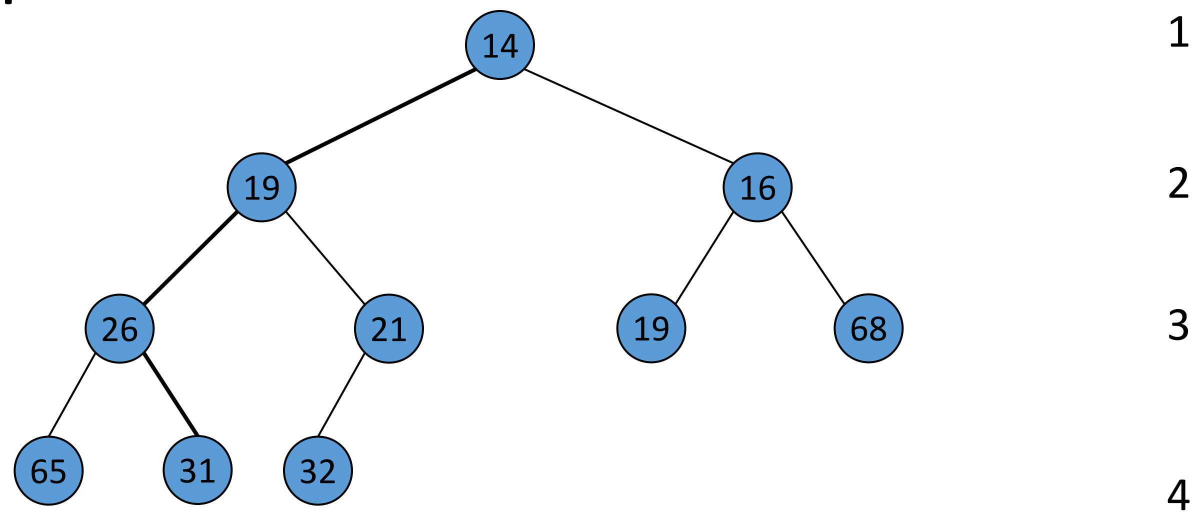
Binary Heaps

`deleteMin()`



	14	19	16	19	21	26	68	65	31	32			
0	1	2	3	4	5	6	7	8	9	10	11	12	13
	1	2		3				4					

”**`percolateDown()`**”

Binary Heaps

	Worst Case	Average
`insert()`	$O(\log N)$	$O(1)$
`deleteMin()`	$O(\log N)$	$O(\log N)$

`buildHeap()` $O(N)$

(Insert elements into the array unsorted, and run `percolateDown()` on each root in the resulting heap (the tree), bottom up)

(The sum of the heights of a binary tree with N nodes is $O(N)$.)

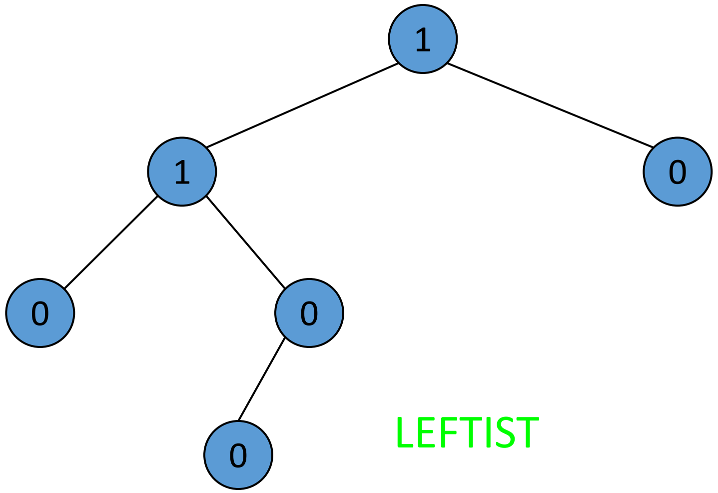
`merge()` $O(N)$

(N = number of elements)

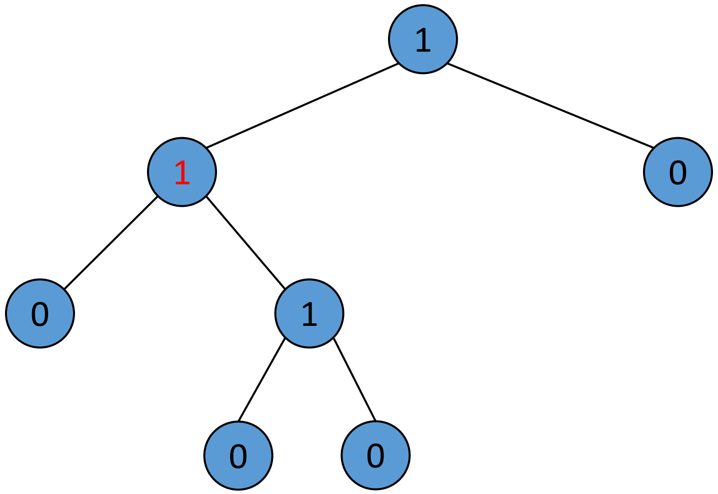
Leftist Heaps

- To implement an efficient `merge()`, we move away from arrays, and implement so-called *leftist heaps* as pure trees.
- The idea behind leftist heaps is to make the heap (the tree) as skewed as possible, and do all the work on a short (right) branch, leaving the long (left) branch untouched.
- A *leftist heap* is still a binary tree with the heap structure (key in root is lower than key in children), but with an extra skewness requirement.
- For all nodes *X* in our tree, we define the *null-path-length*(*X*) as the distance from *X* to a descendant with less than two children (*i.e.* 0 or 1).
- **The skewness requirement is that for every node the null path length of its left child be at least as large as the null path length of the right child.**
- For the empty tree we define the *null-path-length* to be -1, as a special case.

Leftist Heaps



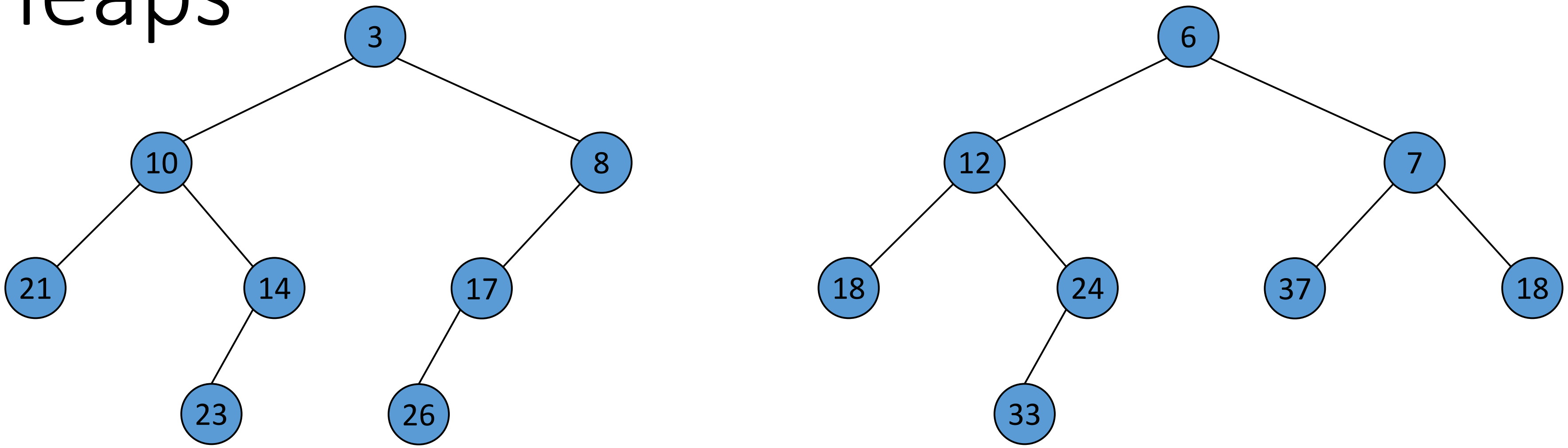
LEFTIST



NOT LEFTIST

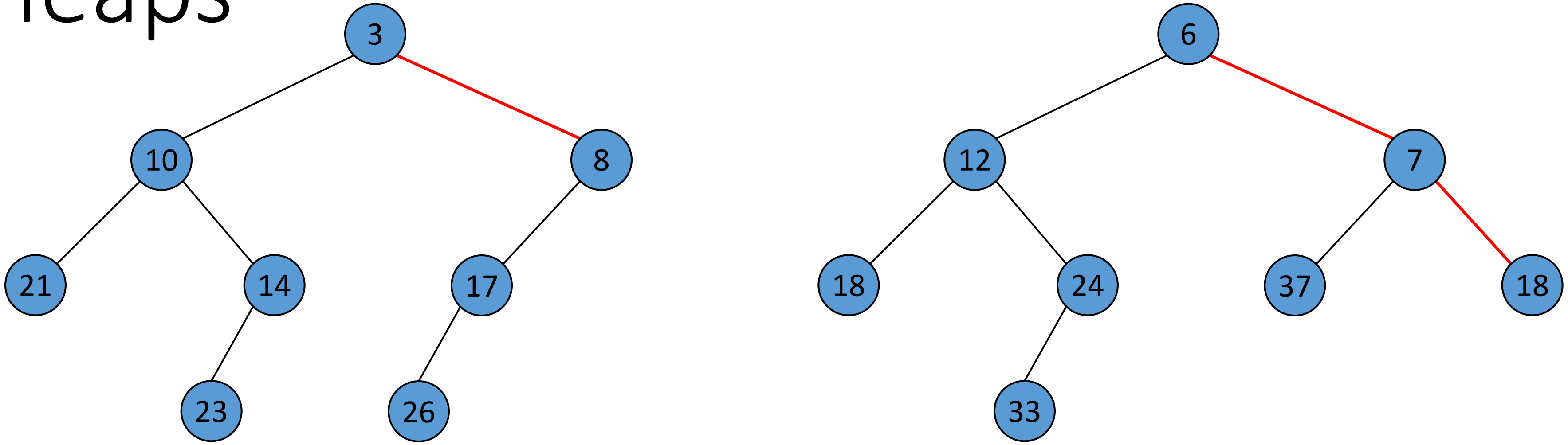
Leftist Heaps

`merge()`



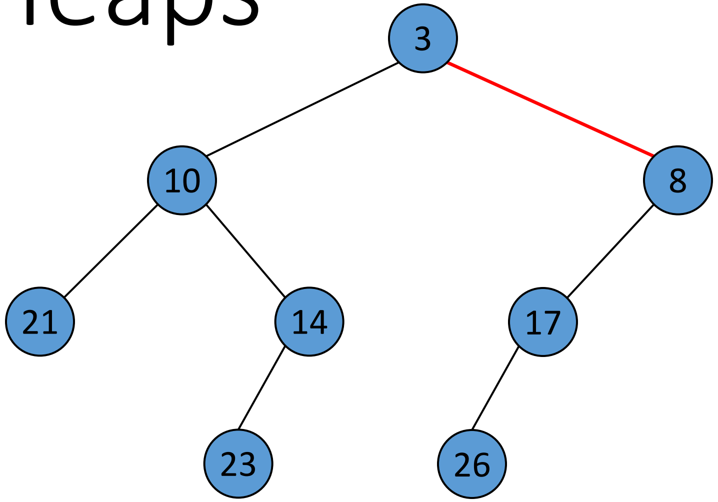
Leftist Heaps

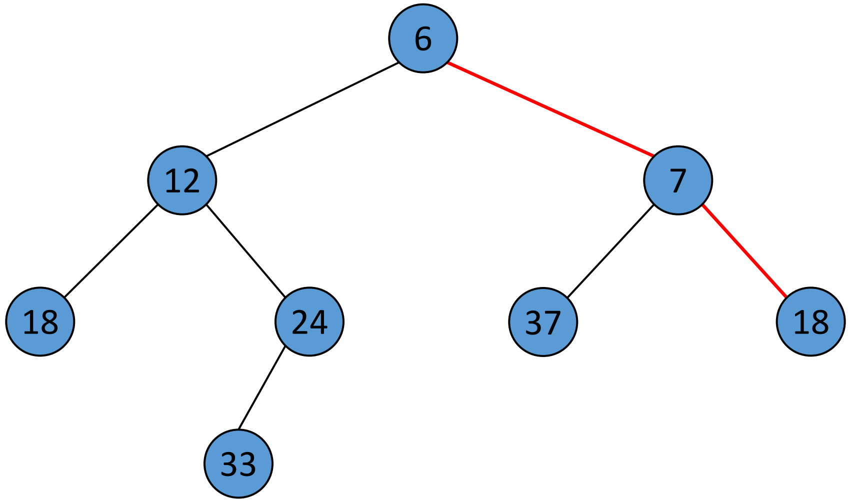
`merge()`

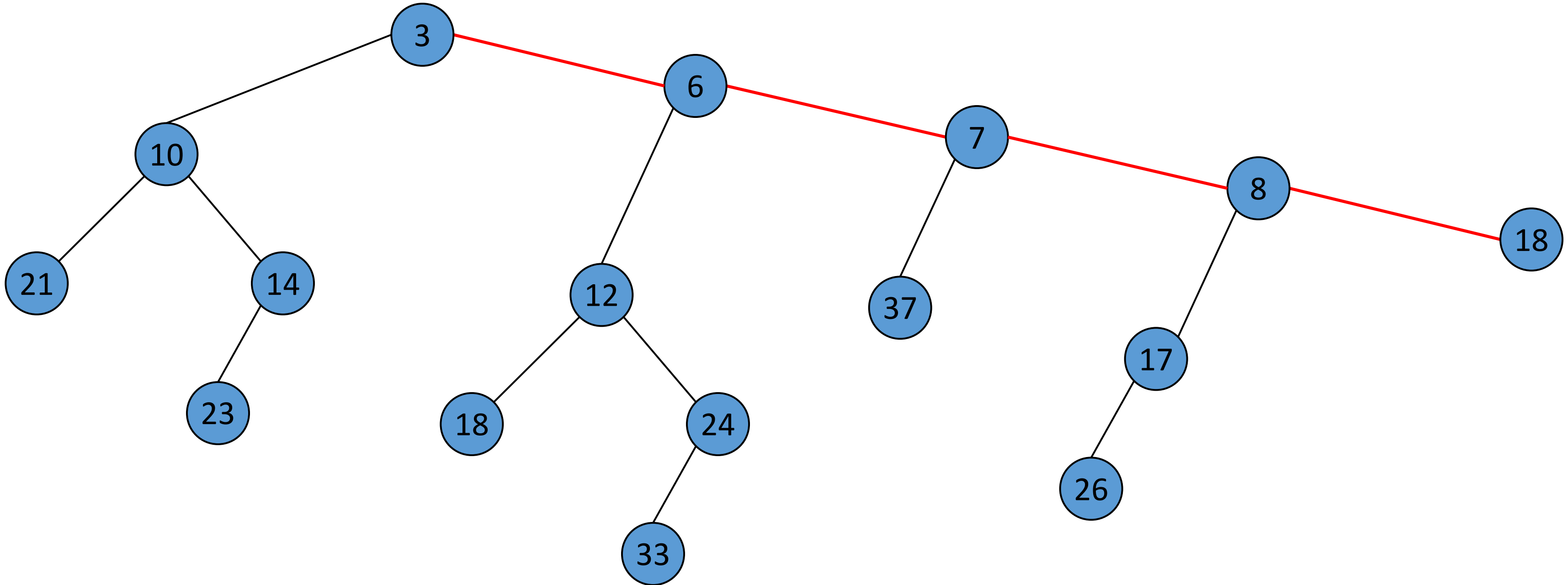


Leftist Heaps

merge()

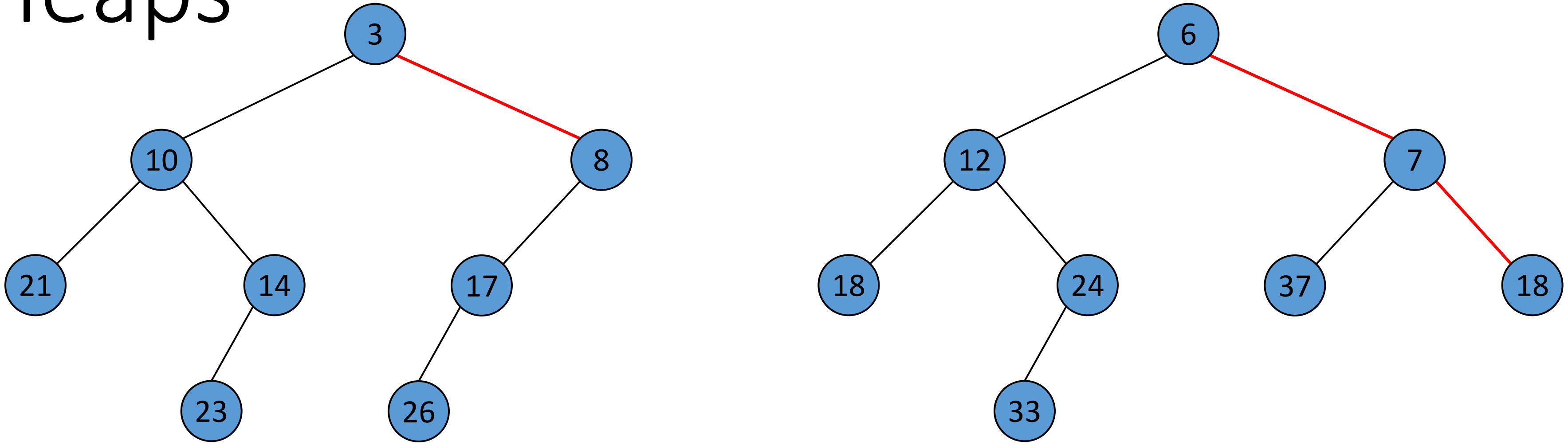


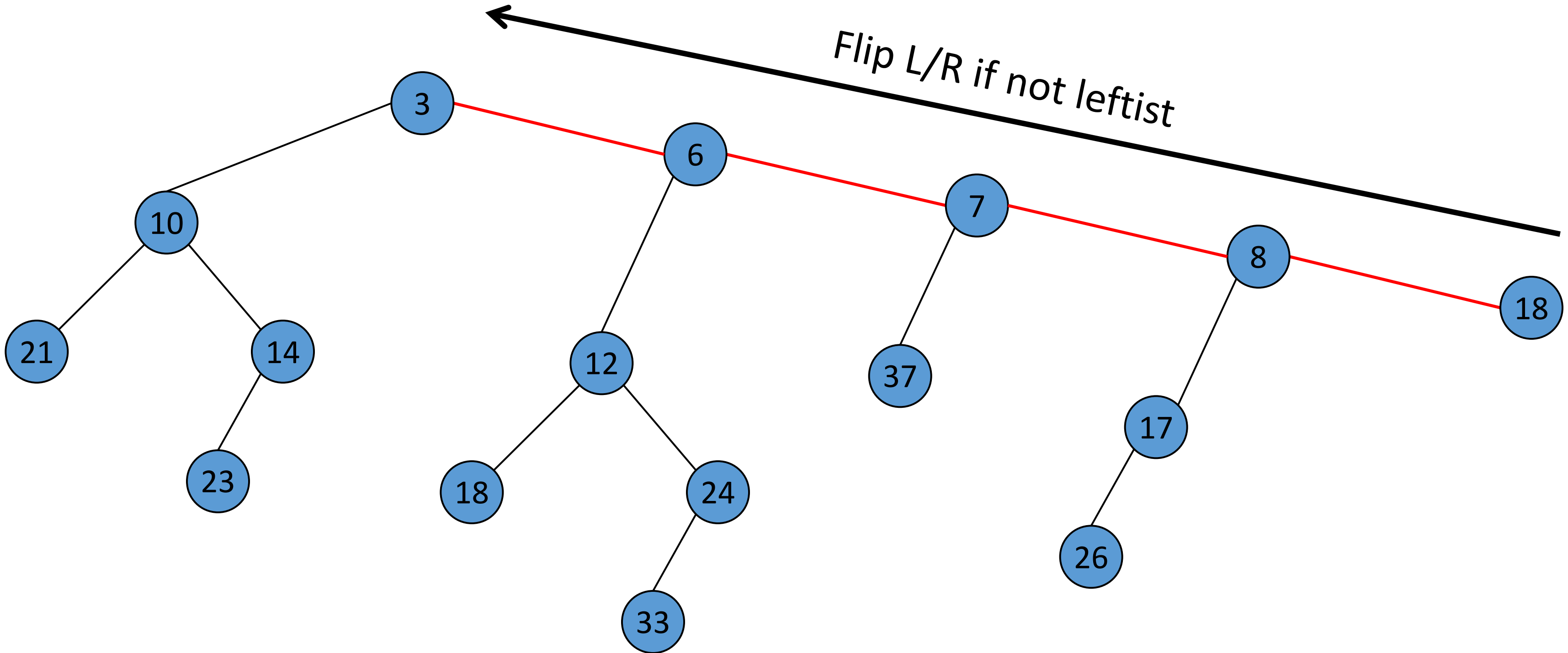




Leftist Heaps

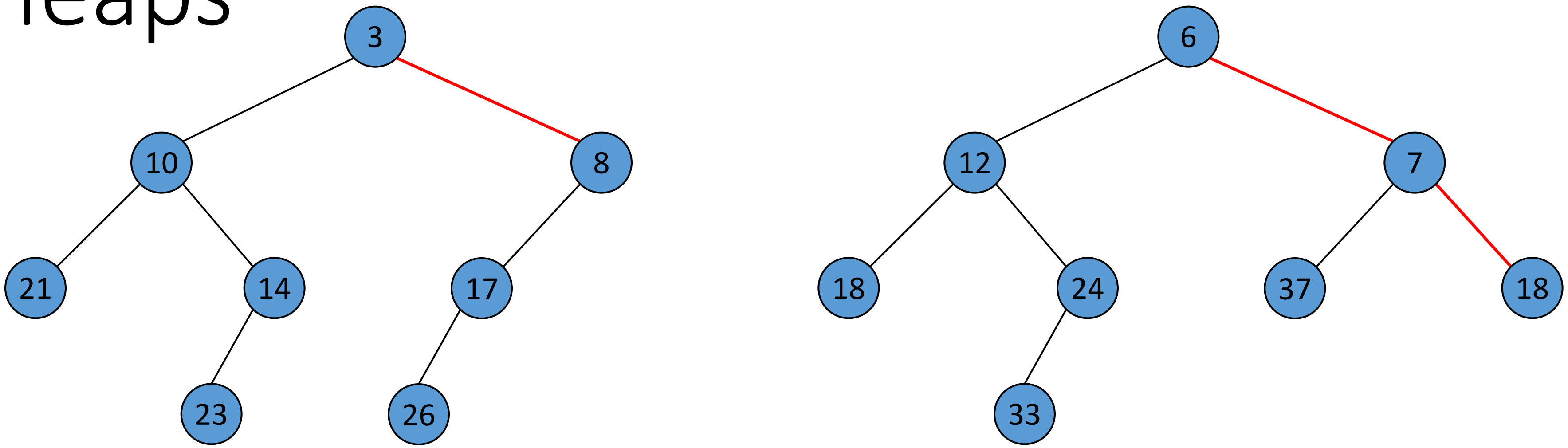
merge()

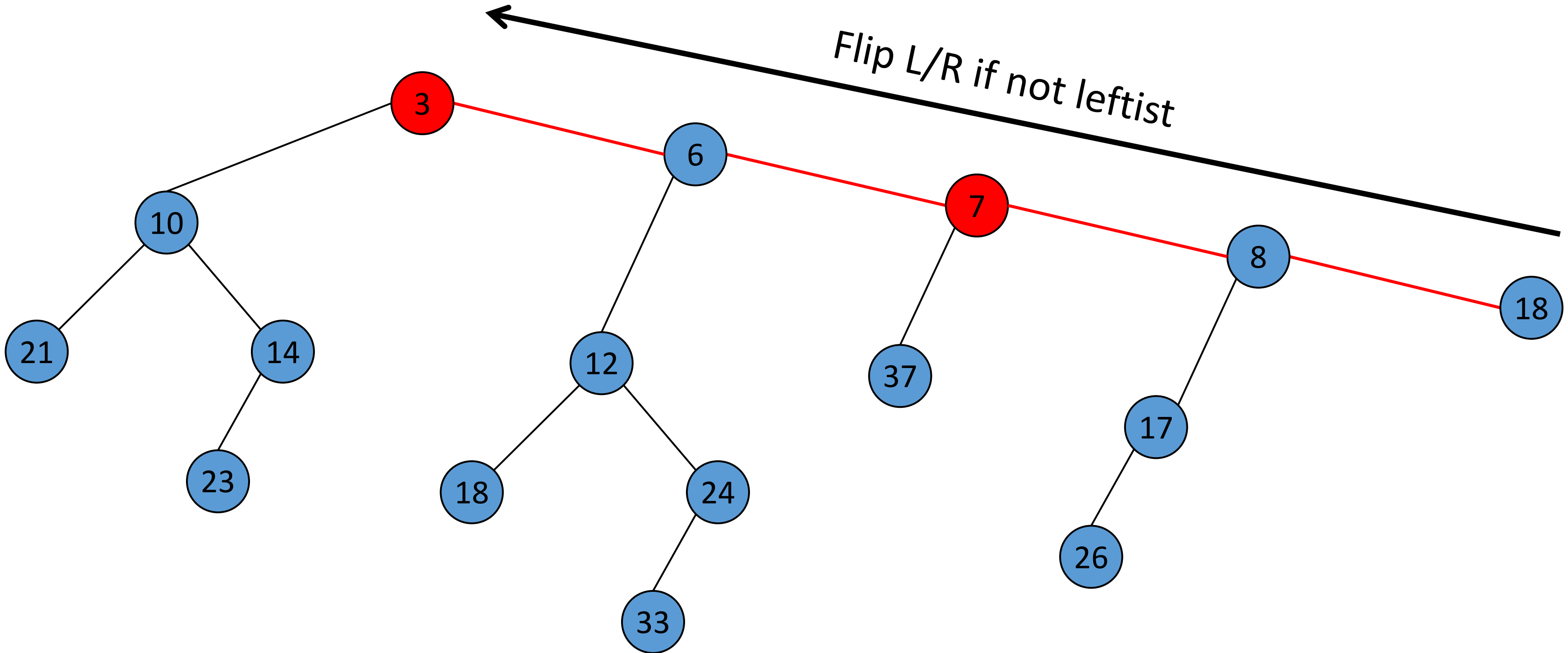




Leftist Heaps

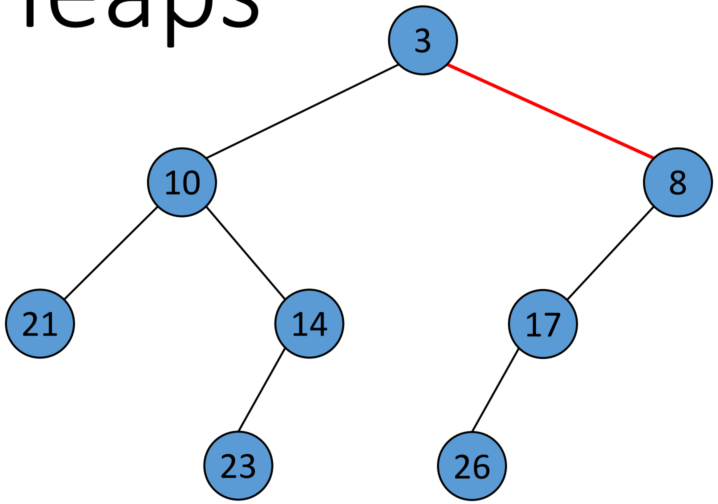
merge()

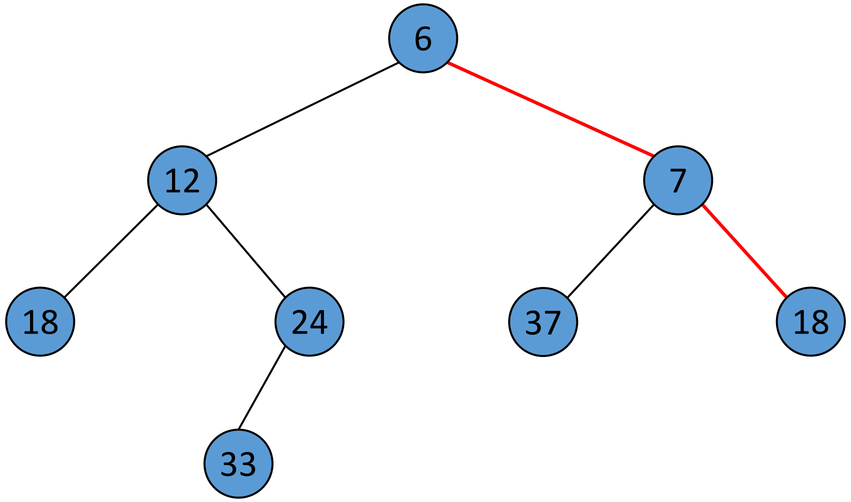


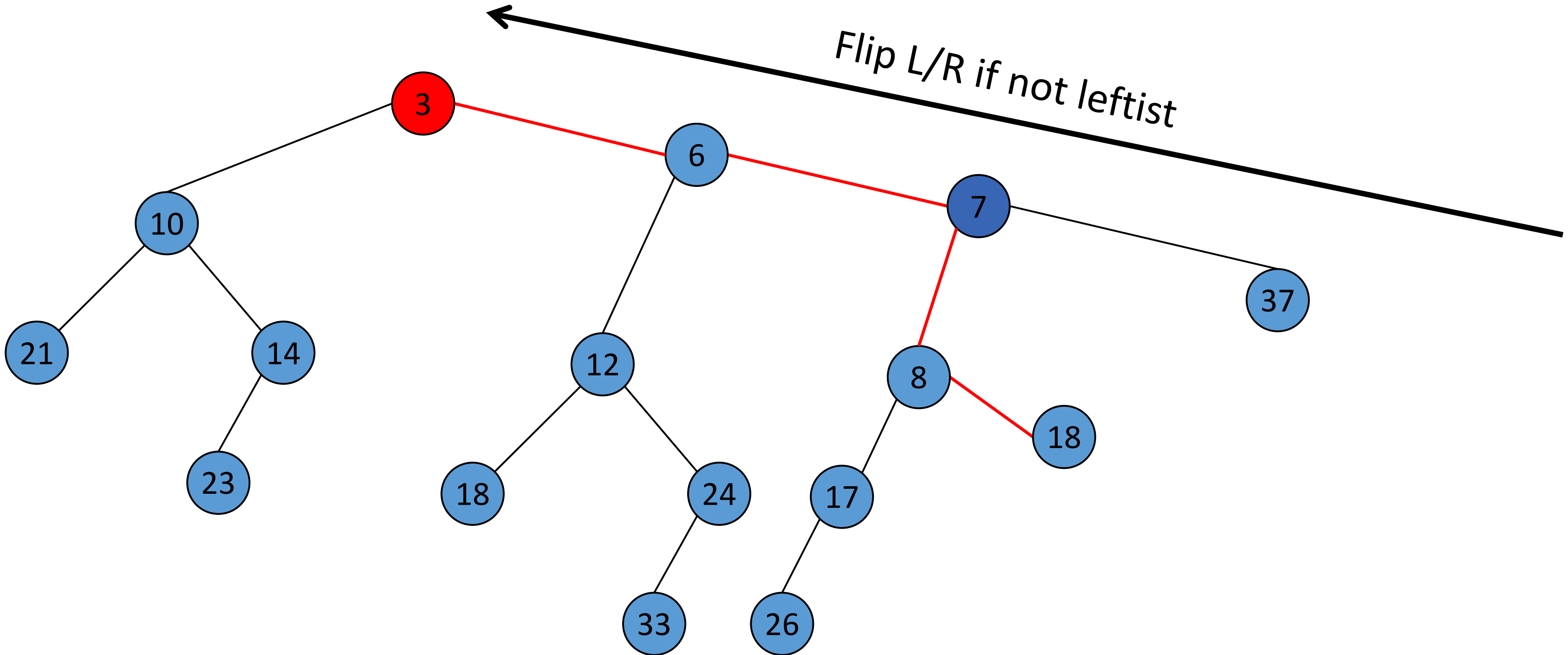


Leftist Heaps

merge()

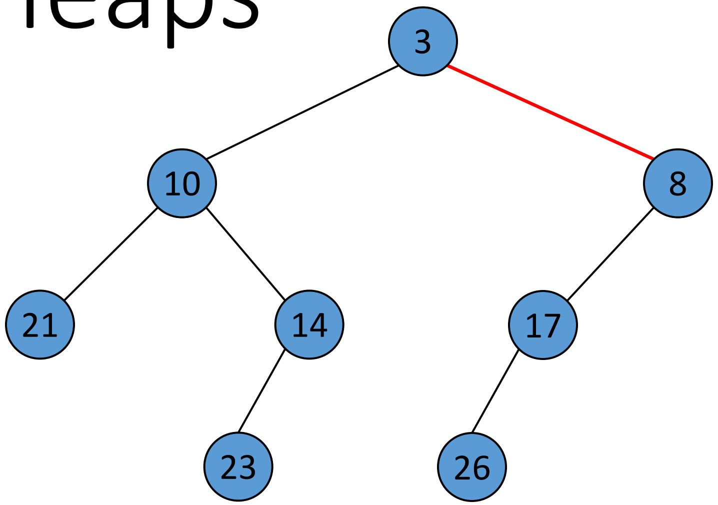


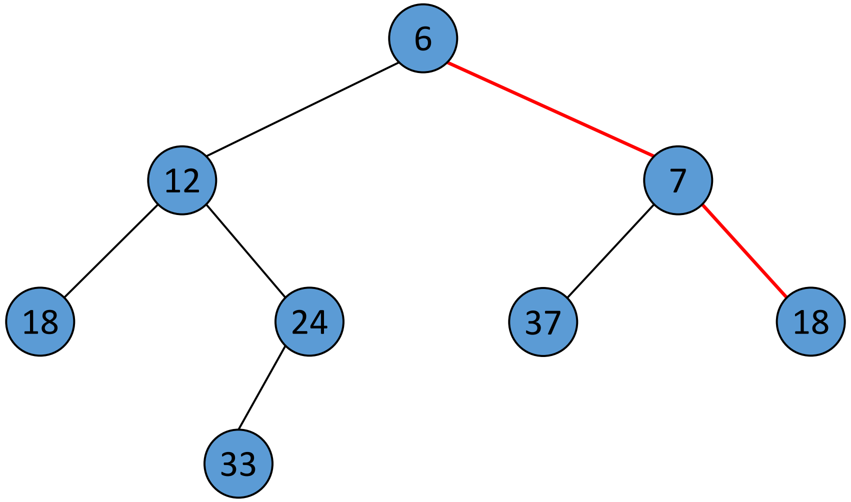


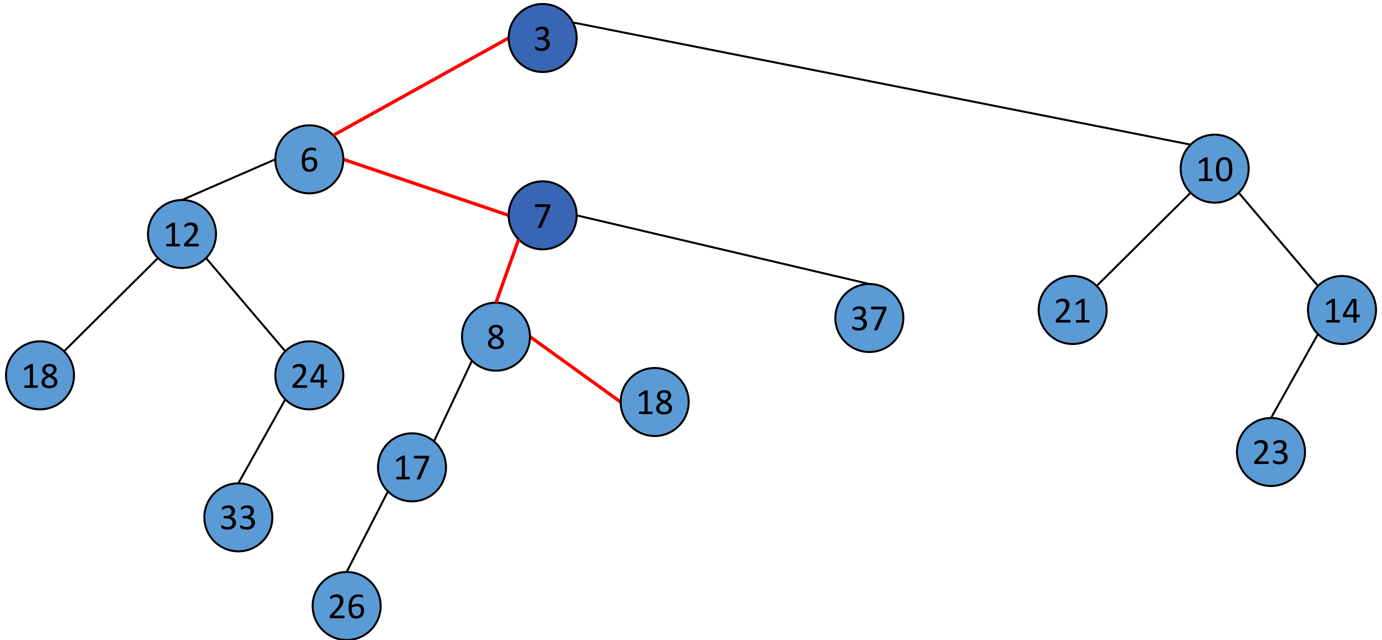


Leftist Heaps

merge()



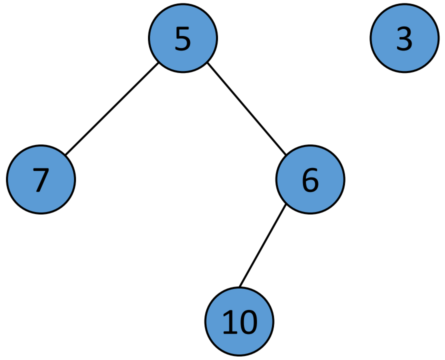




Leftist Heaps

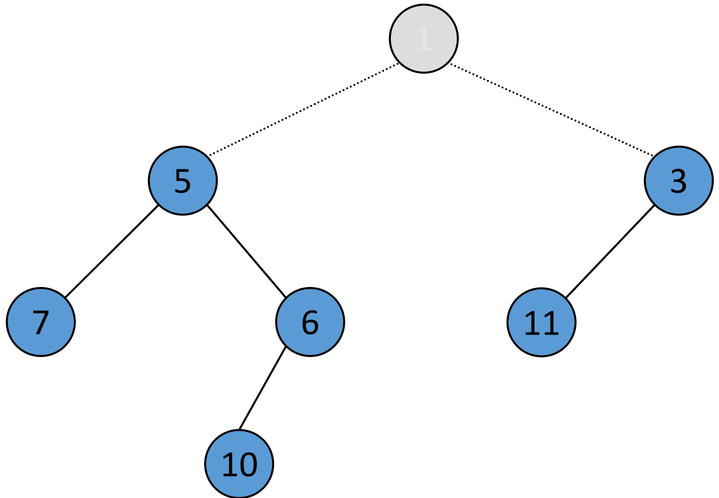
`insert(3)`

`merge()`



`deleteMin()`

`merge()`



Leftist Heaps

	Worst Case
`merge()`	$O(\log N)$
`insert()`	$O(\log N)$
`deleteMin()`	$O(\log N)$
`buildHeap()`	$O(N)$

(N = number of elements)

In a leftist heap with N nodes, the right path is at most $\lfloor \log (N+1) \rfloor$ long.

Binomial Heaps

Leftist heaps:

`merge()`, **`insert()`** and **`deleteMin()`** in $O(\log N)$ time w.c.

Binary heaps:

`insert()` in $O(1)$ time on average.

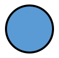
Binomial heaps

`merge()`, **`insert()`** og **`deleteMin()`** in $O(\log N)$ time w.c.

`insert()` $O(1)$ time on average

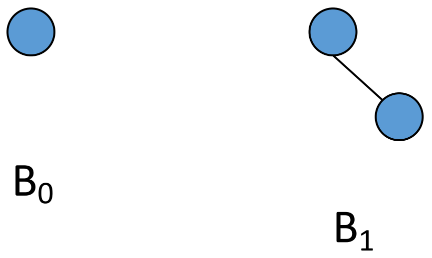
Binomial heaps are collections of trees (sometimes called a forest), each tree a heap.

Binomial Trees

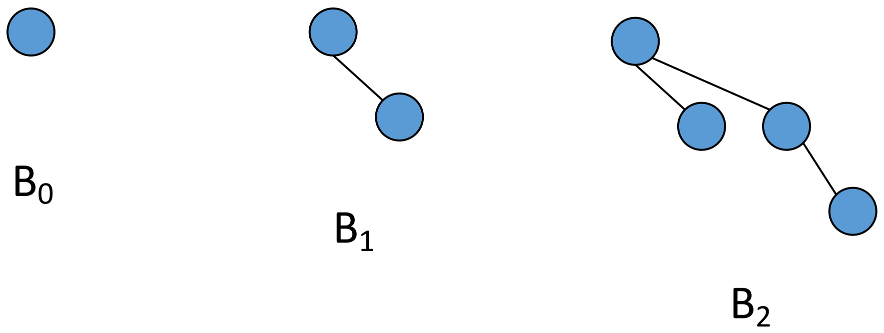


B_0

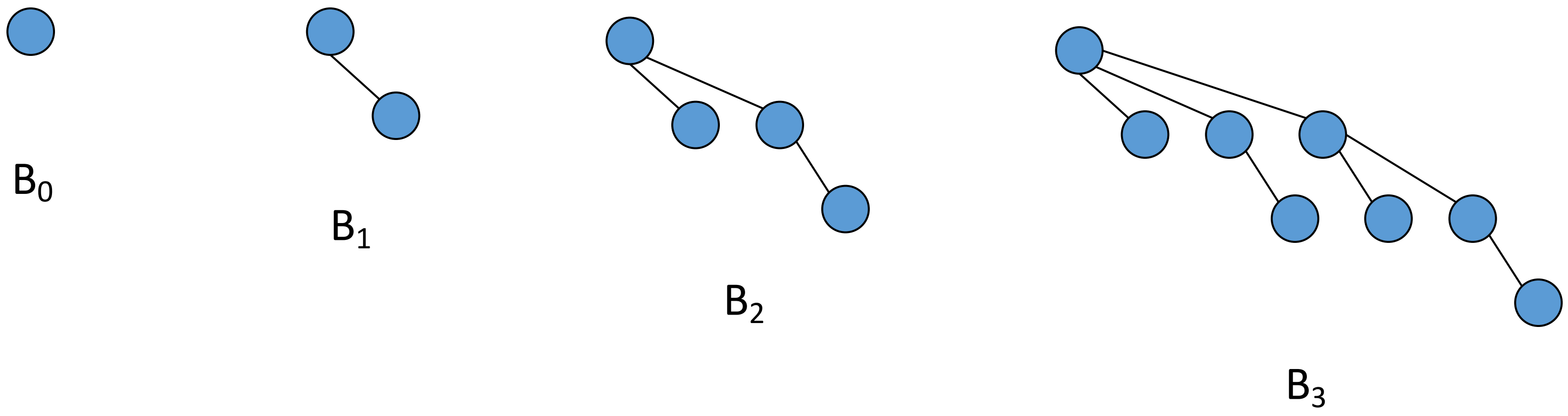
Binomial Trees



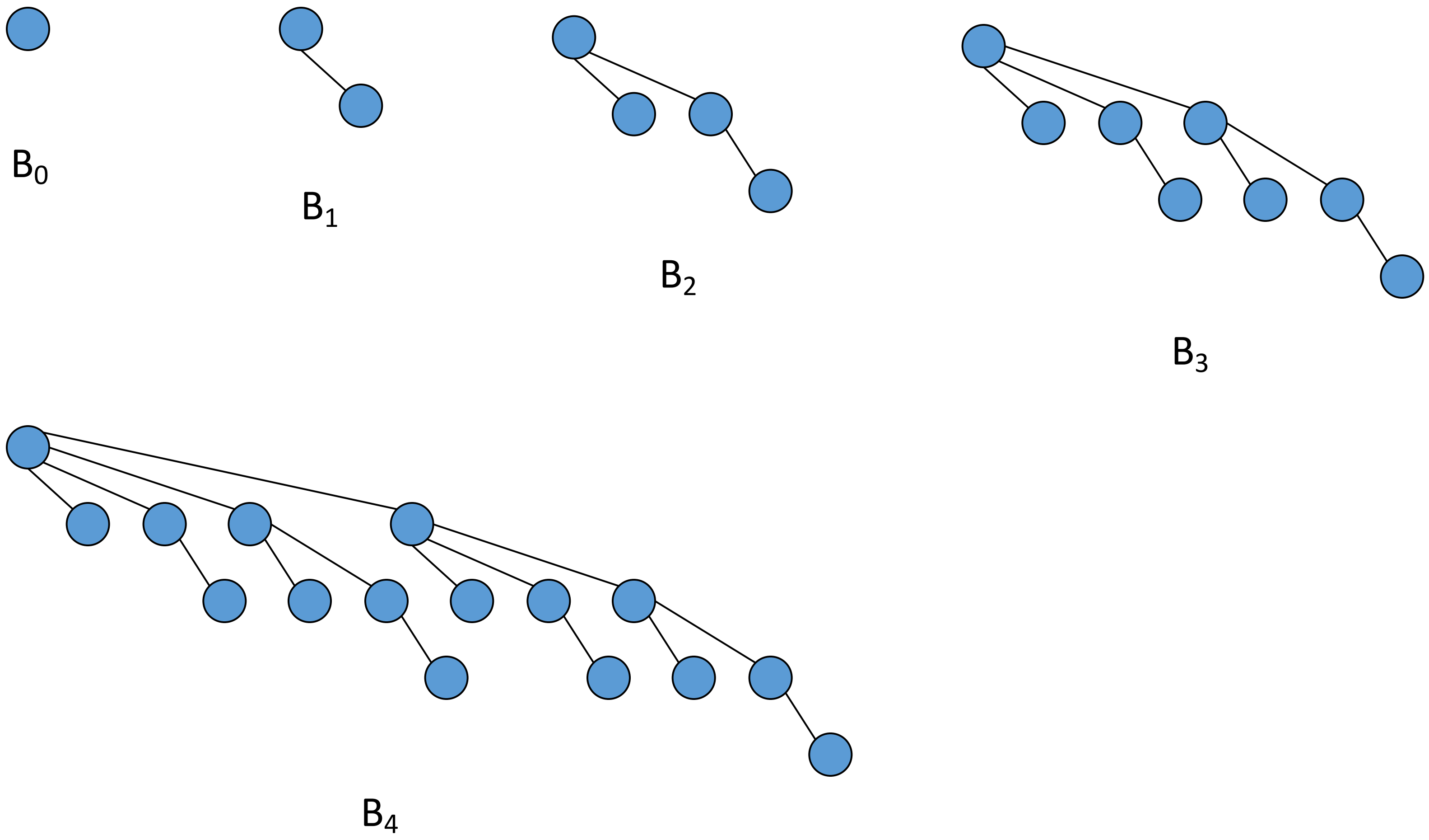
Binomial Trees



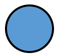
Binomial Trees



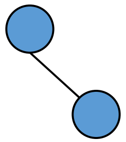
Binomial Trees



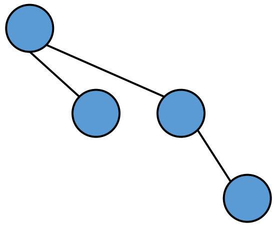
Binomial Trees



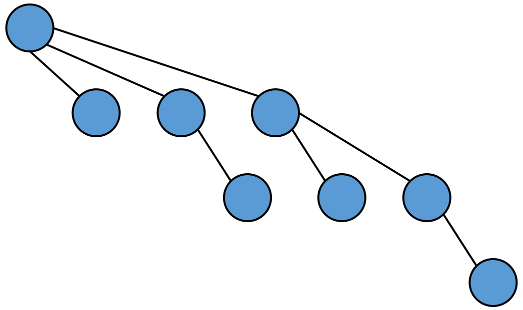
B_0



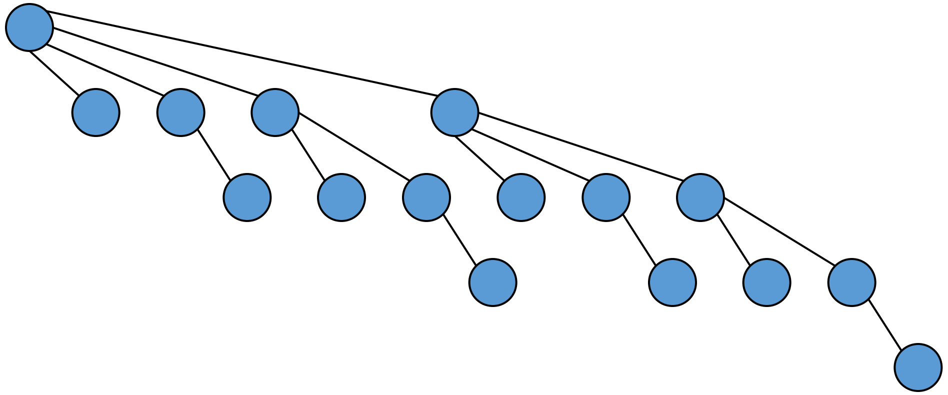
B_1



B_2



B_3



B_4

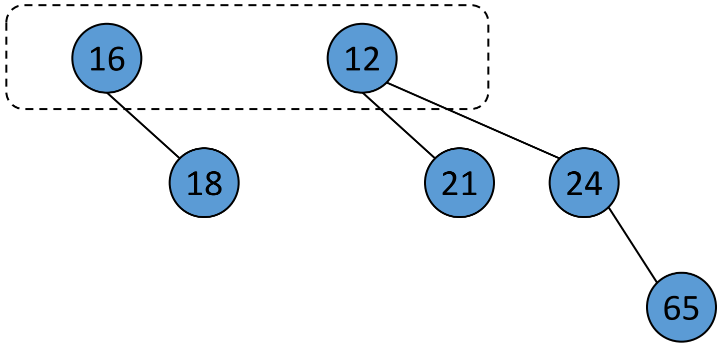
$B_i = 2 \times B_{i-1}$, root of one tree connected as a child of the root of the other tree.

A tree of height k has:

2^k nodes in total,

$\binom{k}{d}$ nodes on level d.

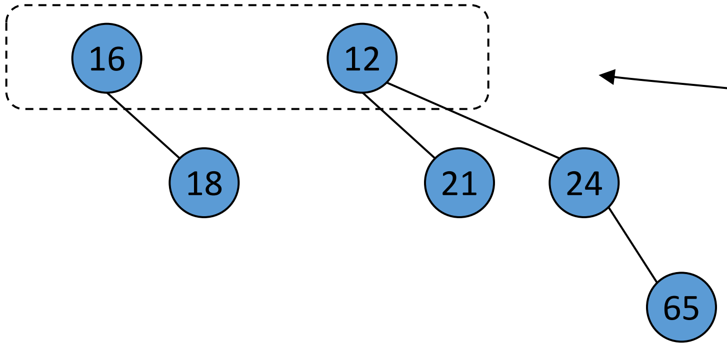
Binomial Heaps



Maximum one tree of each size:

6 elements: 6 binary = 011 (0+2+4) $\cancel{B_0}$ B_1 B_2

Binomial Heaps



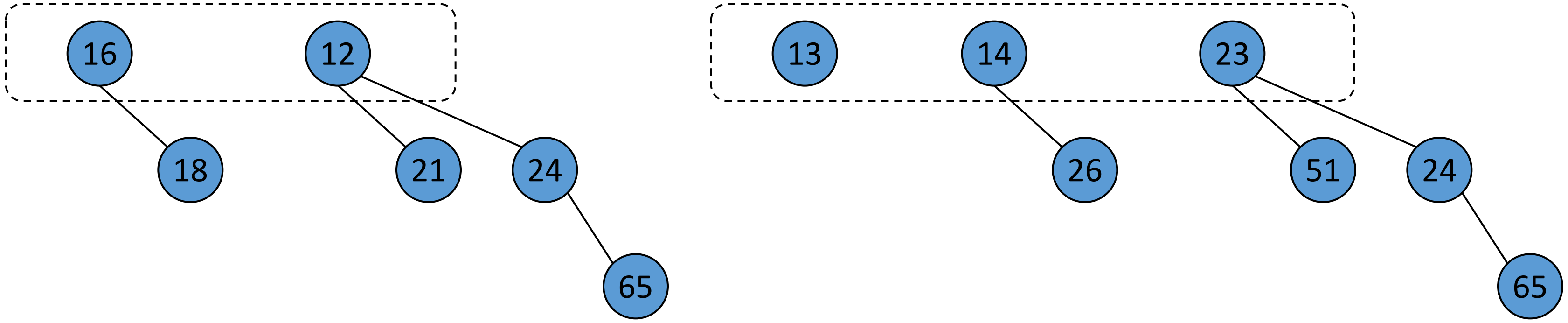
Maximum one tree of each size:

6 elements: 6 binary = 011 (0+2+4) $\cancel{B_0}$ B_1 B_2

The length of the root list in a heap of *N* elements is *O*(log *N*).
(Doubly linked, circular list.)

Binomial Heaps

`merge()`



Binomial Heaps

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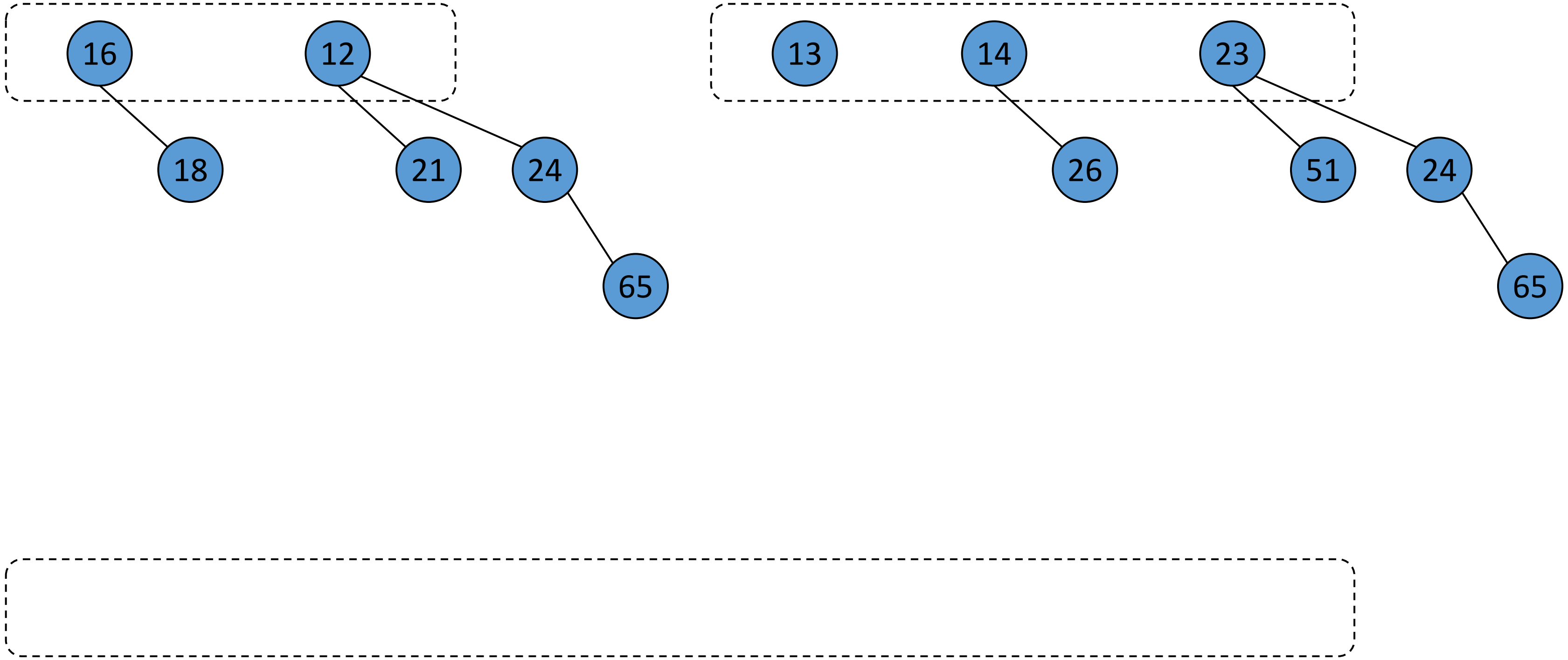

Binomial Heaps

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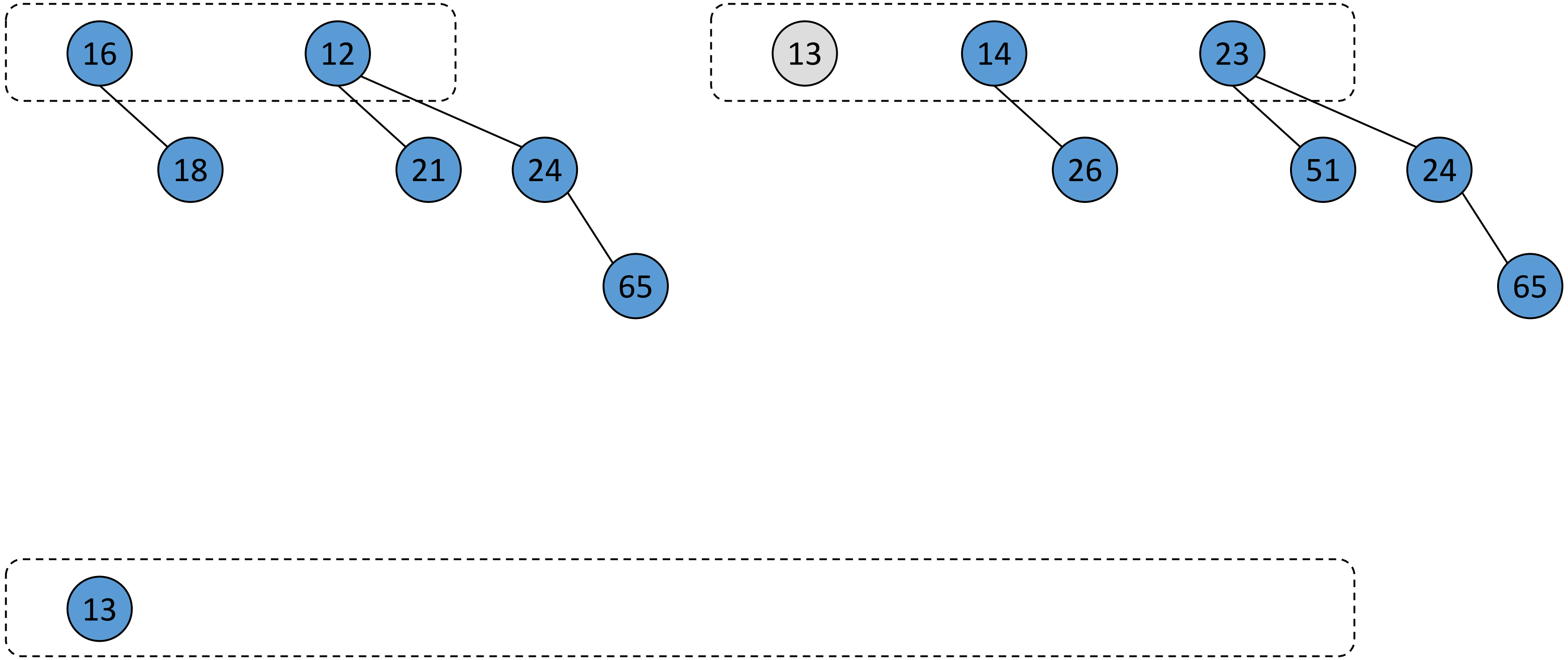

Binomial Heaps

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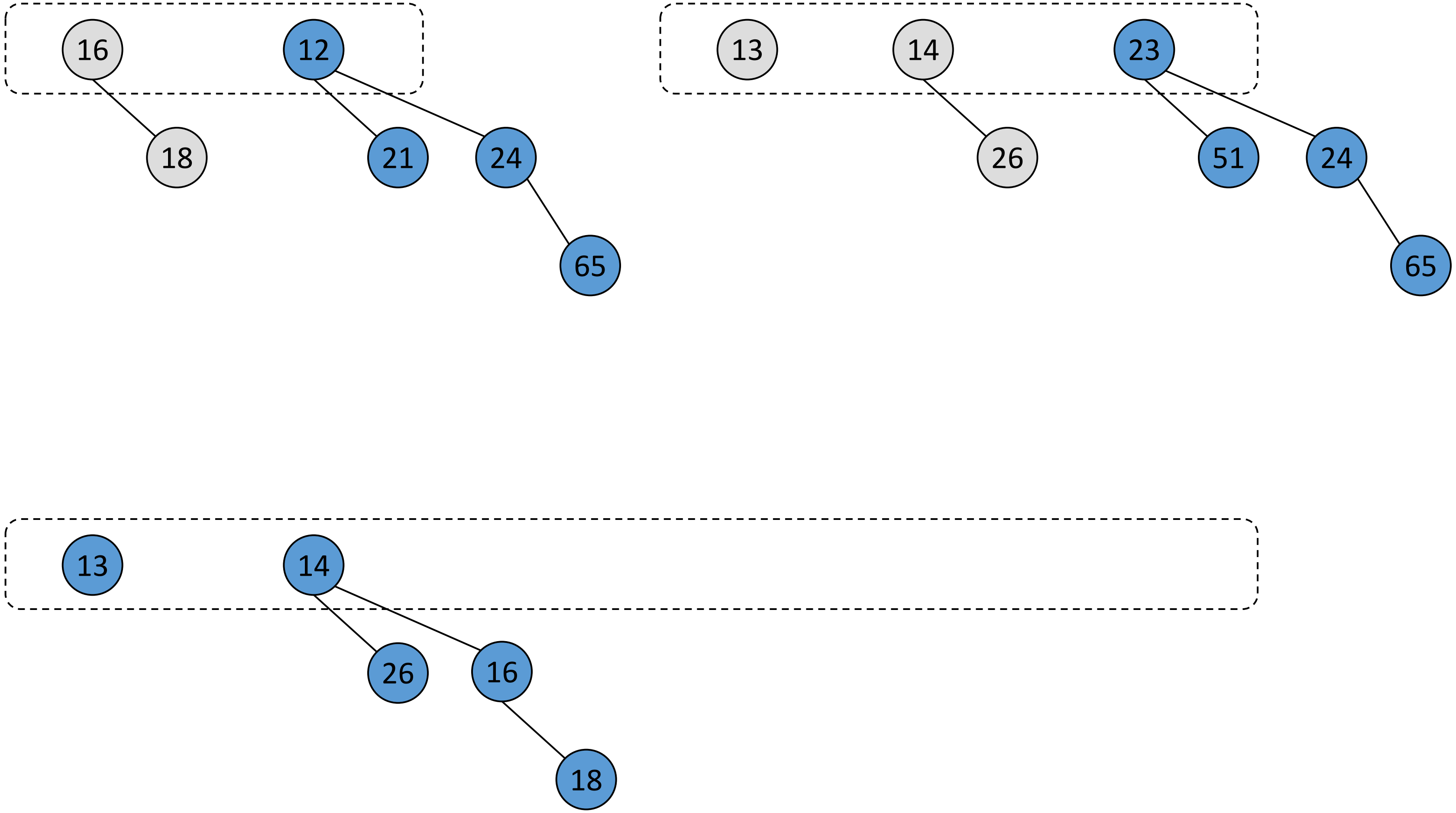
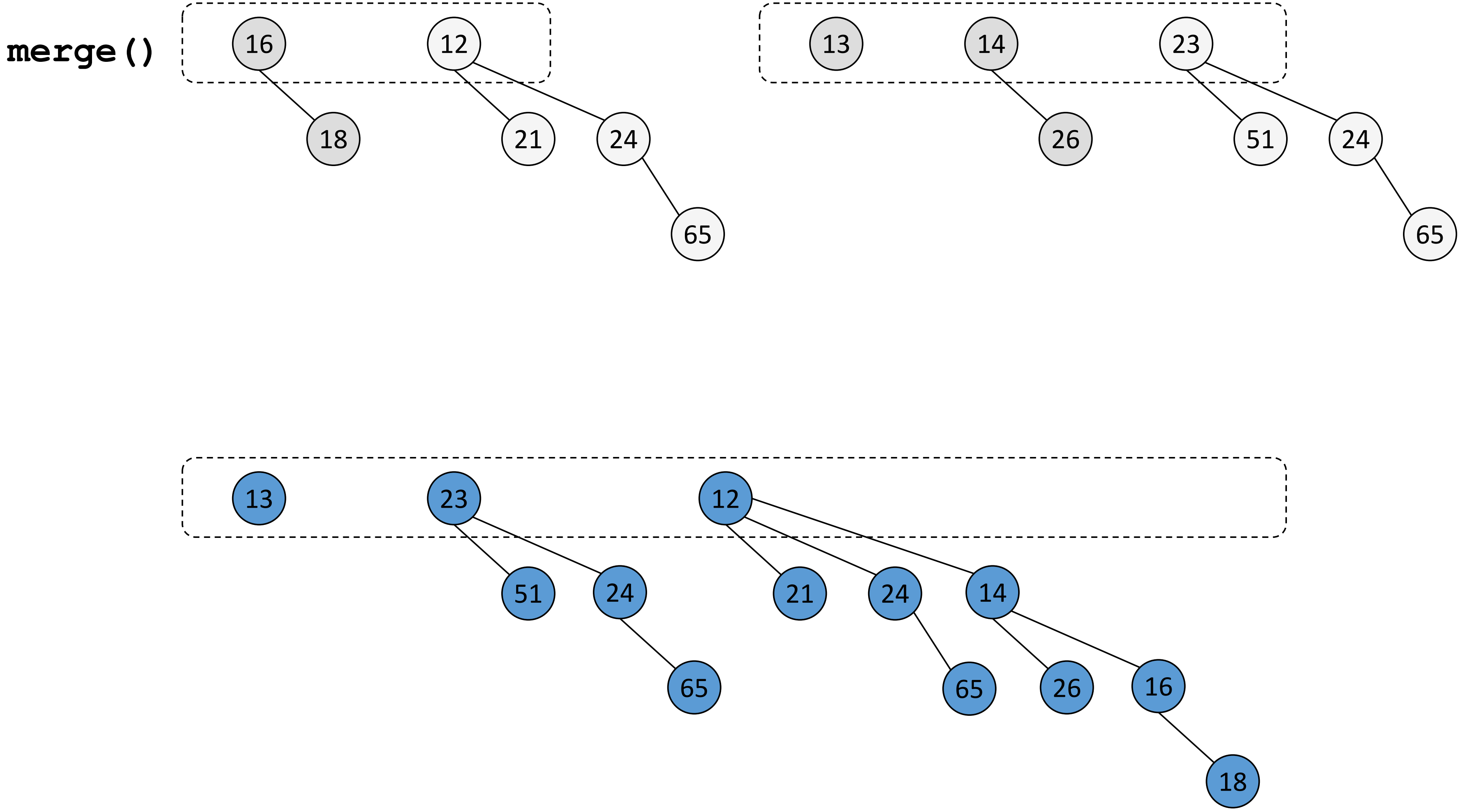


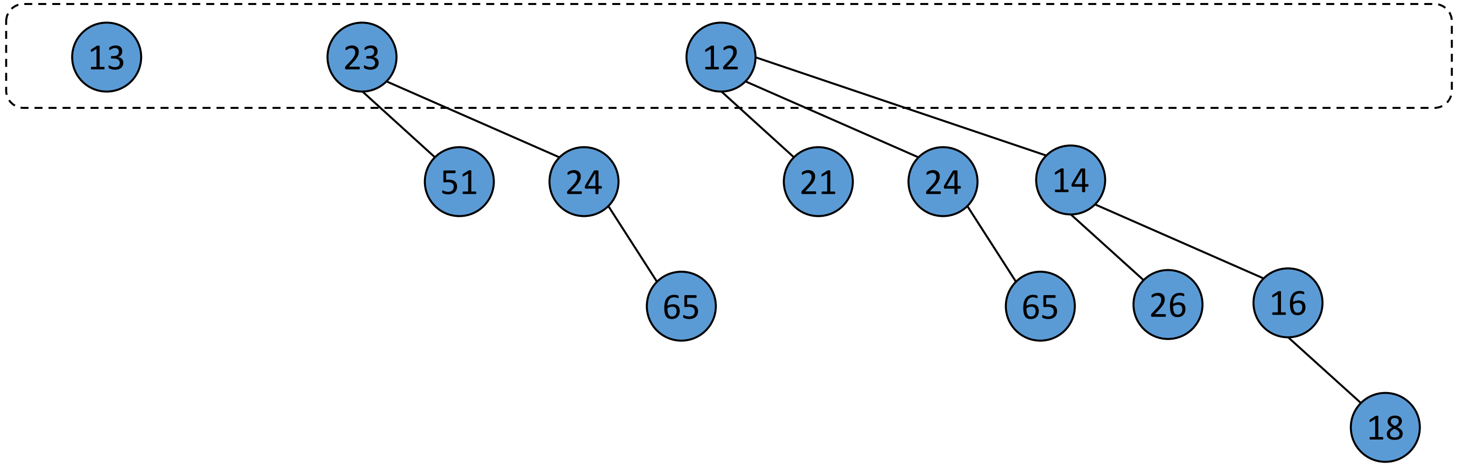
Binomial Heaps



The trees (the root list) is kept sorted on height.

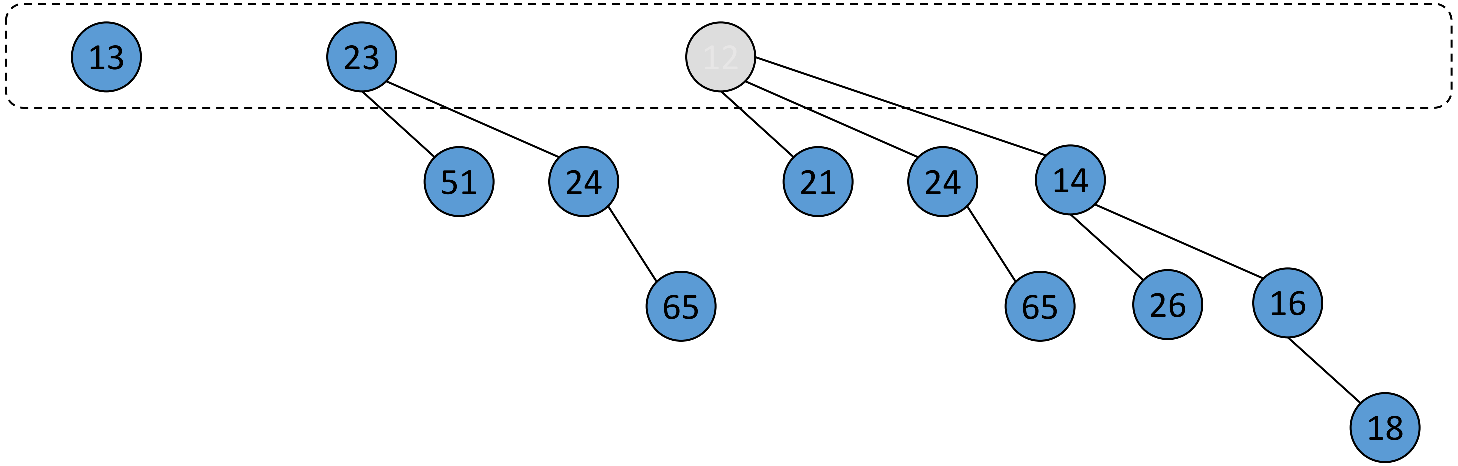
Binomial Heaps

`deleteMin()`

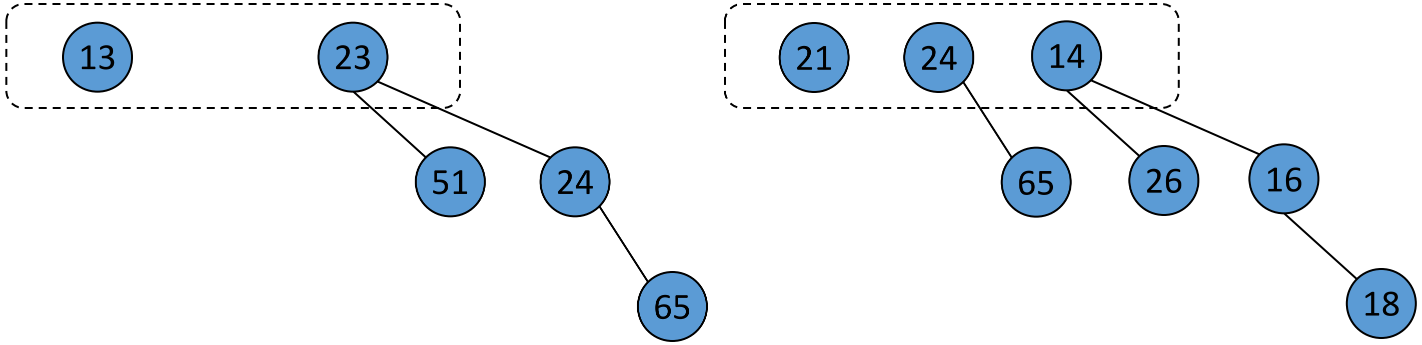


Binomial Heaps

deleteMin()



merge()



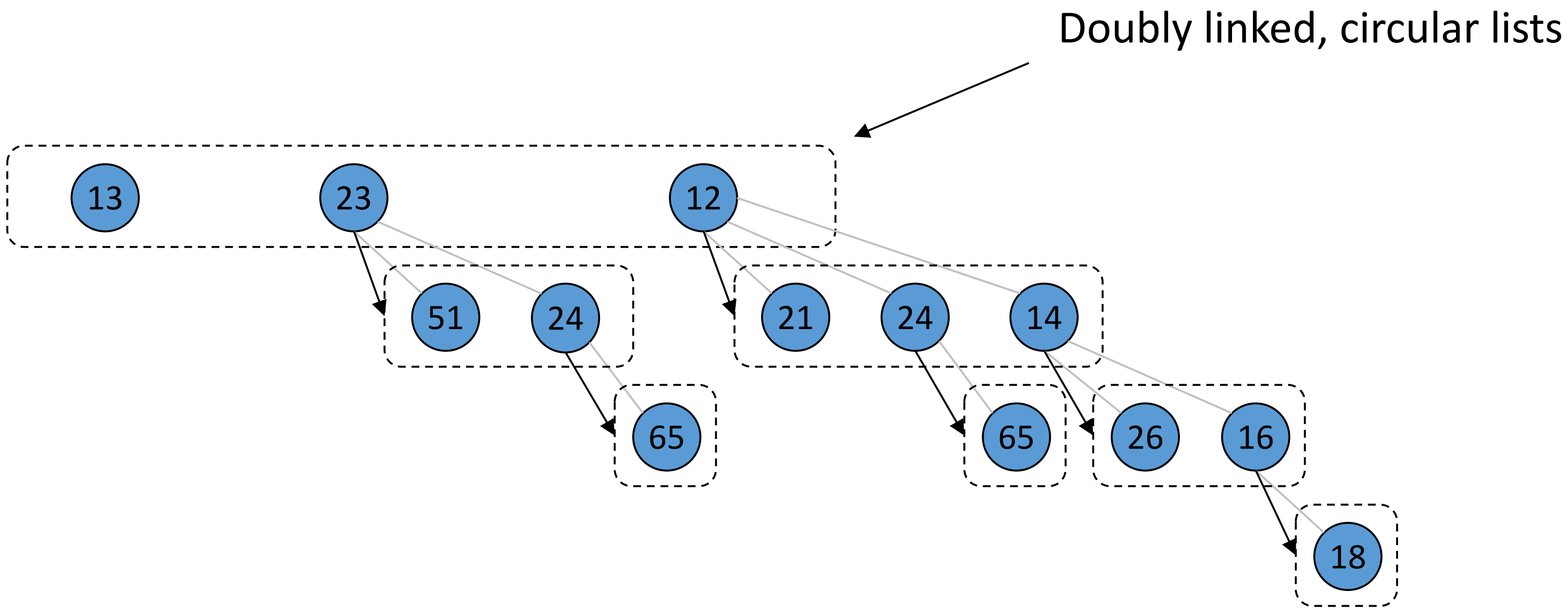
Binomial Heaps

	Worst Case	**Average Case**
`merge()`	$O(\log N)$	$O(\log N)$
`insert()`	$O(\log N)$	$O(1)$
`deleteMin()`	$O(\log N)$	$O(\log N)$
`buildHeap()`	$O(N)$	$O(N)$

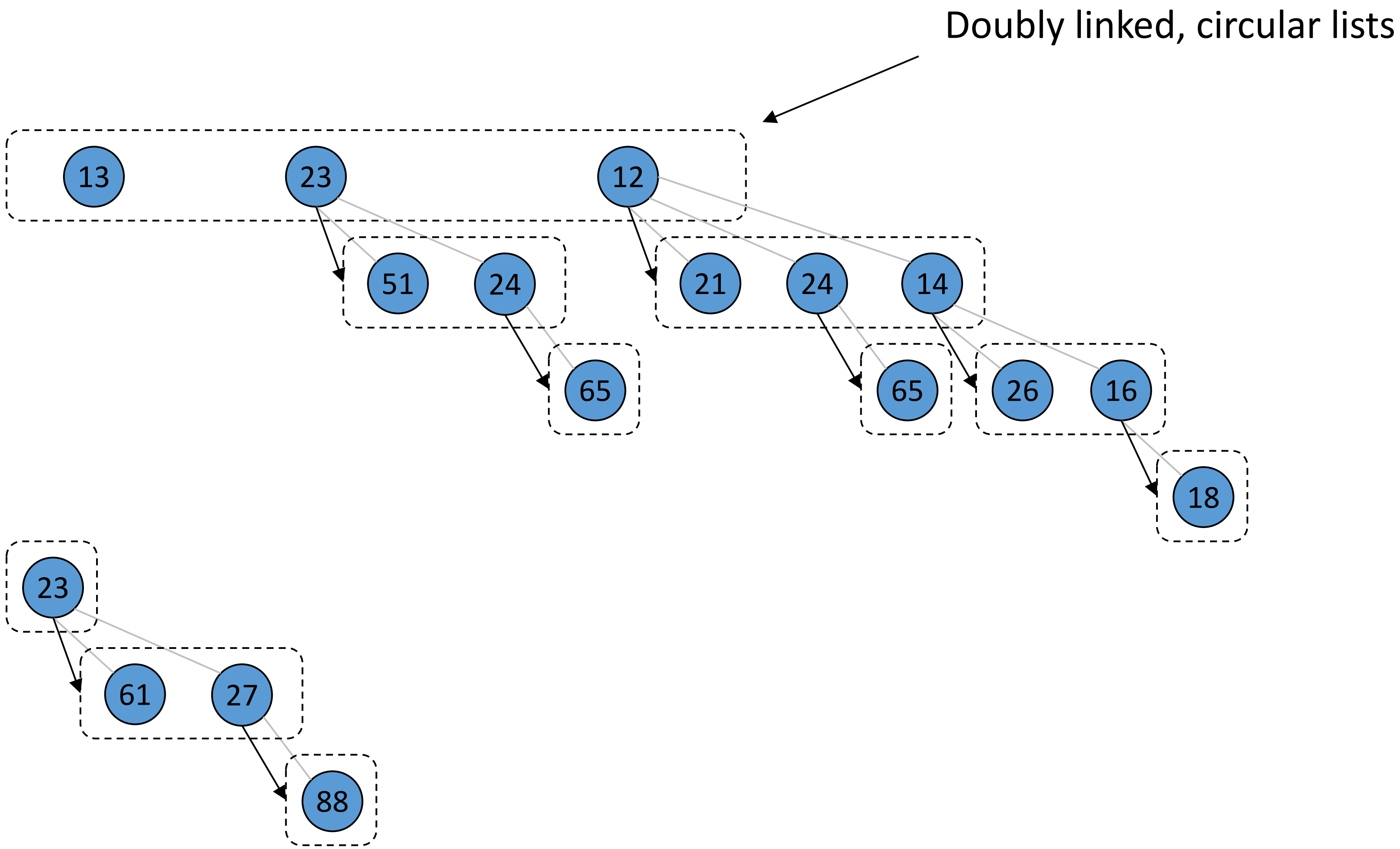
(Run N **`insert()`** on an initially empty heap.)

(N = number of elements)

Binomial Heaps – implementation

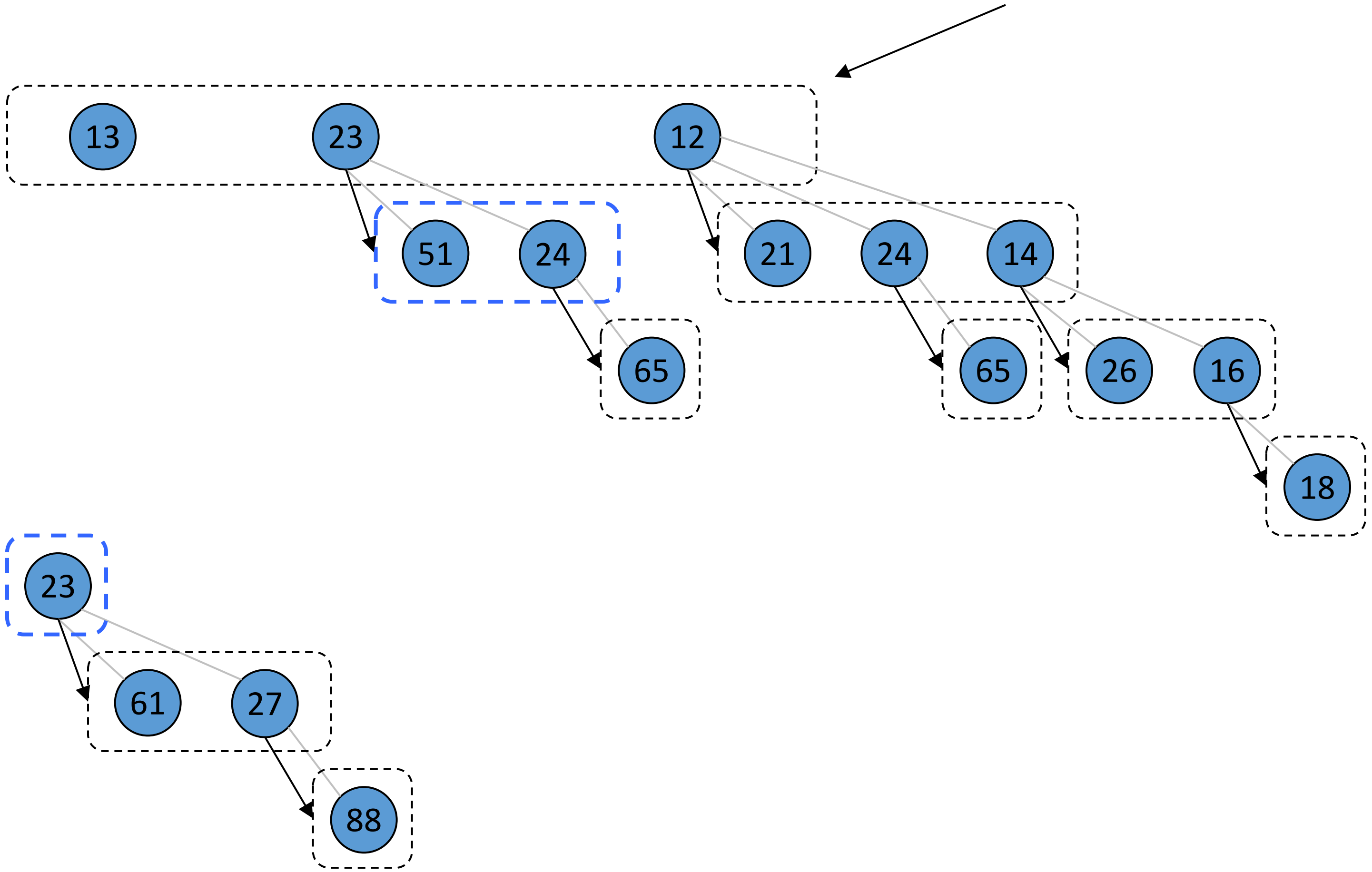


Binomial Heaps – implementation



Binomial Heaps – implementation

Doubly linked, circular lists



Fibonacci Heaps

- Very elegant, and in theory efficient, way to implement heaps: Most operations have $O(1)$ amortized running time. (Fredman & Tarjan ’87)
- **`insert()`**, **`decreaseKey()`** and **`merge()`** $O(1)$ amortized time
- **`deleteMin()`** $O(\log N)$ amortized time

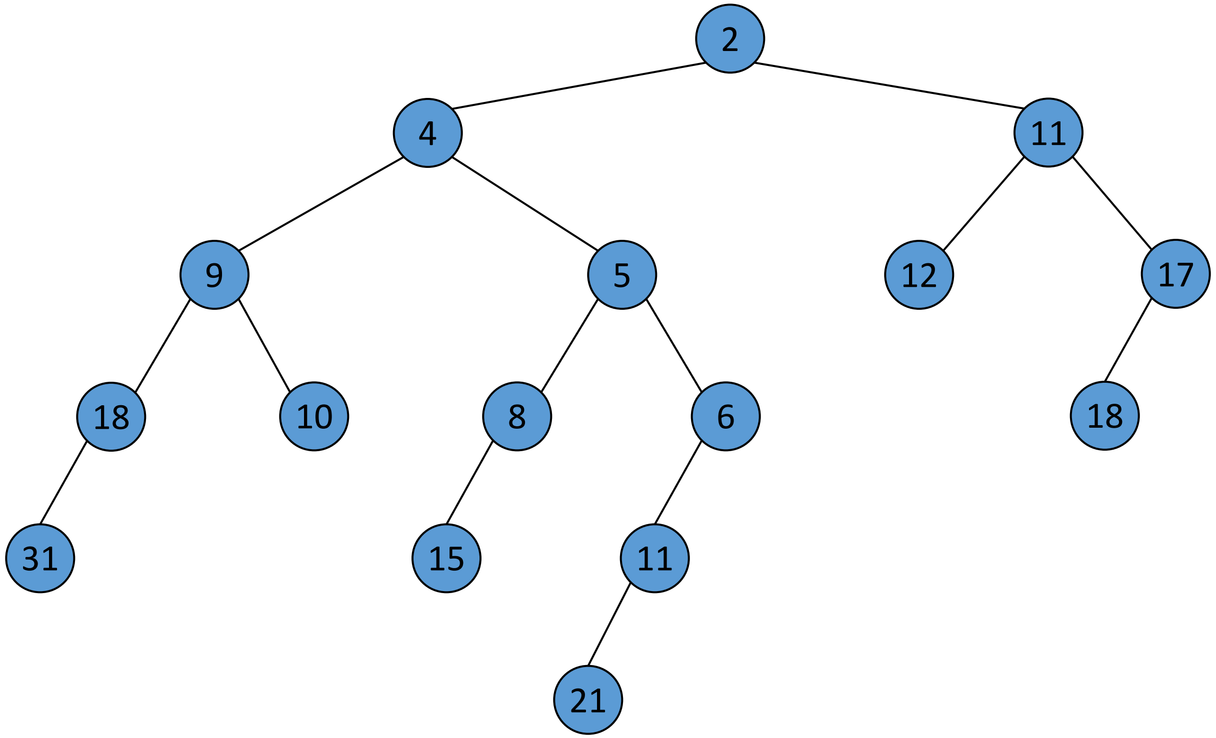
- Combines elements from leftist heaps and binomial heaps.

- A bit complicated to implement, and certain hidden constants are a bit high.

- Best suited when there are few **`deleteMin()`** compared to the other operations. The data structure was developed for a shortest path algorithm (with many **`decreaseKey()`** operations), also used in spanning tree algorithms.

Fibonacci Heaps

We include a smart **`decreaseKey()`** method from leftist heaps.



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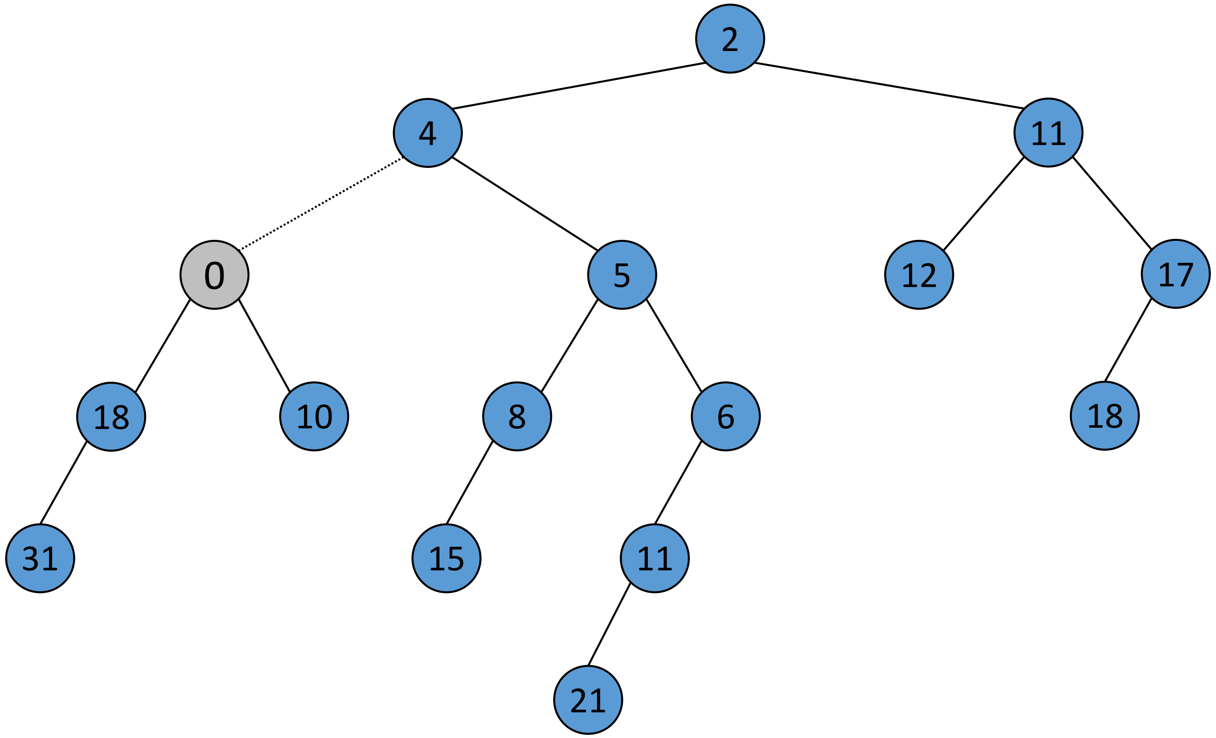

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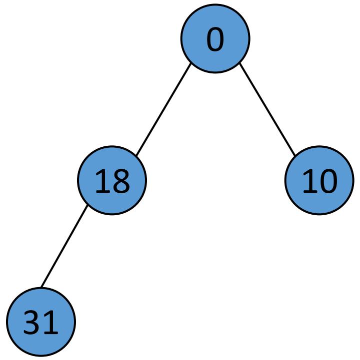
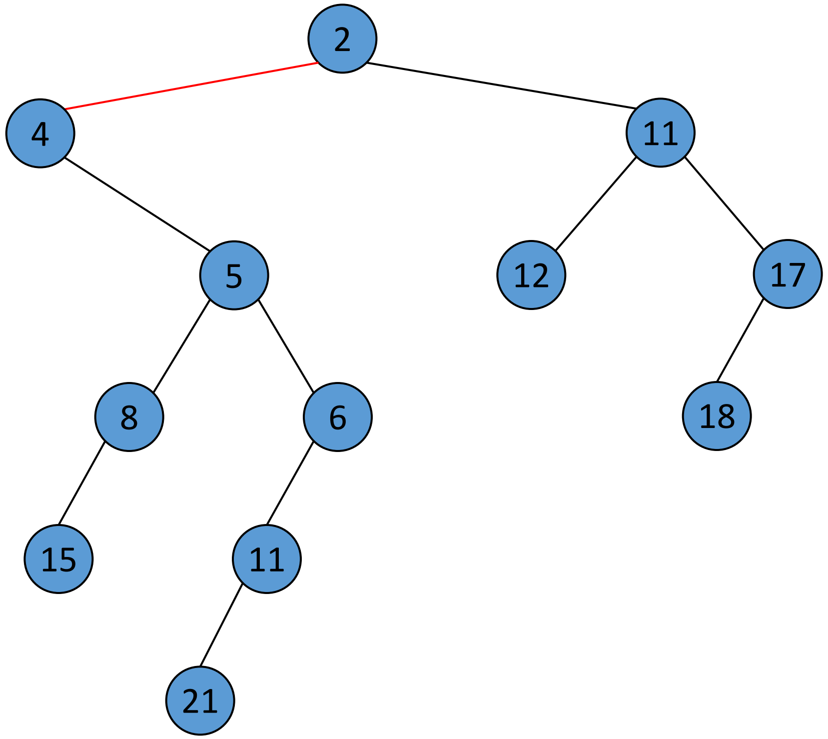


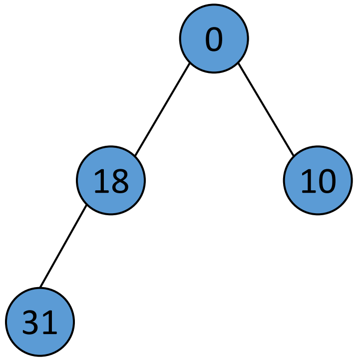
Leftist



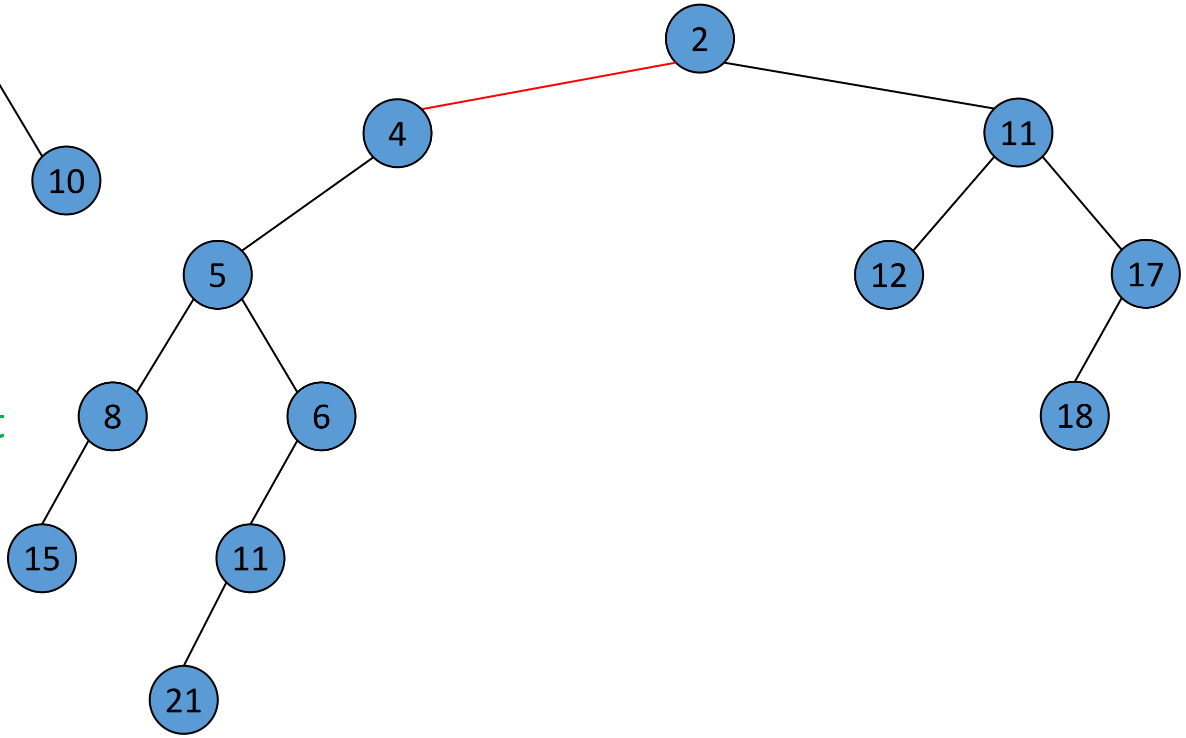
Not leftist

Fibonacci Heaps

We include a smart **`decreaseKey()`** method from leftist heaps.



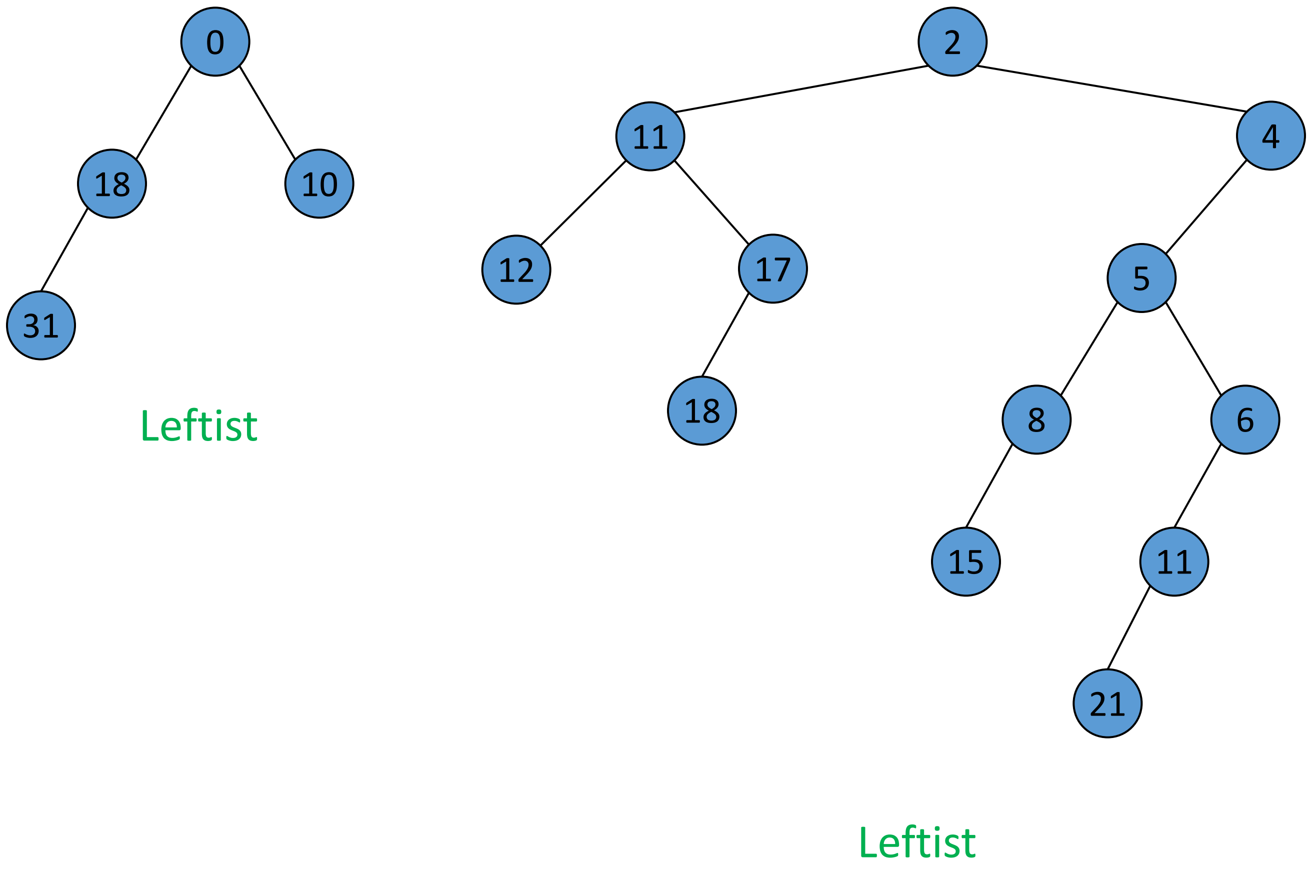
Leftist



Not leftist

Fibonacci Heaps

We include a smart **`decreaseKey()`** method from leftist heaps.



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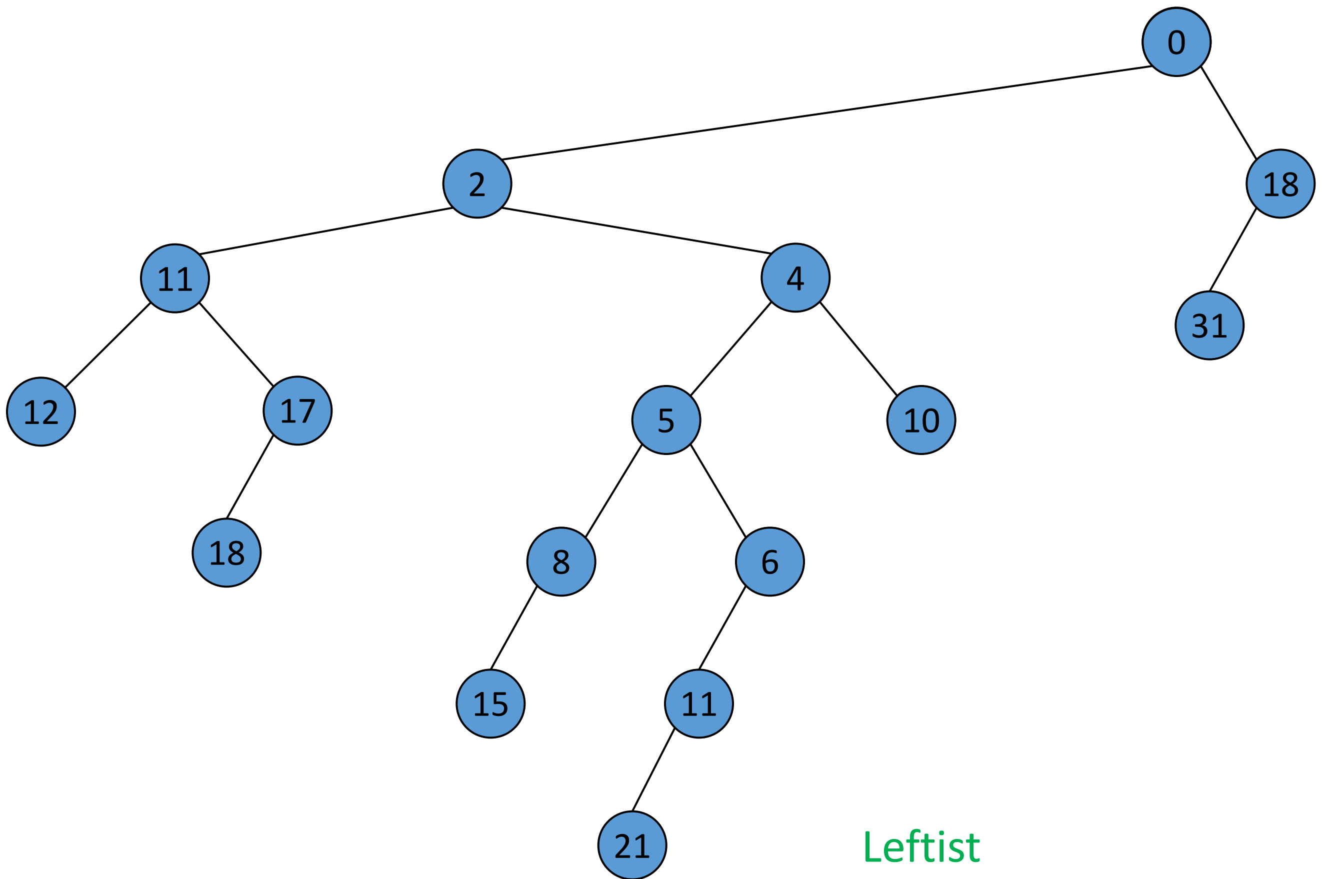
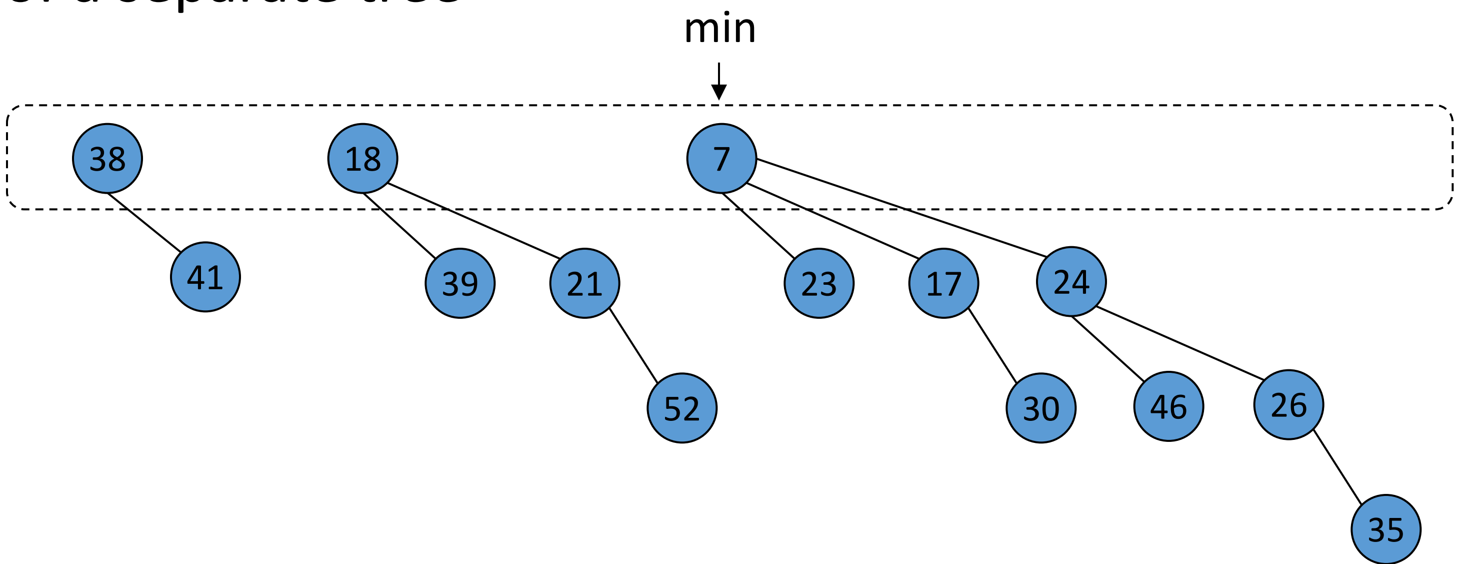


Fibonacci Heaps

We include a smart **`decreaseKey()`** method from leftist heaps.

The method must be modified a bit, as we wish to use trees that are binomial trees, or partial binomial trees.

- Nodes are marked the first time a child node is removed.
- The second time a node gets a child node removed, it is cut off, and becomes the root of a separate tree



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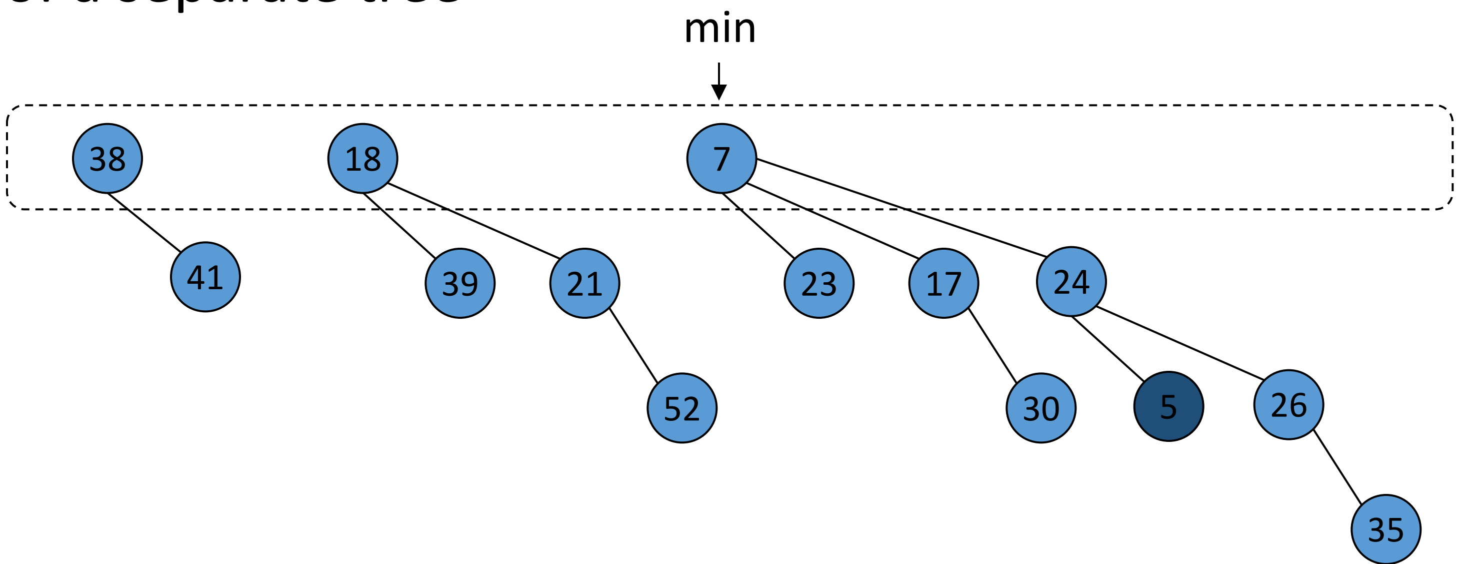

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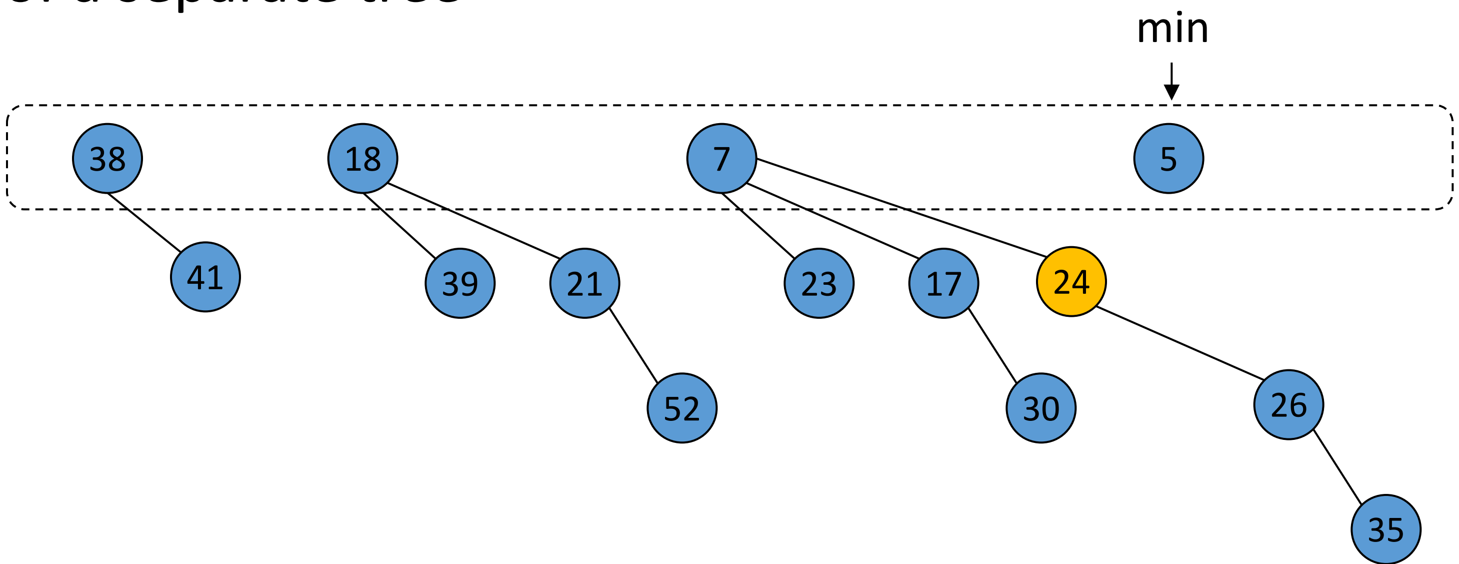

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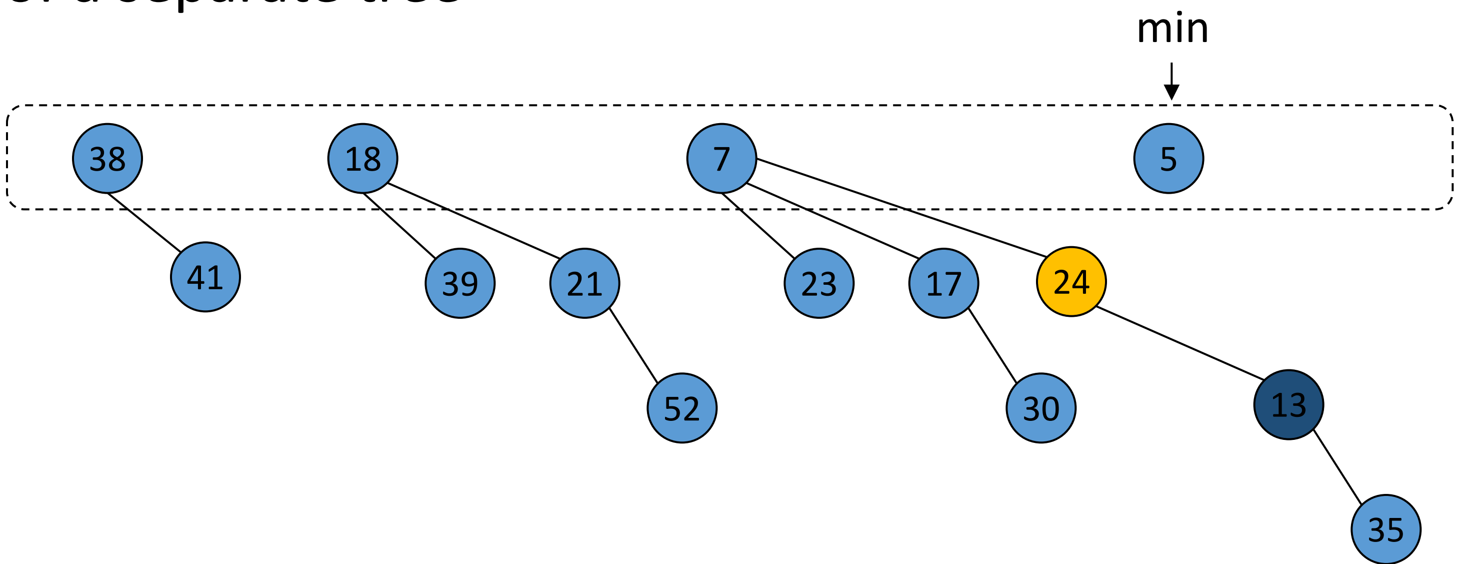

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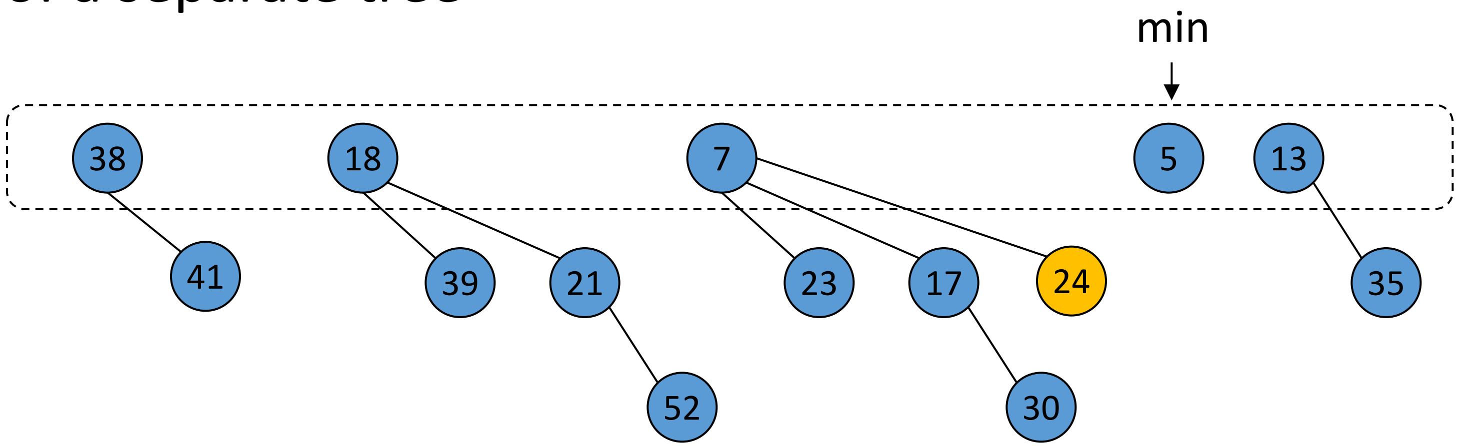

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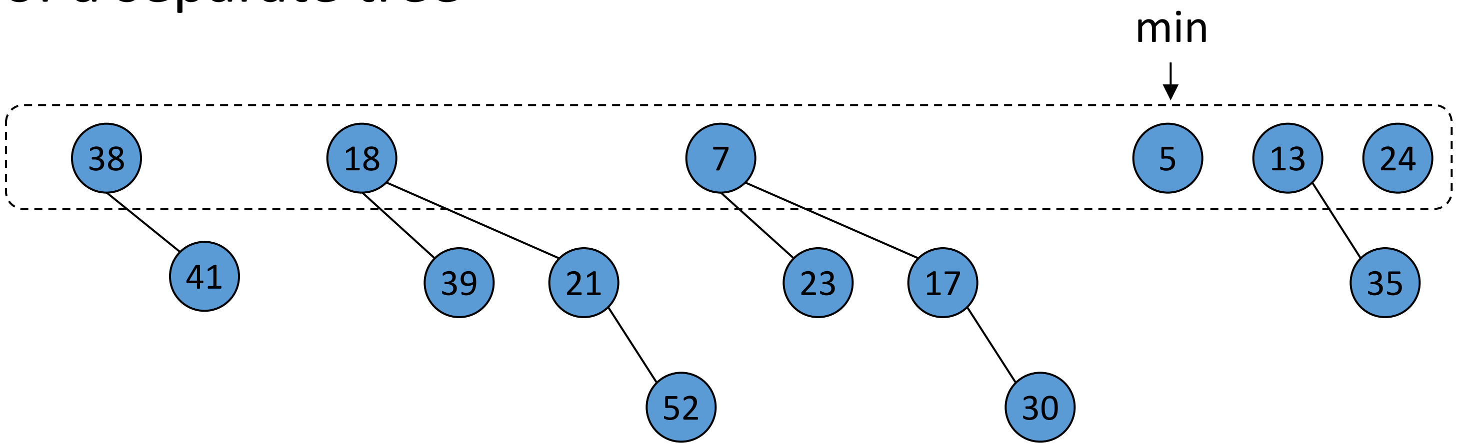
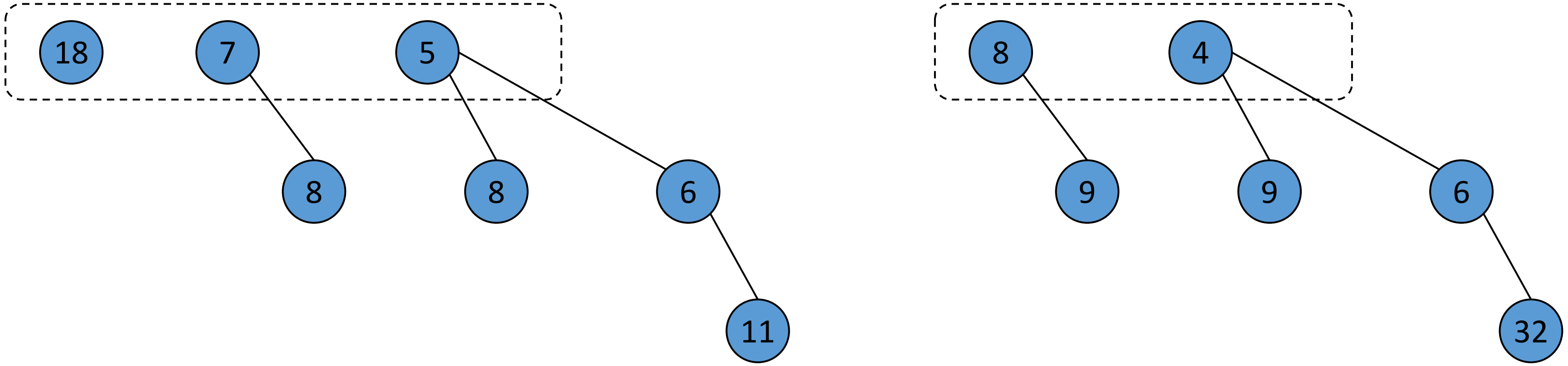


Fibonacci Heaps

We also use *lazy merging / lazy binomial queue*.



Fibonacci Heaps

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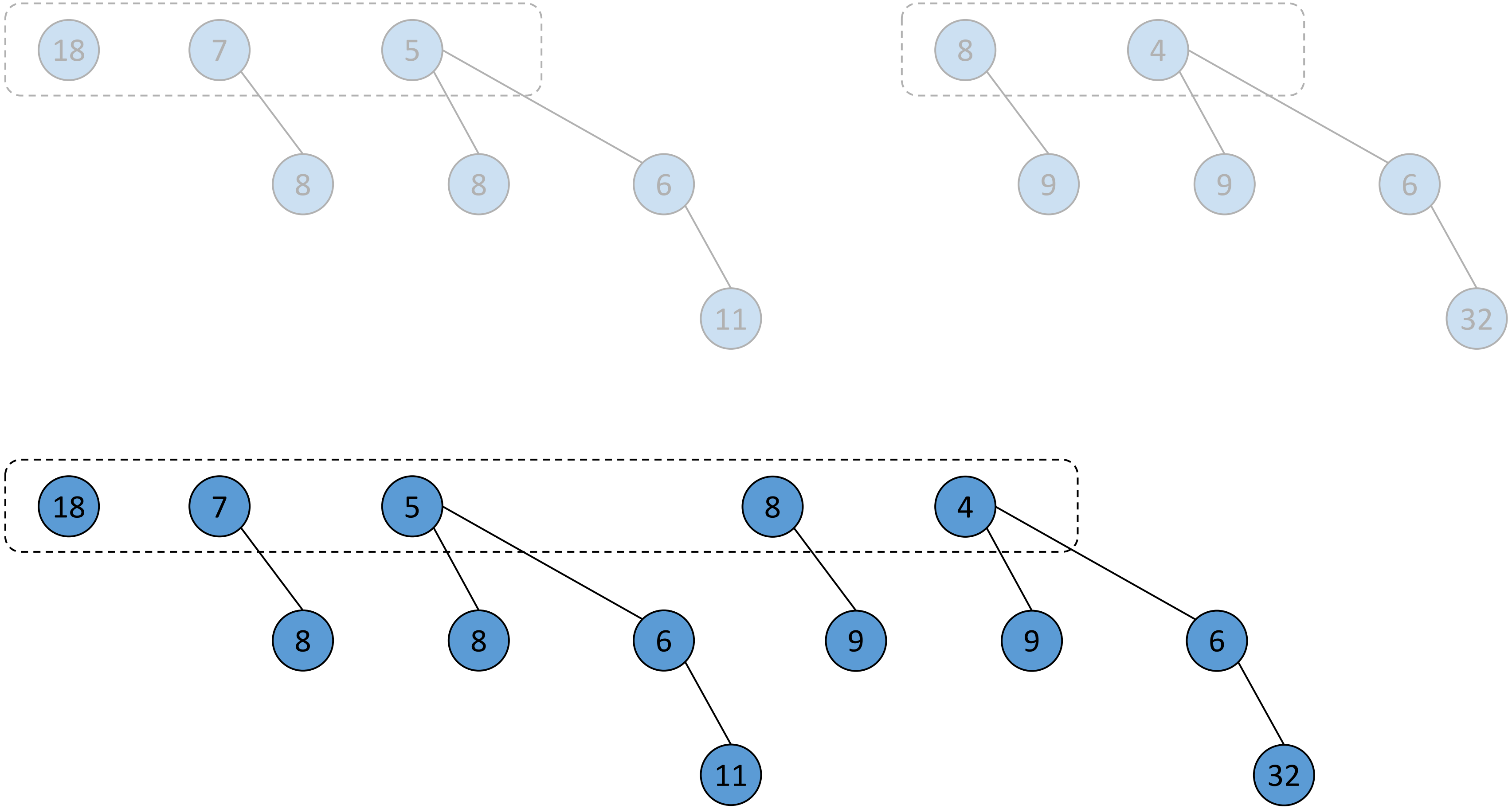

Fibonacci Heaps

The problem with our `decreaseKey()`-method and *lazy merging* is that we have to clean up afterwards. This is done in by the `deleteMin()`-method, which then becomes expensive ($O(\log N)$ amortized time):

- All trees are examined, we start with the smallest, and merge two and two, so that we get at most one tree of each size.
- Each root has a number of children – this is used as the size of the tree. (Recall how we construct binomial trees, and that they may be partial as a result of `decreaseKey()` operations)
- The trees are put in lists, one per size, and we begin merging, starting with the smallest. (As for Binomial heaps.)

Fibonacci Heaps

	Amortized Time
`insert()`	$O(1)$
`decreaseKey()`	$O(1)$
`merge()`	$O(1)$
`deleteMin()`	$O(\log N)$
`buildHeap()`	$O(N)$

(Run N **`insert()`** on an initially empty heap.)

(N = number of elements)