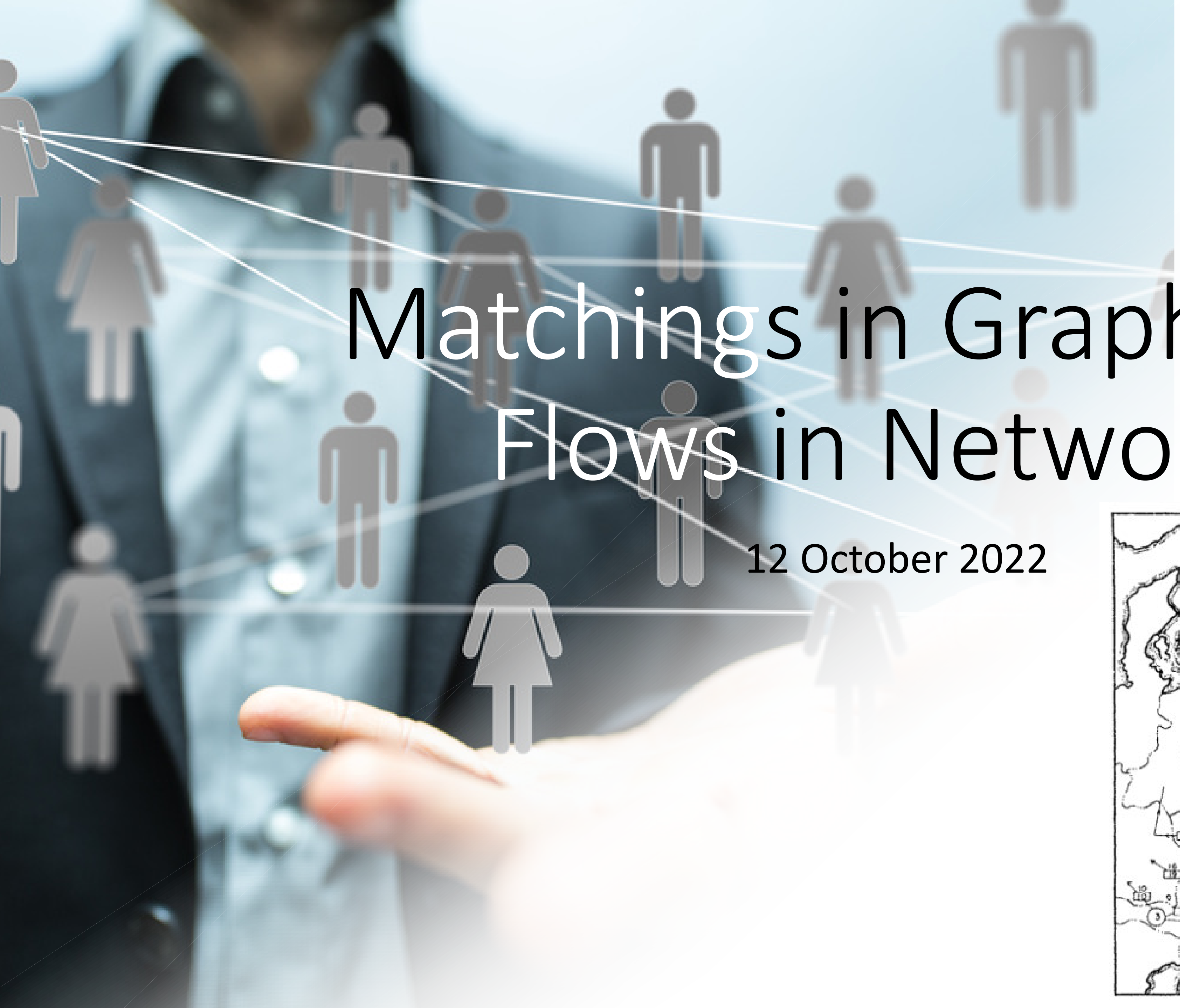
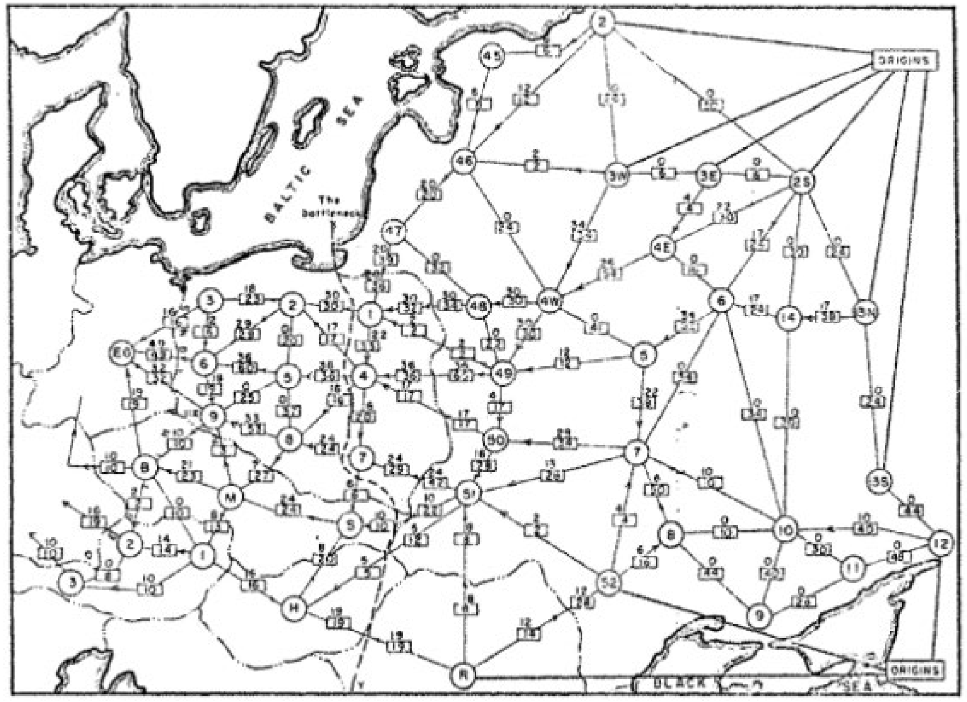


Matchings in Graphs and Flows in Networks

12 October 2022



Matchings and Flows

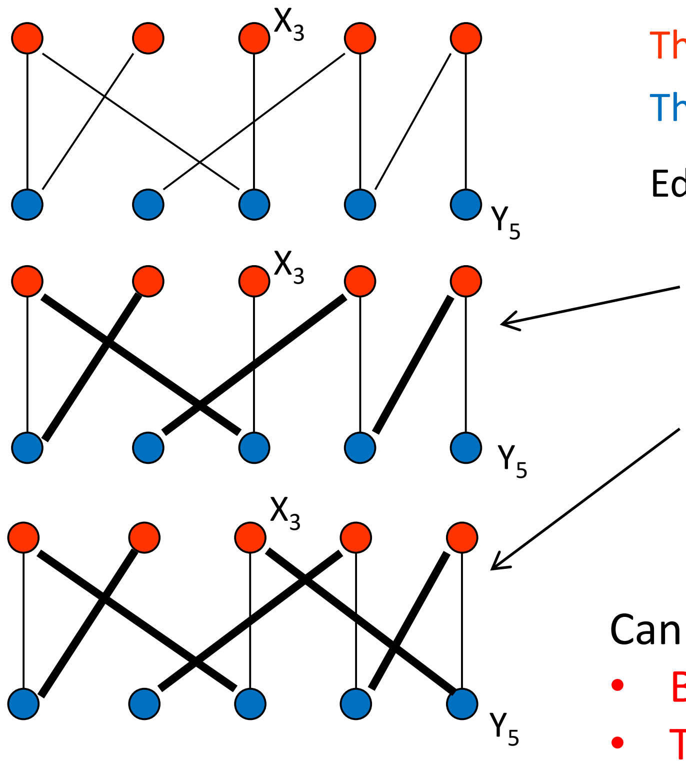
- Matchings
  - Matchings in bipartite graphs
    - Hall´s Theorem
    - The Hungarian Algorithm
  - Matchings in general graphs

- Flows
  - Flows in networks
    - Flows/cuts
    - The Ford-Fulkerson Algorithm
  - Flows vs matchings

Matchings in undirected bipartite graphs

Bipartite graph =
The set of nodes can be partitioned into two sets X and Y, so that each edge has one end in X and the other in Y

It is the same as a *two-colorable graph* or a graph without *odd loops*



The node set X, e.g. workers in a workshop

The node set Y, e.g, the jobs of the day

Edges: Who is competent for the different jobs?

- Here, we are not able to find a **perfect matching**, and thus all jobs cannot be done that day.
- However, if we add the edge $X_3 - Y_5$ we are suddenly able to find a **perfect matching**, so that all jobs can be done.

Can be used in many different areas, e.g.:

- Boys, Girls
- Teaching assistents, Groups
- …, …

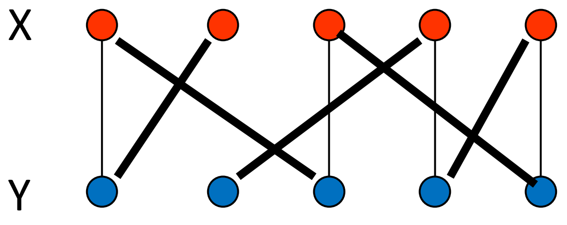
Some variations over the same theme:

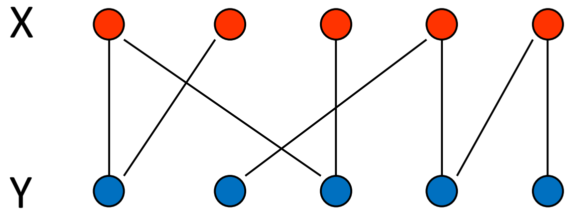
- We might have $|X| \neq |Y|$, and then there is obviously no perfect matching
- Even if there is no *perfect* matching, we are often interested in finding a match that is as large as possible.
- We can have “weights” on the edges, and ask for the matching with max. sum of weights

Hall’s Theorem (1935):

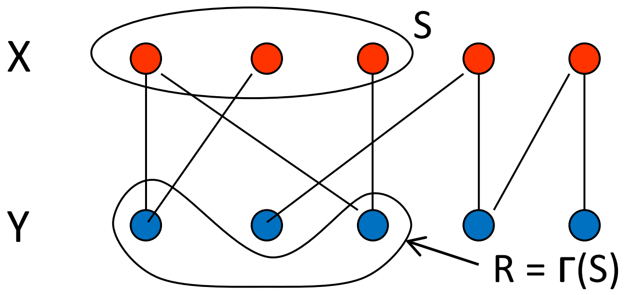
When can we find a perfect matching?

A Bipartite graphs with and without a perfect matching (same as on previous slide)





A subset S of X which is only connected to R in Y, and R has fewer nodes than S.



R = Γ(S) = the set of nodes in Y directly connected to nodes in S

Proof in the easy direction (⇐), as indicated to the left: If there is such an S so that Γ(S) is smaller than S, there is obviously no perfect matching. We are not able to match each node in S with separate nodes in Γ(S).

Proof in the difficult direction (⇒): The Hungarian algorithm will either give a perfect matching, or it will, when it stops without giving a perfect matching, point out an S where |S| > |Γ(S)|

Here we can obviously not find a perfect matching. But this also works the other way around:

Hall’s Theorem: There is **a perfect matching** if and only if there is **no subset S in X so that Γ(S) has fewer nodes than S.**

Algorithms: The naive greedy algorithm doesn't work

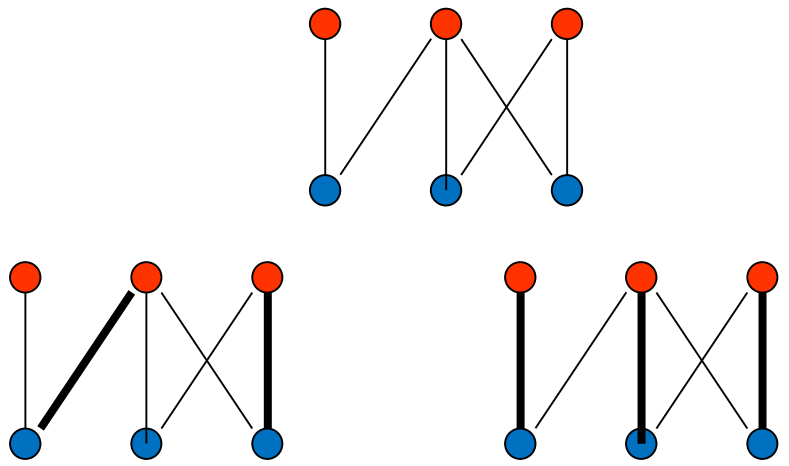
INSTANCE:	Given a bipartite graph.
QUESTION:	Find, if possible, a perfect matching.

We could try a simple *greedy approach,* which could go as follows:

Look repeatedly at the edges of the graph, and include an edge in the matching if it has no node in common with an already included edge.

But the greedy strategy will not work here!!

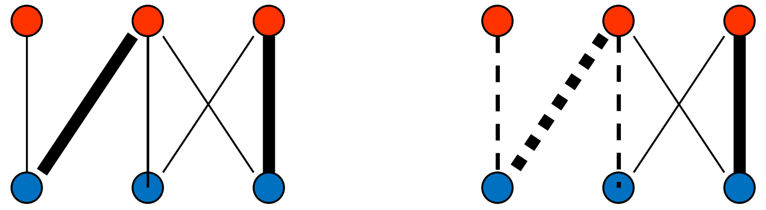
Given the upper bipartite graph to the right. A greedy approach may, after two steps, give the matching to the lower left. However, there exists a matching with three edges (lower right), but we cannot use a simple greedy scheme to extend the left matching to one with three edges.



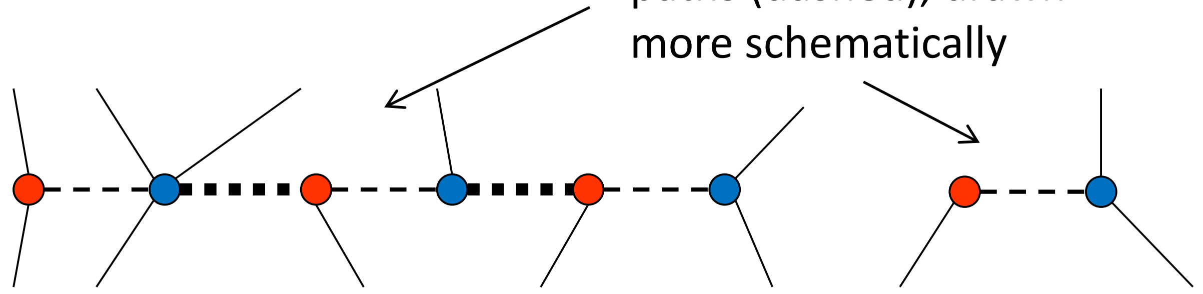
Algorithms: The Hungarian Algorithm for finding a perfect matching

- With the simple greedy strategy, we only looked for “fully independent” edges, but even if we do not find one, there can be larger matchings.
- However, it turns out that if we instead look for **M-*augmenting* paths**, and (if we find one) use that to find a larger matching, the algorithm will work.
  - We will prove this later, and at the same time prove Hall’s Theorem.
- An ***M-augmenting path***:
  - Must first of all be an **M-*alternating* path**, which is a (simple) path where alternating edges are in M and not in M.
  - In addition both end-nodes of the path must be unmatched (and then one end-node will be in X and the other in Y).

A bipartite graph with a matching M, and an M-augmenting path relative to M (dashed):

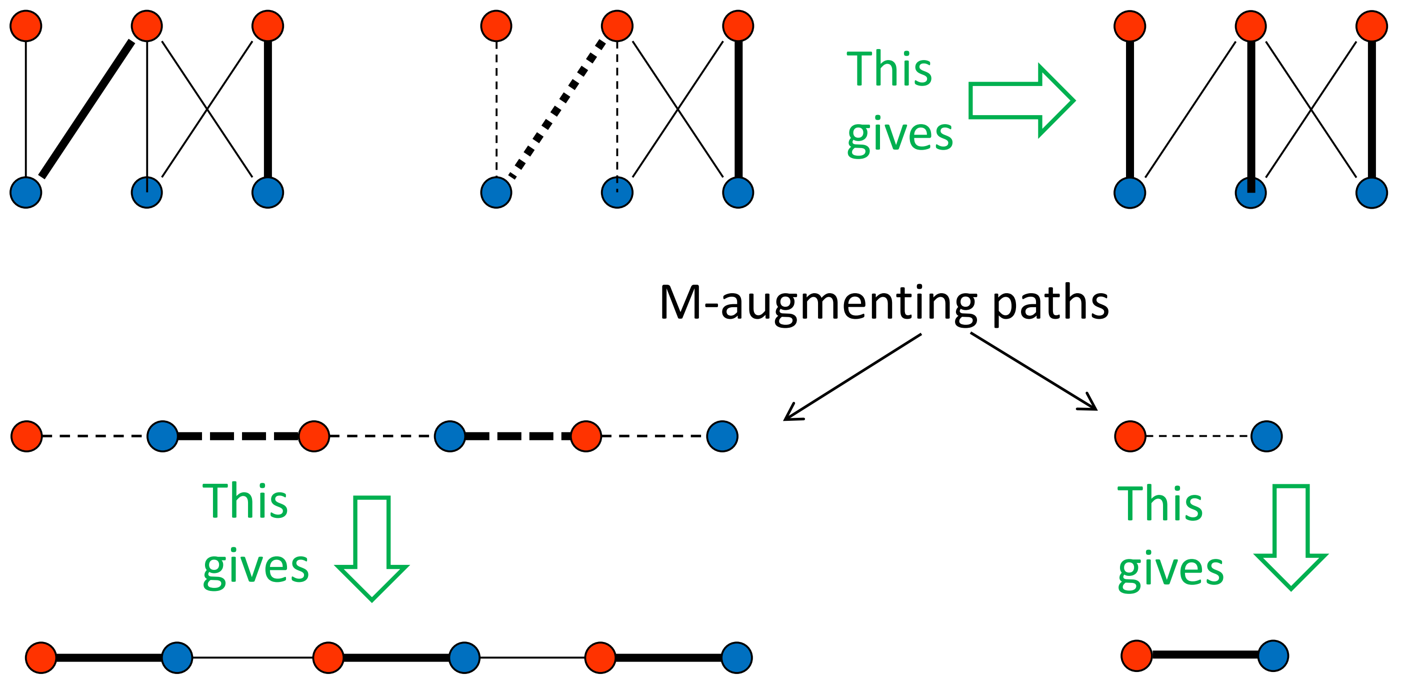


Two M-augmenting paths (dashed), drawn more schematically

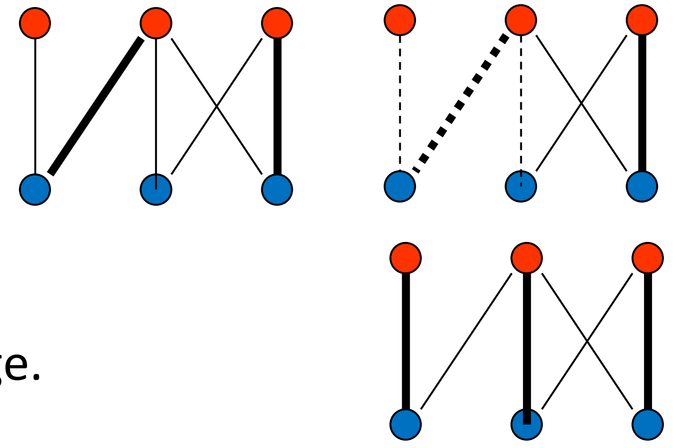


We can use an M-augmenting path to obtain a larger matching

- If we have found an M-augmenting path, we can “obviously use it” to find a matching which is one edge larger (and is written $M \oplus P$).
- This new matching is, for the three dashed M-augmenting paths on the previous slide, as follows:



How can we find possible augmenting paths?

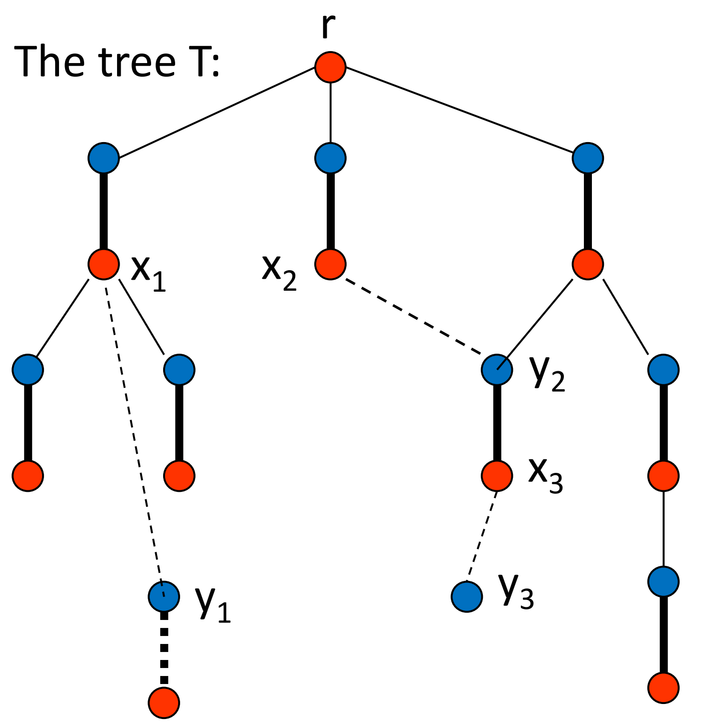


The Hungarian algorithm goes as follows:

- Start with an empty matching, and repeat:
  a) Search for an augmenting path (see next slides)
  b) If you find one, use it to find a matching with one more edge.
- Repeat a) and b) until
  - Either: You have a perfect matching (and you are done!)
  - Or: You cannot find an augmenting path relative to the current match M, by using the **Hungarian tree-building process** described on the next slide.
    - In the last case, the situation will show us a subset S of X where the size of Γ(S) (nodes in Y connected to S by an edge) is smaller than that of S.
    - Thus, if the algorithm stops in this way, we have a proof showing that there can't be any perfect matching in this graph.

Idea: We grow an “alternatig tree”

- The search for an augmenting path is done as follows:
  - Choose an unmatched node ‘r’ in X. This node will be the root in a tree where all paths out from the root are alternating paths.
  - We then grow the tree by adding two and two edges as explained on the next slide, until we have found an augmenting path, or we cannot grow the tree any further.



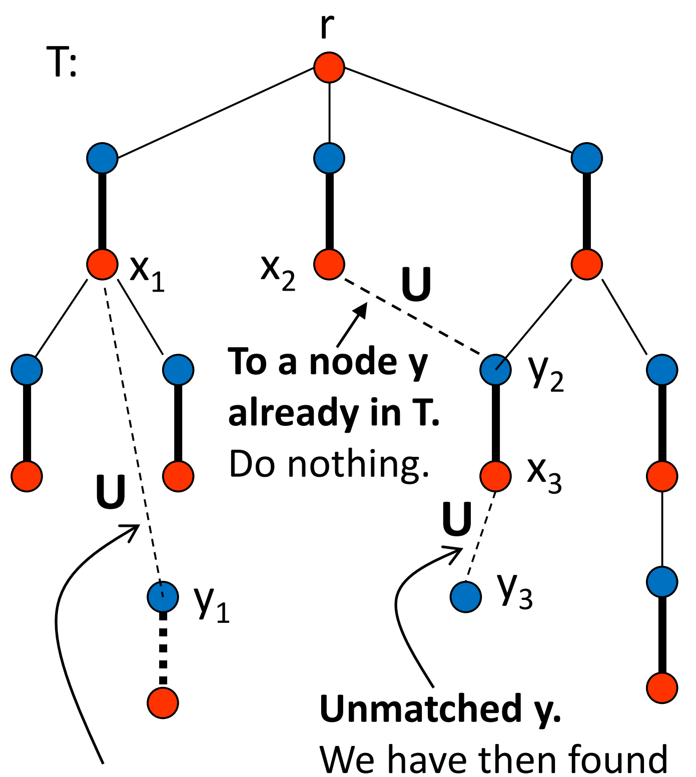
How to grow an "alternating tree"

- We assume that we have a matching M which is *not* perfect, and we search for an augmenting path
- To try to find such a path we will build an *alternating tree* T. At the start the tree will consist only of a root node 'r' in X, which must be unmatched (and such a node can always be found when M is not perfect and |X| = |Y|)
- Building the alternating tree is done by repeating the following steps:

a) We search for an edge U which is *not* in T, but has its red end-node in T. The other end-node y (blue) may be inside or outside T.

b) If we find such an edge, there are three cases:

1. The node y is already in T: Then we do nothing
2. The node y is unmatched. We have then found an M-augmenting path, and we can use this to find a larger M (as seen earlier)
3. The node y is a matched node in Y. We then include in T the chosen edge **U**=(x,y), and the edge adjacent to y in the matching. The tree T will then be extended by two edges/nodes.



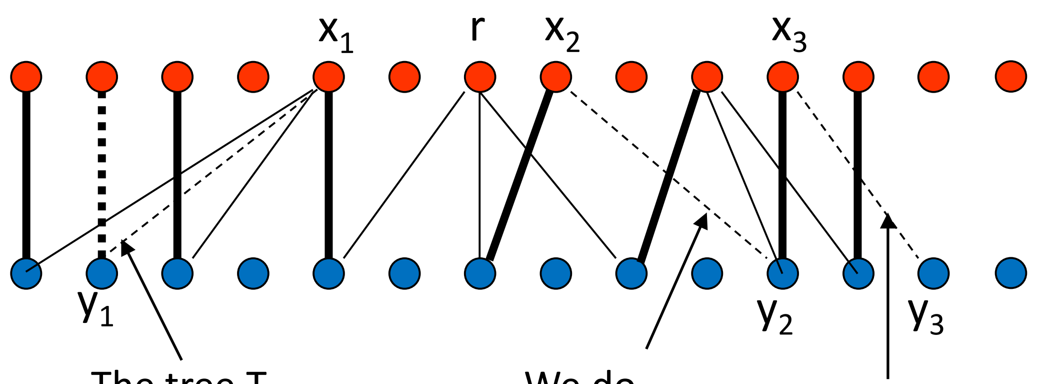
The node y is matched. We then include in T the chosen edge to y and the matched edge adjacent to y.

Unmatched y. We have then found an augmenting path. We use this to obtain a larger matching M. We then trow away the built tree T, and start building a new tree if we don't have a perfect matching

Different drawings of the same half-grown tree

A half-grown tree, as drawn to the right, looks nice and clean. Note that here only the the node and edges of the tree are drawn, and a few potential new ones. There may be a number of other nodes and edges.

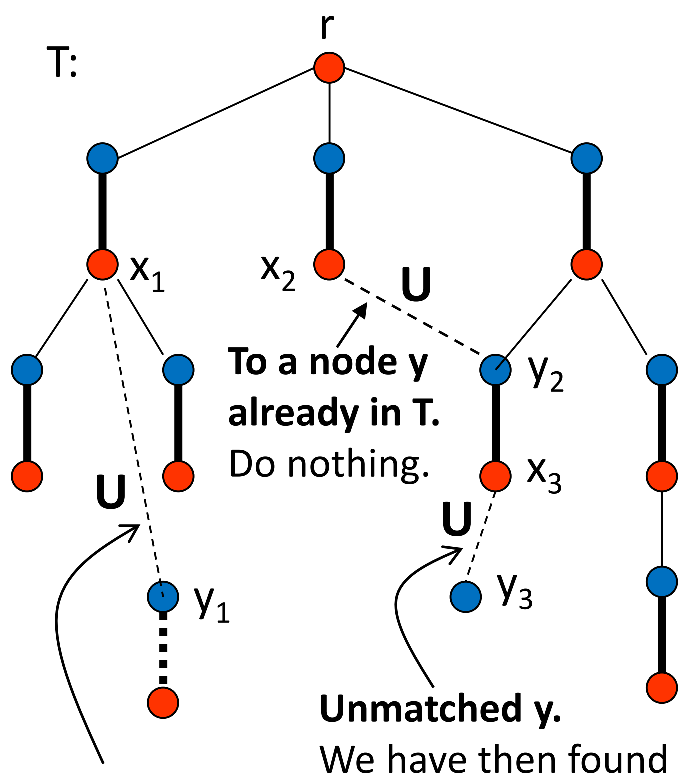
But the tree can obviously also be drawn inside the bipartite graph. Then it will look as shown below (where all nodes of the graph, but only the edges currently of interest, are drawn). However, it is easier to get an overview in the picture to the right



The tree T grows with an umatched and a matched edge

We do nothing with this edge

Augmentig path is found



To a node y already in T. Do nothing.

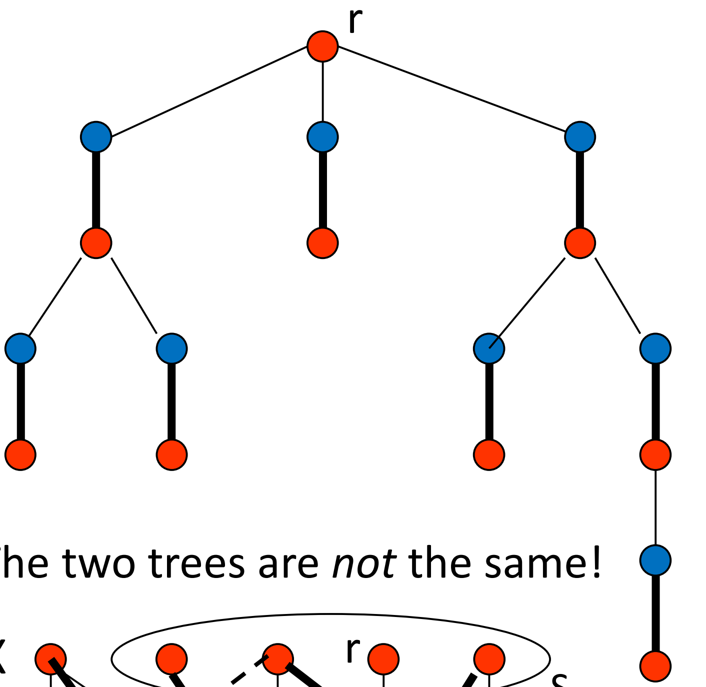
The node y is matched. We then include in T the chosen edge to y and the matched edge adjacent to y.

Unmatched y. We have then found an augmenting path. We use this to obtain a larger matching M. We then trow away the built tree T, and start building a new tree if we don’t have a perfect matching

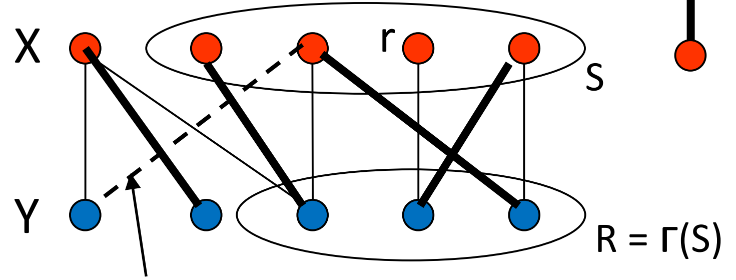
Termination of the Hungarian Algorithm

The case when we cannot extend the tree

- Assume that, when we are growing a tree, and that the algorithm stops because we can't find any edge between a red node in the tree and a blue node outside the tree. Then at least this search did not find any augmenting path. Our hope is then that this stopping situation will point out a "**Hall-situation"** which shows that no perfect matching can be found at all:
- We want: A subset S of X such that the set of nodes R = Γ(S) in Y is smaller than S.
- For this we simply choose S to be the red nodes in the tree T. The nuber of nodes in S is then one larger than the number of blue nodes in the current tree (node r makes up an extra red node)
- We then claim that the only nodes in Y connected to a node in S are the blue nodes in T.



NB: The two trees are *not* the same!



No such edge exists
(or we would have used it!)

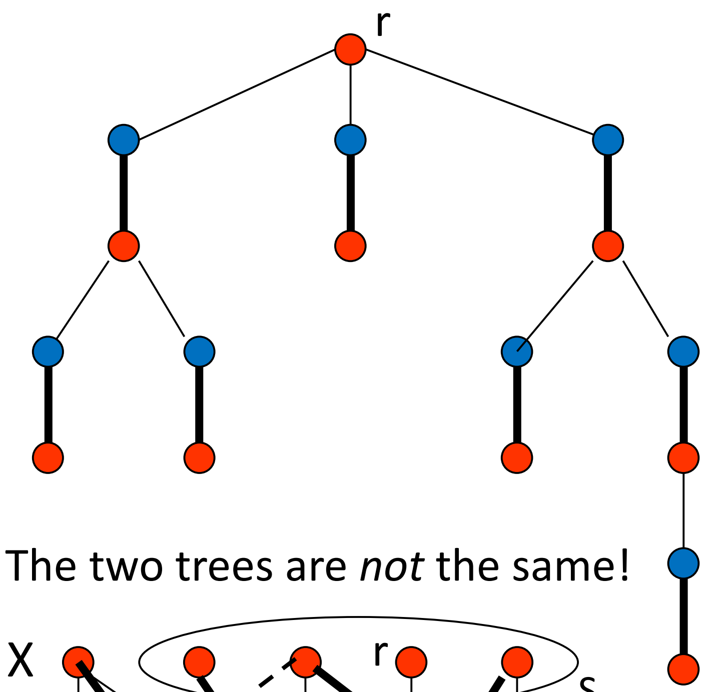
Proof:
If there were an edge from S to Y-R, then the algorithm would not have stopped.

Termination of the Hungarian Algorithm

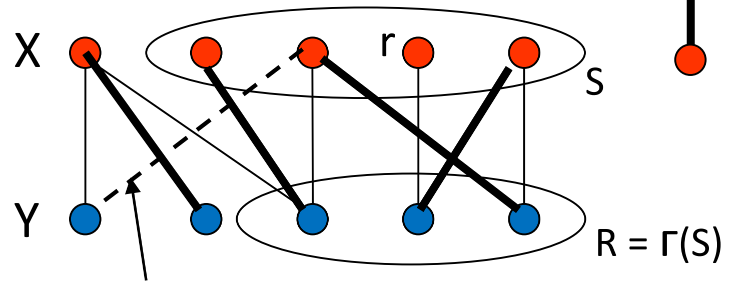
Hall’s Theorem

This proves Hall’s Theorem

The Hungarian algorithm can be run on any bipartite graph with $|X| = |Y|$, and it will either go on until it has found a perfect matching or it will stop and point out a set S in X so that $|\Gamma(S)| < |S|$, proving that no perfect matching exists.



NB: The two trees are *not* the same!



No such edge exists
(or we would have used it!)

Proof:
If there were an edge from S to $Y-R$, then the algorithm would not have stopped.

Variations of the matching problem

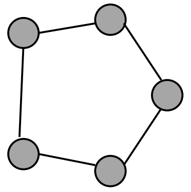
- Studied until now:
  - Find a perfect matching in a bipartite graph with |X| = |Y|, or show that no one exists.
  - A sketch of a program for this algorithm is given at page 422/423 in B&P.

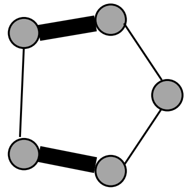
- Variants of the problem (which can also be solved in similar ways)
  - Find a matching with as many edges as possible
    (and then X and Y don’t have to be of the same size).
    - We shall look at this as an exercise.

  - Given «weights» on the edges: Find a perfect matchingt with as high weight as possible.
    - Is described in B&P but is not part of the curriculum

Matchings in graphs that are *not* bipartite

May have odd loops:





These are ”difficult” with respect to matchings.

The matching problem for general graphs:

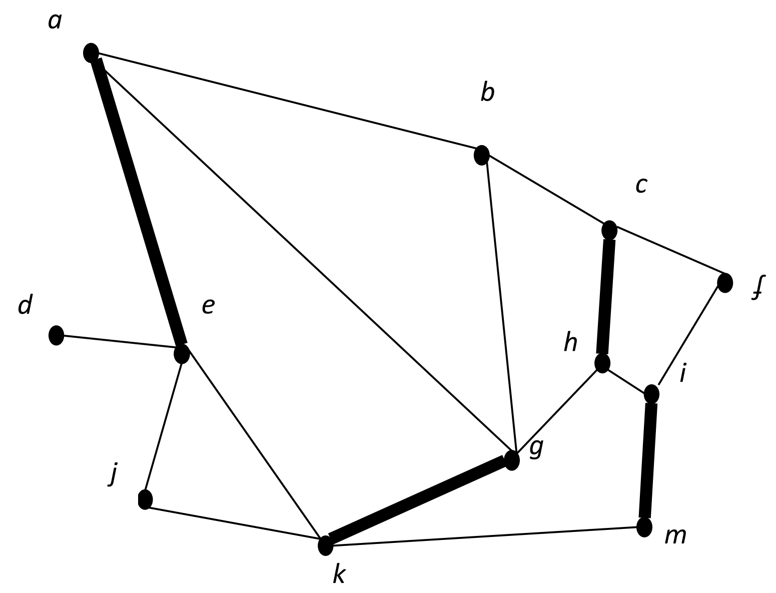
- Pose the same questions as for bipartite graphs:
  - Find a perfect matching (or show that no one can be found)
  - Find a matching with as many edges as possible

- These problems can also be solved in polynomial time!

- We will look at an algorithm for finding largest matching in general graphs:
  - This algorithm is only *slightly* more complicated to describe
  - But it is *much more complicated to prove that it really works*
  - It is part of this years curriclium to know the algorithm itself, but *not* how it it can be proven correct (we won’t even look at that)
  - The algorithm is a generalization of that for the bipartite case, with one more case in a few places.

Example: A non-bipartite graph with a non-perfect matching

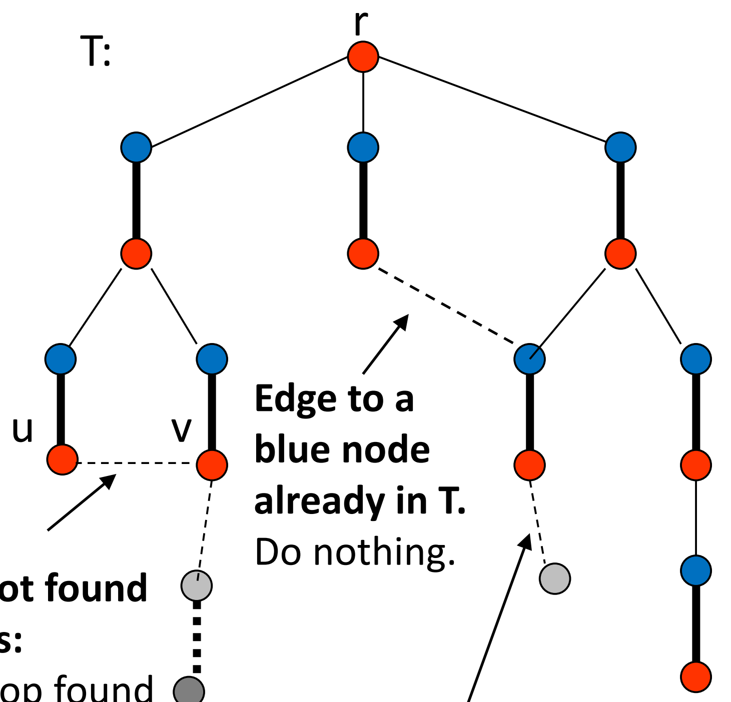
- Is there larger matching?
- If so, will there also be an augmenting path?
  - (In fact, and without proof:
    If there is a larger matching, there will, also for general graphs, exist an augmenting path. One can e.g. try to find one between d and f.)



The main step in the "Extended Hungarian Algorithm"

New elements in the extended algorithm:

- There should be *no* node colors at the outset
- Each tree building starts with an unmatched node. We color it red, and it will be the root of the new tree
- As the graph is not bipartite, there may be edges from red to red nodes, like the edge (u,v) in the figure to the right. This will form an odd loop with the rest of the tree. (there may also be blue-to-blue edges, but we don't care about those!)
- This loop is treated by simply "collapsing" it (including its internal edges) to one red node.
- If growing a tree stops without finding an augmenting path, *start with another unmatched* node as root

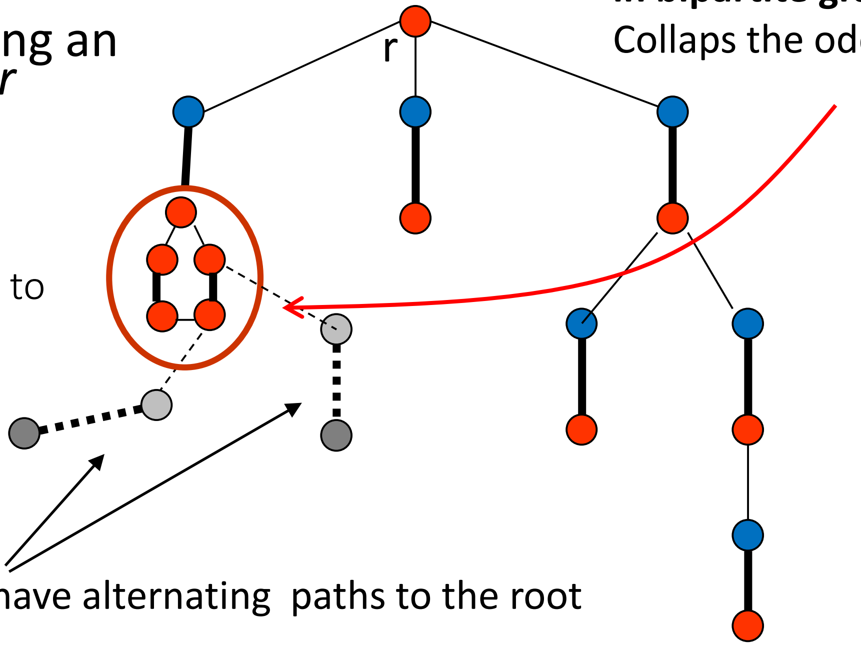


Edge to a blue node already in T. Do nothing.

New edge type, not found in bipartite graphs: Collaps the odd loop found

Edge to a matched uncolored node: We color the node blue, and the corresponding matched node red, and include both nodes in the tree

Edge to uncolored and unmatched node: We have then found an augmenting path, and we can use it to get a larger matching.

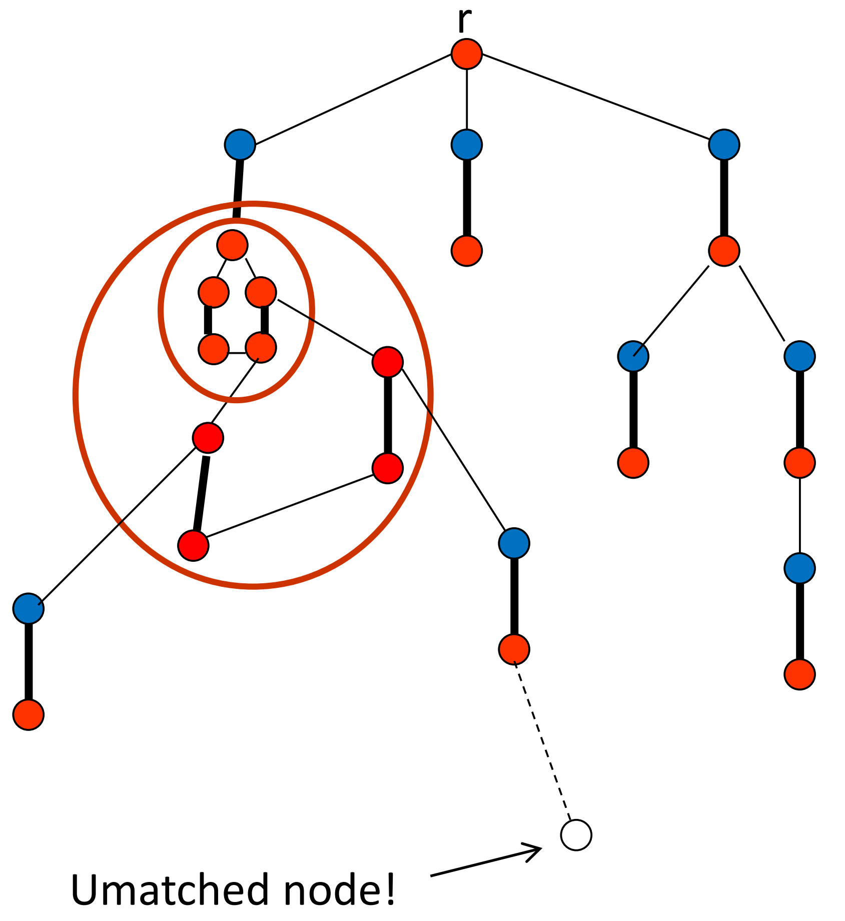


Red collapsed nodes: They all have an alternating path back to the root, stating with a matched node

Important: Both of these extensions have alternating paths to the root

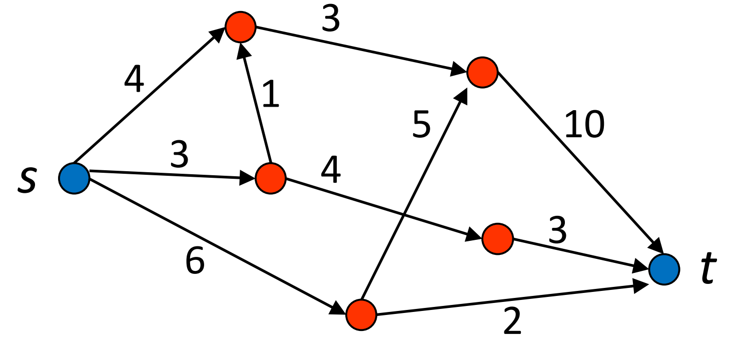
The end of a treebuilding in the Extended Hungarian Algorthm

- If we find an augementing path to an unmatched node in the graph with some collapsed nodes:
  - We go backwards along the alternating path, and along the way we unpack the collapsed (red) nodes, and find the alternating path through them.
  - We thereby get an alternating path in the original graph back to the root.
  - We can use this to find a matching that is one edge larger than the one we have.
- Otherwise the treebuilding stops because there are (1) no unexplored edge from a red node, and (2) no more unmatched nodes that can be the root of a new tree.
  - Then no larger matching will exist!!
  - But this is more complex to prove, and the proof is not part of the curriculum



Flows in Networks

- The use of the word “Network” is simply a tradition in this area. It is the same as directed graphs, usually with some weight, capacity etc. for edges and/or nodes, and some special, named nodes (“source” and “sink”).
- There are a lot of practical problems that can be seen as flow problems in networks.
  - Data nets, where there is a flow of data packages through the edges.
  - Different types of pipe-networks where fluid or gas can flow, and where each pipe has a *capacity.*
  - Networks of railroads or roads with different capacities, where cars are “flowing” on the roads.
- The networks we shall study here have:
  - A capacity *c* on each of the edges,
  - One *source* node *s* and one *sink* node *t,*
  - And the goal is usually to find a largest possible flow from *s* to *t.*

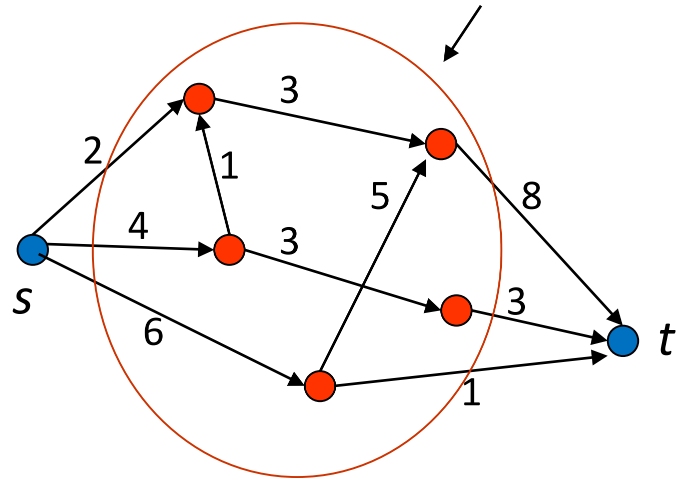


A network with capacities

Flows in Networks

- A flow *f* in a network is composed of one flow *f(e) ≥ 0* for each edge *e*, with the following properties:
  - **Flow conservation:** For each node except *s* and *t*, the sum of flow into the node is equal to the sum of the flow out of the node (where *into* and *out of* is defined according to the directions of the edges).
  - **In networks with capacities:** Each edge has a capacity *c(e) ≥ 0* , and the flow *f(e)* must be in the interval *0* to *c(e).*
- We assume
  - There are no edges leading into *s* or out of *t*.
  - *val(f)* is by definition the sum of the flow out of *s*.

A network *without* capacities.
A legal flow is given:



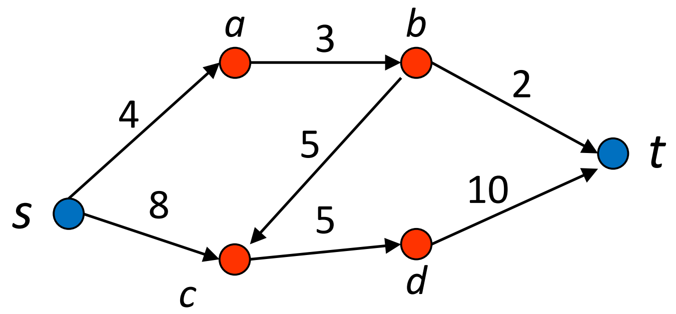
Lemma: The sum of the flow into *t* is the same as *val(f)*

- Can be proved by summation of the flow in and out of all «internal» (red) nodes (which sum to zero!)

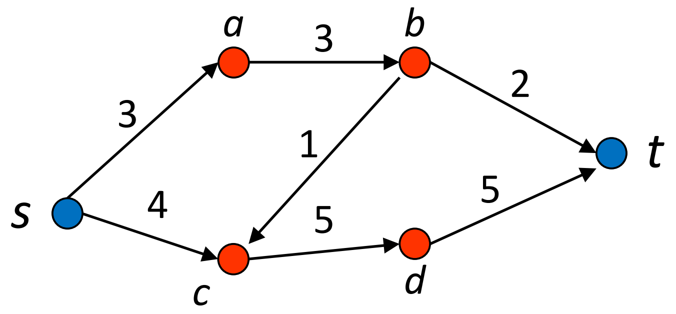
Flows in networks with capacities

- Each edge has a capacity *c(e),* and the flow *f(e)* must be between 0 and *c(e).*
- Our goal:
  - Given a network with capacities
  - We want to find edge flows *f(e)* that
    - Satisfy the capacity requirement $0 \leq f(e) \leq c(e)$
    - Forms a maximum flow (there are no legal flow with larger *val(f)*)
- The example below: To the left is a network with given capacities.
  - We can easily find a legal flow of 7 (given on the right). Can we find any larger flows?

A network with capacities

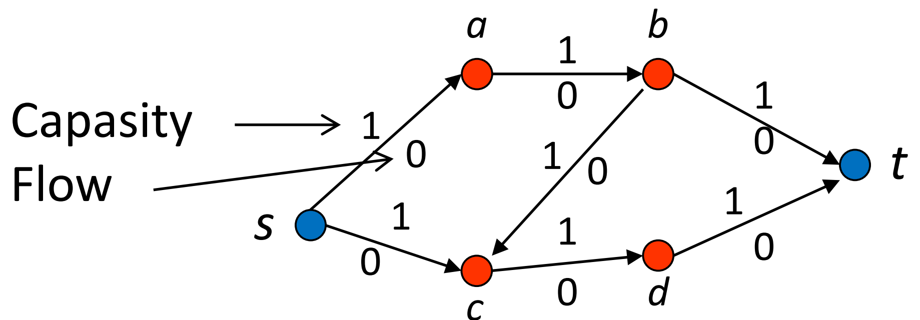


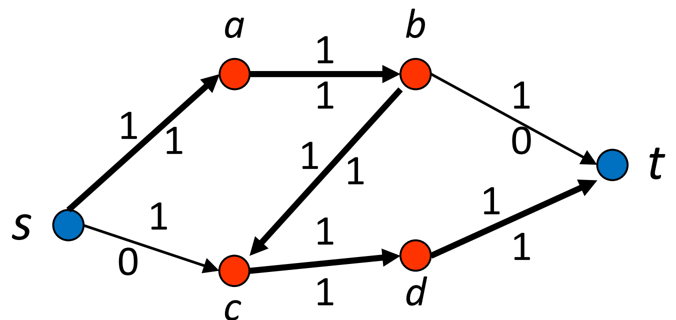
A flow *f* with *val(f)* = 7. Is this a maximum flow?



The naive greedy algorithm will again not work !

- The naive greedy algorithm (that in fact doesn’t work!) could be described as follows:
  - The step:
    - Find a directed, simple path from s to t where all *f(e)* are *non-negative* and *f(e)* ≤ *c(e)*
    - Increase the flow along this path as much as possible (dictated by the edge that has the smallest *c(e) – f(e)* along the path)
  - Repeat this step until no such pathes can be found.
- In the figures below the capacity is given *above* the edges (all *c*(*e*) =1) and the current flow is given below the edge (initially zero everywhere)
  - We first find a simple flow-increasing path, e.g. *s-a-b-c-d-t*. We can increase each edgeflow along this path with 1, and get the situation to the right.
  - Now *val(f)*= 1. But this is not a maximum flow, as we can easily find a flow with *val(f)*=2
  - BUT, there is no flow-increasing path in this network that can bring us to a flow with value 2. Thus this simple scheme won’t bring us to a maximum flow.

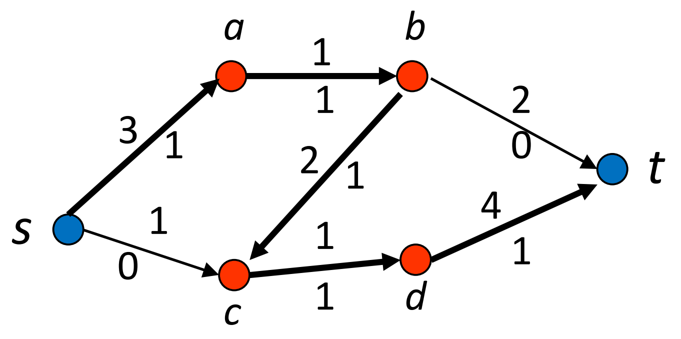




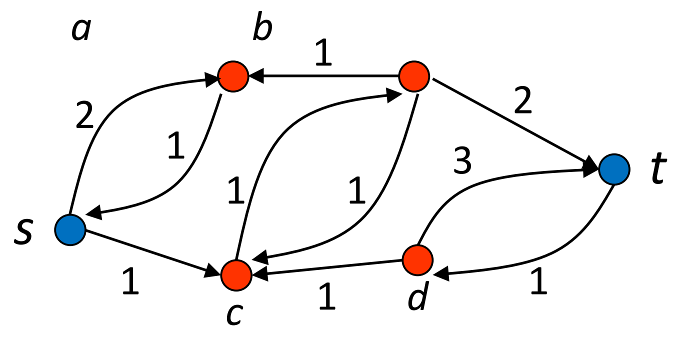
The *f*-derived network *N(f)*

- What we haven't taken into account on the previous slide, is that we, while searching for a larger flow, can also *decrease* the flow for edges with nonzero flow. And by utilizing this, we will in fact get a working algorithm!
- To get an overview of the ways we can *change* the current flow on each edge we can construct the "*f-derived network*" often referred to as ***N**$_f$*, ***Nf***, or ***N(f)***. We shall here use ***N(f)***.

(Note: the capacities below are different from those on the previous slide)



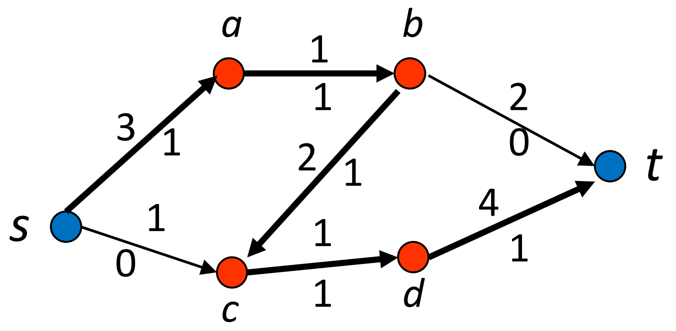
A network with capacities (the edges) and a flow (under)

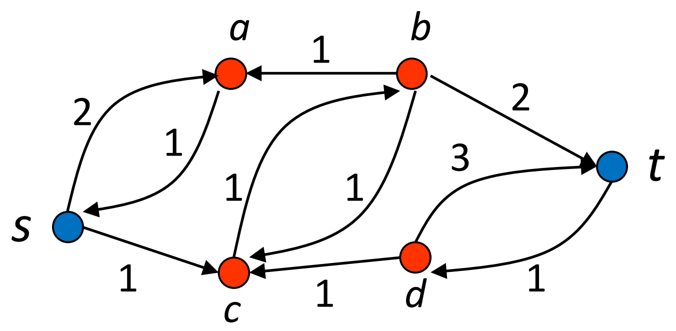


The *f*-derived network for the situation to the left

Augmenting paths

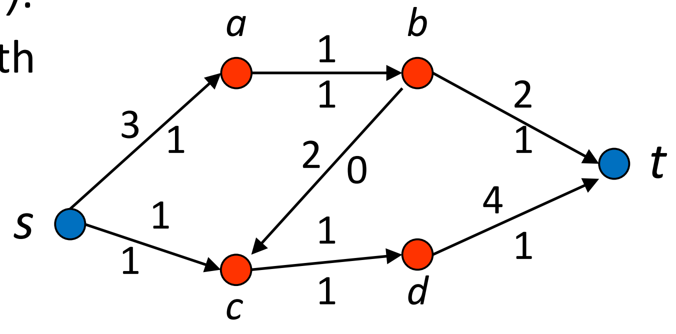
The network, capacities, flow and *N(f)* is as in the previous slide:





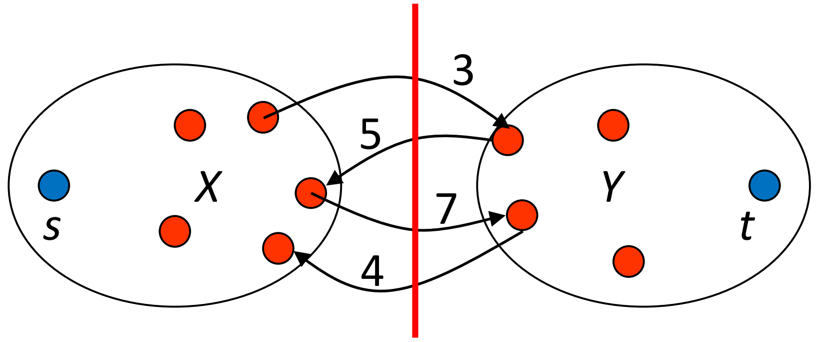
- We search for paths from *s* to *t* in the *f*-derived network *N(f)*
  - Such paths are called *f*-augmenting paths
  - The search for such paths can be done e.g. with *bredth-firs*t or *depth-first* in *N(f)*.
  - We can e.g choose the path *P = s-c-b-t*. The max. increase in flow along this path
  - is here 1 (called ***h*** = mininimum of possible increment over all the path edges)
- We then obtain the corresponding flow increase by, for all edges of P:
  - If the edge-direction in *N* is the same as in *P:* Increase the flow with ***h,***
  - If the edge direction in *N* is opposite to that of *P*: Reduce the flow with ***h***.
- We then forget the old *f*-derived network, and build a new one relative to the new flow.

New flow:



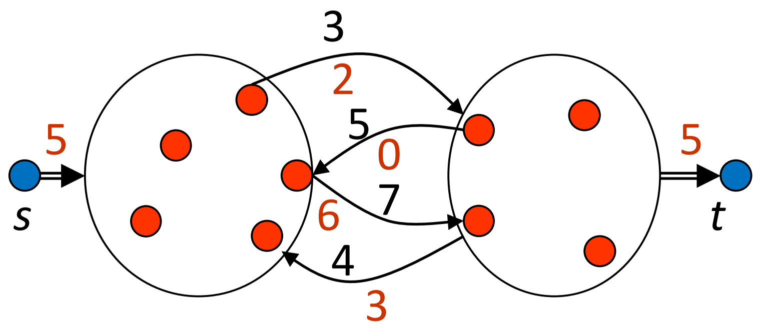
Cuts in networks

- A *cut* in a network is defined by a set *X* of nodes containing *s* but not *t*. The set of the rest of the nodes is then called *Y*, and we know that Y contains *t.*



- The *capacity* of a cut *K=(X,Y),* written *cap(K),* is the sum of the capacities of all edges leading from a node in X to a node in Y (disregarding edges from Y to X)
- In the figure above, the capacity of the cut is 3 + 7 = 10
- Thus, the capacity of the edges leading from *Y* to *X* do not influence the capacity of the cut.

More about cuts in networks



Lemma: Given a legal flow *f* and a cut *K = (X,Y).* Then *val(f) ≤ cap(K).*

This can be shown as follows:

- By adding together the flow in/out of all nodes in *X' = X–s*, we find that
  (flow out of s) + (flow backwards over *K*) = (flow forwards over *K*)
- As we by definition know that: (flow out of s) = *val(f)* we know that
  val(f) = (flow forward over *K*) – (flow backwards over *K*)
- The right hand side of the above equality is called the *flow over K*. As the last term is non-negative we know that the first term will obey: *val(f) ≤ cap(K).*
  This is true for any cut K.
- In the figure above: *val(f) = 5 = 2 + 6 – 0 – 3 ≤ cap(K) = 3 + 7 = 10*

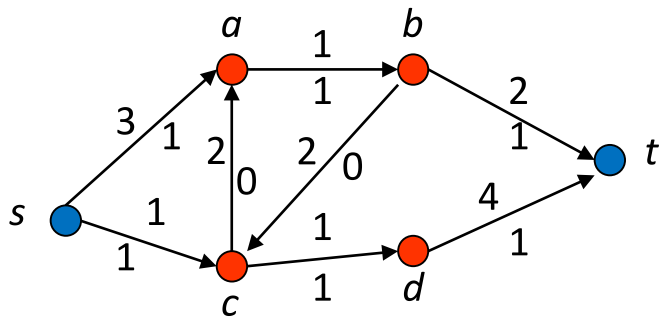
This gives us a way to decide whether a given flow is optimal
If we have a flow *f* and a cut *K* so that *val(f) = cap(K)*,
then we have a maximum flow, and there is no *cut with smaller capacity*!

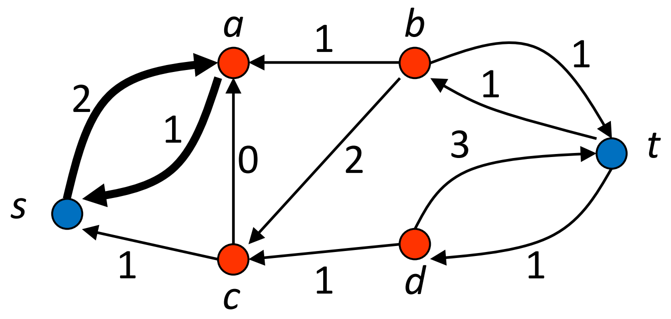
The Ford-Fulkerson Algorithm

The FordFulkerson-algorithm goes as follows:

- Start with zero flow (which is always a legal flow)
- The main step (and at the start of this we generally have any legal flow):
    - Find the *f*-derived network *N(f)* (that shows all possible changes for the edgesflows)
    - Find, if possible, an *f*-augmenting path from *s* to *t*, and find the maximum increase it allows (before any of the edgeflows exceed the capacity or will go under zero).
    - Do the changes that this *f*-augmenting path indicate.

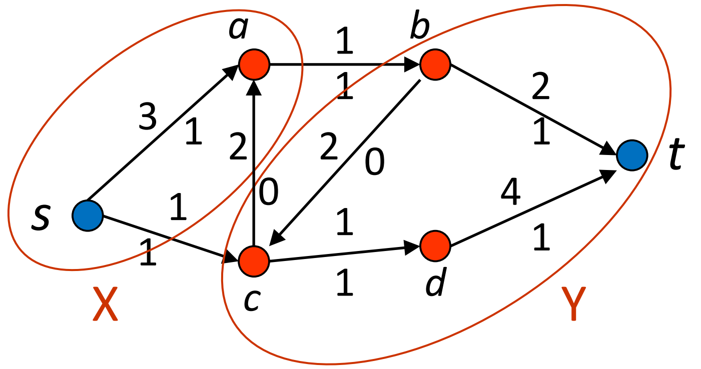
- Repeat this step until we can no longer find an *f*-augmenting path in *N(f).*
    - The algorithm stops when there are no directed path from s to t in *N(f).*
    - A proof showing that we now have a maximum flow, is that we can now point out a cut with capasity equal to the current flow. Thus, there can be no larger flow!

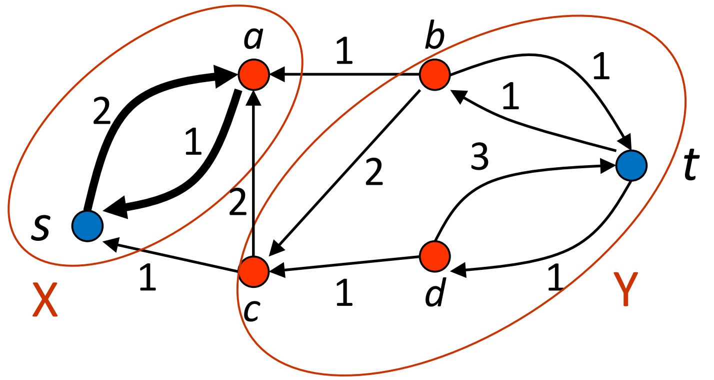




Termination of the Ford-Fulkerson Algorithm

The FordFulkerson-algorithm stops when there is no connection from s to t in *N(f).*



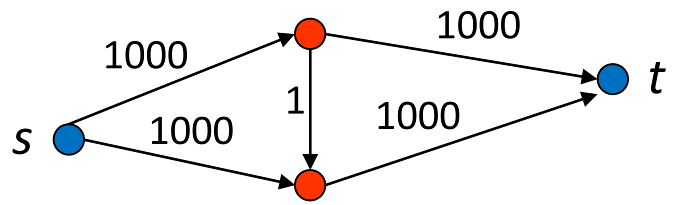


- As indicated: To show that we now have a maximum flow, we will show that we can construct a cut *K* with capacity equal to the current flow. That is: *cap(K)=val(f).*
- It turns out that such a cut is easy to find: Let X be the set of nodes reachable from *s* in *N(f)*, and let Y be the rest of the nodes (including *t*).
- As no edge in *N(f)* leads from X to Y, we know by the def. of *N(f):*
  - All edges in *N* (the original network) from X to Y are used to its full capacity.
  - All edges in *N* leading from Y to X have flow *f* = 0
- This means that *cap(K)* equals the current flow over K, which again is *val(f).*
- Thus, we know we have a maximum flow, and we have proven the following Theorem:

Theorem (Max-flow, min-cut): In a network with capacities we can find a flow *f* and a cut *K* so that *val(f)=cap(K).* Then we know that we have a maximum flow, and that no cut has lower capacity.

Variations of the Ford-Fulkerson algorithm

- The **Ford-Fulkerson Algorithm** says nothing about which *f*-augmenting path should be chosen in each step, if there is more than one
- If we do not decide anything about the choice of *f*-augmenting paths, we know:
  - If all capacities are (positive) intergers, then the number of steps can be as large as the size of the largest flow (but indeed, no larger, as each incrrement is an integer!):



n = number of nodes

m = number of edges

  - If the capacities are real numbers, the algorithm can in theory loop for ever.

Proposal 1: All the time, choose the *f*-augmenting path that gives that largest possible increment in the flow. (This path can be found by an algorithm similar to a shortest path algorithm)

- This gives a worst-case-time: $O(\ m \log(n) \log(\ max\text{-}flow\ )\ )$

Proposal 2: (Edmonds og Karp) All the time, choose the *f*-augmenting path that has the smallest number of edges (can be found by a bredth-first search)

- This gives a worst-case-time: $O(n\ m^2)$
  This is independent of the max. flow, and it thereby shows that there exists a polynomial time algorithm that solves this problem! Thus, **the problem is in P**.

Variations of the problem of max. flow

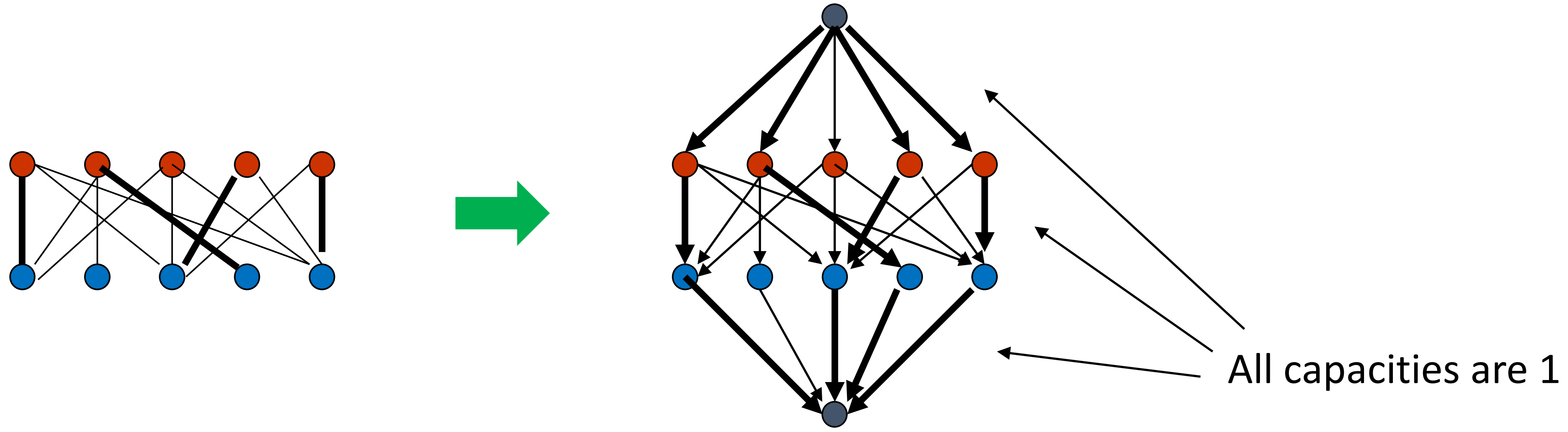
- First of all, there are alternatives to the Ford-Fulkerson algorithm
  - Dinic has designed an algorithm with time O(n2 m)
  - Goldberg and Tarjan («preflow push algorithm», time $O(n^2\sqrt{m})$)
- We may also have a minimal flow for each edge
  - Then it is an interesting problem just to find a possible flow
  - But after that you can proceed as in Ford-Fulkerson
- We may also have a price on each edge, saying how much a flow of one will cost over this edge. We want to minimize total cost for getting a certain total flow through the network.
  - On old algorithm here is the «Out-of-kilter algorithm» (not polynomial), but many later algorithms runs in polynomial time.
- We can also have multiple sources and/or multiple sinks, with different requirements to the flow in and out of these
- We may also have different "commodities" that should flow in the network (cars, busses, trucks, etc. in a rail or street network) , and the edges may have a different capacity and cost for each commodity.
  - This is a field of active research, in connection with e.g. traffic planning, routing in communication networks, etc.

A connection between flow in networks and matching in bipartite graphs

A simple but important lemma, which is obvious from the Ford-Fulkerson Algorihtm and the max flow-min-cut Theorem:

1. If we have interger capacities, then we can always find an interger max. flow.
2. And thus the Lemma: When all the capacities are 1, we can find a max. flow where all edgeflows are either 0 or 1.

Such a flow can be seen as pointing out a subset of the edges (those with flow 1)



Concerning the above picture, we will as an exercise look at:

- That searching for an M-augmenting path in the bipartite graph to the left, corresponds to searching for an *f*-augmenting path to the right.