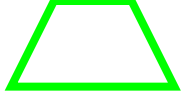
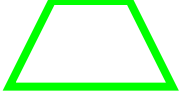
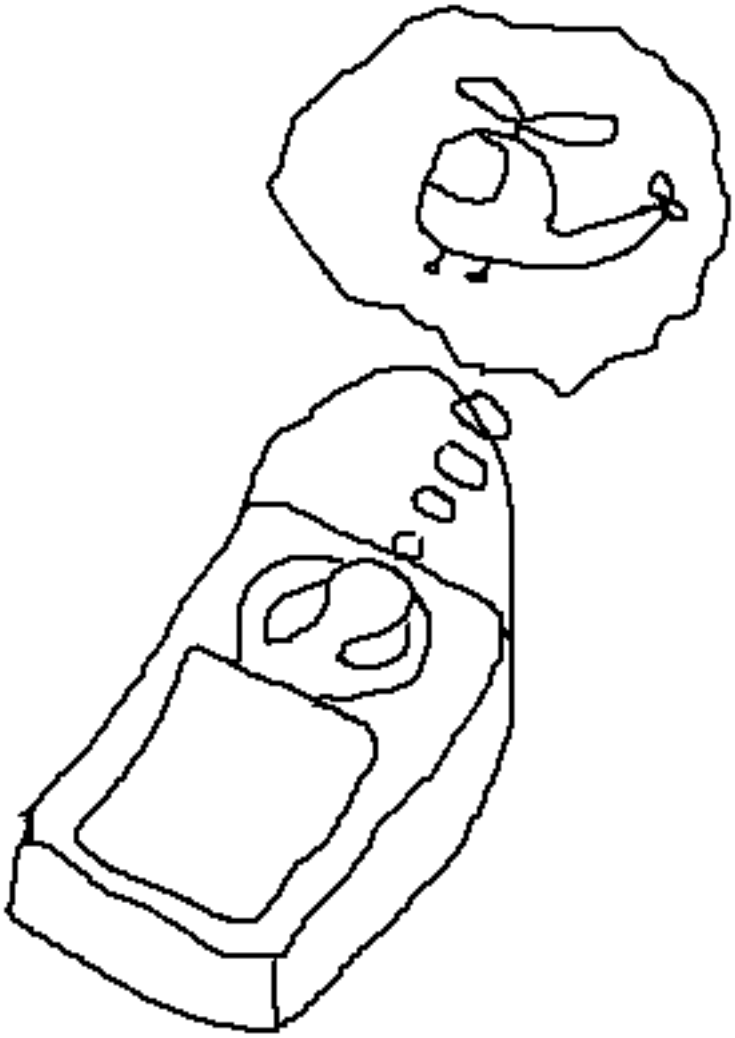


Alternative approaches to algorithm design and analysis

- **Problem:** Exhaustive search gives typically $\mathcal{O}(n!) \approx \mathcal{O}(n^n)$-algorithms for $\mathcal{NP}$-complete problems.
- So we need to get around the **worst case / best solution** paradigm:
  - worst-case $\rightarrow$ average-case analysis
  - best solution $\rightarrow$ approximation
  - best solution $\rightarrow$ randomized algorithms



Average-case analysis & algorithms



~~Worst case~~

- **Problem = (L, P_r)** where P_r is a probability function over the input strings: $P_r : \sum^* \rightarrow [0, 1]$.
- $\sum_{x \in \sum^*} P_r(x) = 1$ (the probabilities must sum up to 1).
- **Average time** of an algorithm:

$$T_A(n) = \sum_{\{x \in \sum^* \mid |x|=n\}} T_A(x) P_r(x)$$

- **Key issue:** How to choose P_r so that it is a realistic model of reality.
- Natural solution: Assume that all instances of length n are equally probable (uniform distribution).

Random graphs

Uniform probability model (UPM)

- Every graph G has equal probability
- If the number of nodes = n, then $P_r(G) = \frac{1}{\#\text{graphs}} = \frac{1}{2^{\binom{n}{2}}}$, where $\binom{n}{2} = \frac{n(n-1)}{2}$
- UPM is more natural for interpretation

Independent edge probability model (IEPM)

- Every possible edge in a graph G has equal probabilility p of occuring
- The edges are independent in the sense that for each pair (s,t) of vertices, we make a new toss with the coin to decide whether there will be an edge between s and t.
- For $p = \frac{1}{2}$ IEPM is identical to UPM:

$$P_r(G) = \left(\frac{1}{2}\right)^m \cdot \left(\frac{1}{2}\right)^{\binom{n}{2}-m} = \frac{1}{2^{\binom{n}{2}}}$$

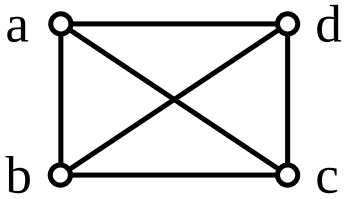
- IEPM is easier to work with

Example: 3-COLORABILITY

In 3-COLORABILITY we are given a graph as input and we are asked to decide whether it is possible to color the nodes using 3 different colors in such a way that any two nodes have different colors if there is an edge between them.

Theorem 1 3-COLORABILITY, *which is an $\mathcal{NP}$-complete problem, is solvable in* ***constant average (expected) time*** *on the IEPM with $p = 1/2$ by a branch-and-bound algorithm (with exponential worst-case complexity).*

Proof:
Strategy (for a rough estimate): Use the indep. edge prob. model. Estimate expected time for finding a proof of non-3-colorability.



K_4 (a clique of size 4) is a proof of non-3-colorability.

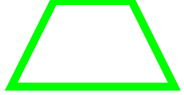
- The probability of 4 nodes being a K_4:
  $P_r(K_4) = 2^{-\binom{4}{2}} = 2^{-6} = \frac{1}{128}$
- Expected no. of 4-vertex sets examined before a K_4 is found:

$$\sum_{i=1}^{\infty} i\left(1-2^{-6}\right)^{i-1}2^{-6} = 2^{-6}\sum_{i=1}^{\infty} i\left(1-2^{-6}\right)^{i-1}$$
$$\stackrel{*}{=} 2^{-6}\frac{1}{\left(1-\left(1-2^{-6}\right)\right)^2}$$
$$= 2^{-6}\frac{1}{\left(2^{-6}\right)^2} = \frac{2^{12}}{2^6} = 2^6 = 128$$

  — $\left(1-2^{-6}\right)^{i-1}2^{-6}$ is the probability that the first K_4 is found after examining exactly i 4-vertex sets.

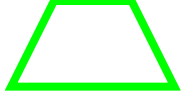
  — $(*)$ is correct due to the following formula ($q = 1-2^{-6}$) from mathematics (MA100):

$$\sum_{i=1}^{\infty} iq^{i-1} = \frac{\delta}{\delta q}\left(\sum_{i=1}^{\infty} q^i\right) = \frac{\delta}{\delta q}\left(\frac{q}{1-q}\right)$$
$$= \frac{1}{(1-q)^2}$$



Conlusion: Using IEPM with $p = \frac{1}{2}$ we need to check 128 four-vertex sets on average before we find a K_4.

Note: Random graphs with constant edge probability are very dense (have lots of edges). More realistic models has p as a function of n (the number of vertices), i.e. $p = 1/\sqrt{n}$ or $p = 5/n$.



0-1 Laws

as a link between probabilistic and deterministic thinking.

Example: "Almost all" graphs are

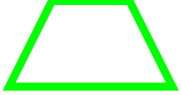
- not 3-colorable
- Hamiltonian
- connected
- ...

Def. 1 *A property of graphs or strings or other kind of problem instances is said to have a* ***zero-one law*** *if the limit of the probability that a graph/string/problem instance has that property is either* 0 *or* 1 *when* n *tends to infinity* ($\lim_{n\to\infty}$).

Example: HAMILTONICITY

a linear expected-time algorithm for random graphs with $p = 1/2$.

- **Difficulty:** The probability of non-Hamiltonicity is too large to be ignored, e.g. $P_r(\exists \text{ at least } 1 \text{ isolated vertex}) = 2^{-n}$.
- The algorithm has 3 phases:
  - **Phase 1:** Construct a Hamiltonian path in linear time. Fails with probability $P_1(n)$.
  - **Phase 2:** Find proof of non-Hamiltonicity or construct Hamiltonian path in time $\mathcal{O}\left(n^2\right)$. Unsuccessful with probability $P_2(n)$.
  - **Phase 3:** Exhaustive search (dynamic programming) in time $\mathcal{O}\left(2^{2n}\right)$.
- Expected running time is
  $\leq \mathcal{O}(n) + \mathcal{O}\left(n^2\right) P_1(n) + \mathcal{O}\left(2^{2n}\right) P_1(n)P_2(n)$
  $= \mathcal{O}(n)$ if $P_1(n) \cdot \mathcal{O}\left(n^2\right) = \mathcal{O}(n)$
  and $P_1(n)P_2(n) \cdot \mathcal{O}\left(2^{2n}\right) = \mathcal{O}(n)$
- Phase 2 is necessary because $\mathcal{O}\left(2^{-n}\right) \cdot \mathcal{O}\left(2^{2n}\right) = \mathcal{O}\left(2^n\right)$.
- After failing to construct a Hamiltonian path fast in phase 1, we first reduce the probability of the instance being non-Hamiltonian (phase 2), before doing exhaustive search in phase 3.

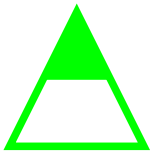


Randomized computing

Machines that can **toss coins** (generate random bits/numbers)

- Worst case paradigm
- ~~Always~~ give the correct (best) solution

Randomized algorithms

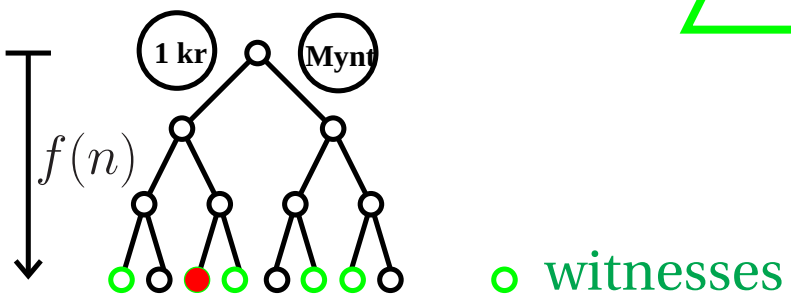


Idea: Toss a coin & simulate non-determinism

Example 1: Proving polynomial non-identities

$$(x+y)^2 \stackrel{?}{\neq} x^2 + 2xy + y^2$$
$$\stackrel{?}{\neq} x^2 + y^2$$

- What is the "classical" complexity of the problem?
- Fast, randomized algorithm:
  - Guess values for x and y and compute left-hand side (LHS) and right-hand side (RHS) of equation.
  - If LHS $\neq$ RHS, then we know that the polynomials are different.
  - If LHS $=$ RHS, then we suspect that the polynomials are identical, but we don't know for sure, so we repeat the experiment with other x and y values.
- Idea works if there are many witnesses.



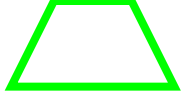
Let $f(n)$ be a polynomial in n and let the probability of success after $f(n)$ steps/coin tosses be $\geq \frac{1}{2}$. After $f(n)$ steps the algorithm either

- finds a witness and says "Yes, the polynomials are different", or
- halts without success and says "No, maybe the polynomials are identical".

This sort of algorithm is called a **Monte Carlo algorithm**.

Note: The probability that the Monte Carlo algorithm succeeds after $f(n)$ steps is **independent of input** (and dependent only on the coin tosses).

- Therefore the algorithm can be repeated on the same data set.
- After 100 repeated trials, the probability of failure is $\leq 2^{-100}$ which is smaller then the probability that a meteorite hits the computer while the program is running!



Metaheuristics

Simulated Annealing

- Analogy with physical annealing
- 'Temperature' T, annealing schedule
- 'Bad moves' with probability $\exp(-\delta f/T)$

Genetic algorithms

- Analogy with Darwinian evolution
- 'individuals', 'fitness', 'cross breeding'

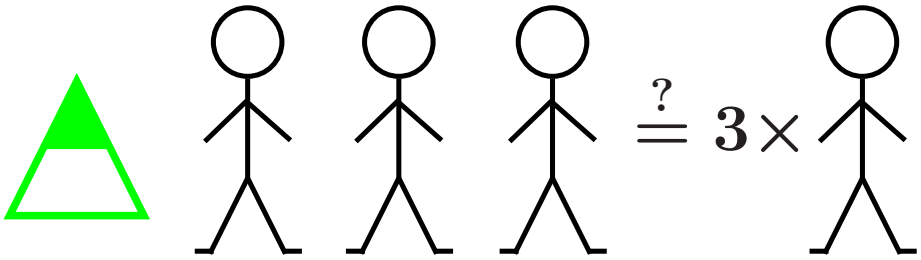
Neural Networks

- Analogy with human mind
- 'neurons', 'learning'

Taboo search

- Analogy with culture
- adaptive memory, responsive exploration

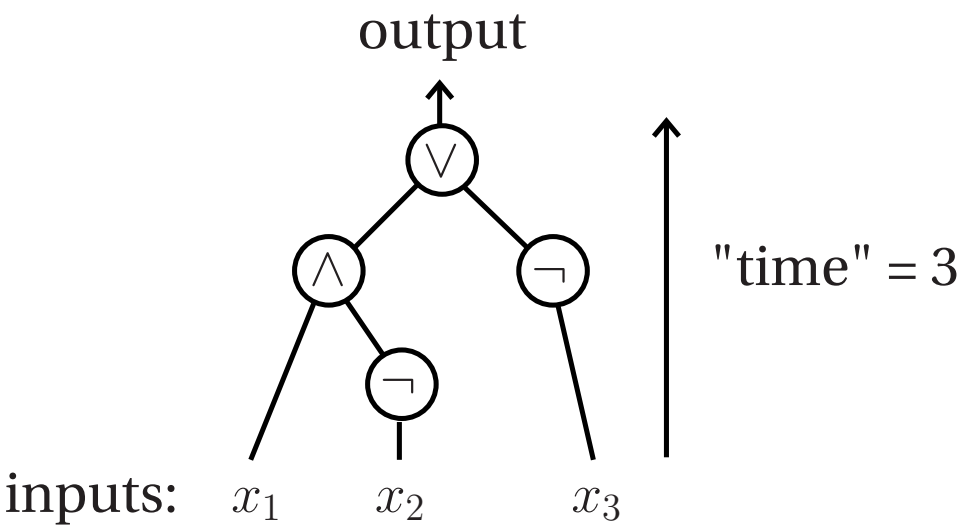
Parallel computing



- some problems can be efficiently parallelized
- some problems seems inherently sequential

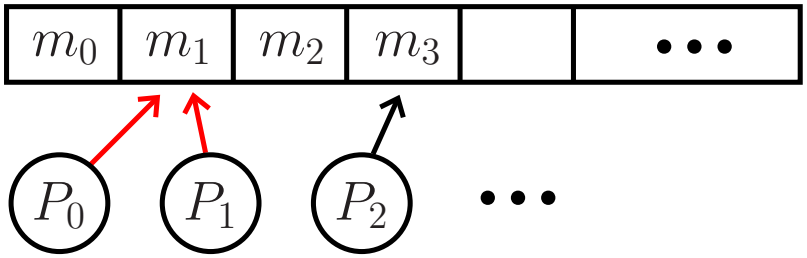
Parallel machine models

- **Alternating TMs**
- **Boolean Circuits**



— Boolean Circuit complexity: **"time"** (length of longest directed path) and **hardware** (# of gates)

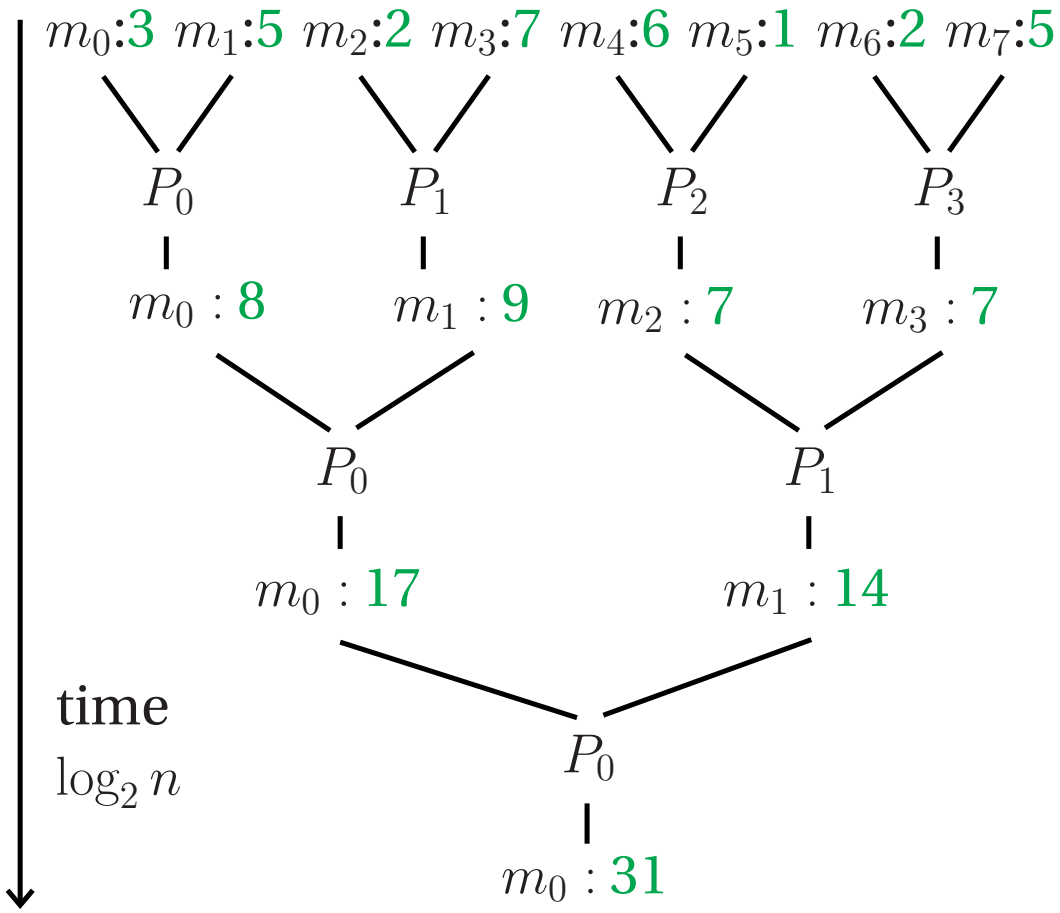
- **Parallell Random Access Machines (PRAMs)**



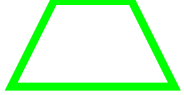
— Read/Write conflict resolution strategy

— PRAM complexity: **time** (# of steps) and **hardware** (# of processors)

Example: Parallel summation in time $\mathcal{O}(\log n)$



Result: Boolean Circuit complexity = PRAM complexity.



Limitations to parallel computing

Good news

parallel time $\leftrightarrow$ sequential space

Example: HAMILTONICITY can easily be solved in parallel polynomial time:

- On a graph with n nodes there are at most $n!$ possible Hamiltonian paths.
- Use $n!$ processors and let each of them check 1 possible solution in polynomial time.
- Compute the the OR of the answers in parallel time $\mathcal{O}\left(\log(n!)\right) = \mathcal{O}\left(n \log n\right)$.

Bad news

Theorem 2 *With polynomial many processors*

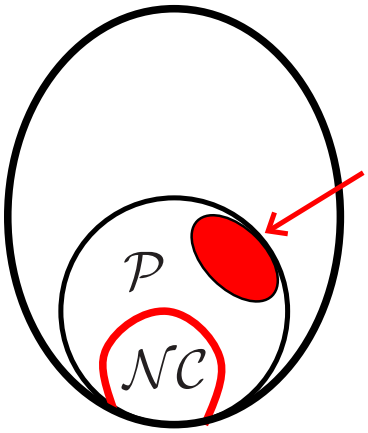
parallel poly. time = sequential poly. time

Proof:

- 1 processor can simulate one step of m processors in sequential time $t_1(m) = \mathcal{O}\left(m\right)$
- Let $t_2(n)$ be the polynomial parallel time of the computation. If m is polynomial then $t_1(m) \cdot t_2(n) =$ polynomial.

Parallel complexity classes

Def. 2 *A language is said to be in class $\mathcal{NC}$ if it is recognized in polylogarithmic, $\mathcal{O}\left(\log^k(n)\right)$, parallel time with uniform polynomial hardware.*



$\mathcal{P}$-hard, Ex: CIRCUIT VALUE

- $\mathcal{P} \stackrel{?}{=} \mathcal{NC}$