Game Trees and Strategies for Two-player Games

2nd November 2022

Ch. 23.5: Games, game trees and strategies

- We have looked at "one player games" (= search) and their decision trees, earlier (Ch 23.1 – 23.4)
  - This is search for a goal node that everybody agrees is "good"
- Then you can for instance use A*-search for e.g.:
  - Solve the 15-puzzle from a given position
  - Find the shortest path between nodes in a graph (better than plain Dijkstra)

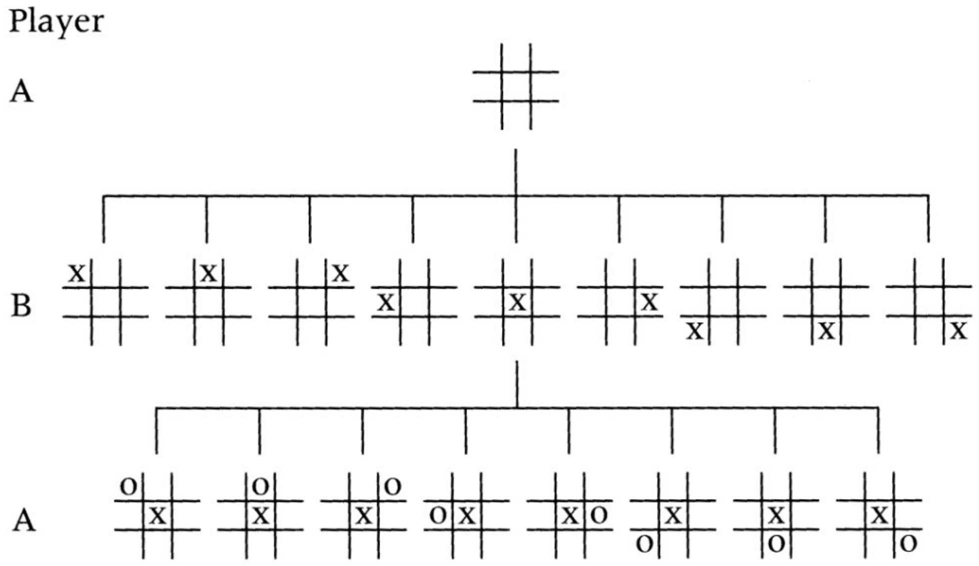
BUT:

- When two players are playing against each other, things get very different. What is *good* for one player is *bad* for the other
  - The tree of possible plays is often enormous. For chess it is estimated to have ca 10^{100} nodes, and can therefore never (?) be searched exhaustively!
- We look at *"zero-sum" games*. This roughly means:
  - If, during a move, the "chances to win" is increased for one of the players, then it is decreased by the corresponding amount for the other

Example: A game tree for Tic-Tac-Toe

- The board has 3 x 3 squares.
- The game: Repeat the following moves
  - Player A chooses an unused square and writes **X** in it
  - Player B does the same, but writes **O**
- Player A (always) starts
- When a player has three-in-a-row, he/she has won
- The game stops when A or B wins, or when all squares are filled (maybe with a “draw” = neither A nor B has three-in-a-row)

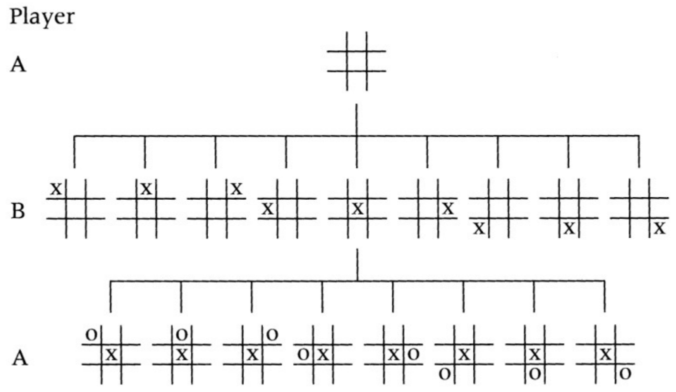
The start/root node of the *game tree* for Tic-Tac-Toe.



A final situation without a winner →

o	o	x
x	x	o
o	x	x

Number of nodes in a fully expanded tree for Tic-Tac-Toe



1 node

9 nodes

9*8 = 72 nodes

9*8*7 = 504 nodes . . . .

9*8*7*6 = 3024 nodes . . . .

9*8*7*6*5 = 15120 nodes . . . .

……

9*8*7*6*5*4*3*2*1 = 9! (“factorial”) = 362 880 nodes

Comment: By searching depth-first in this tree, you never need to store more than 9 nodes, but it will take some time to go through all 362 880 nodes.

(And for “interesting games” there are usually *a lot more*!)

The same situation may occur many places in the tree. We may represent each situation by *one* node

1 node

9 nodes

9*8 = 72 nodes

9*8*7 = 504 nodes

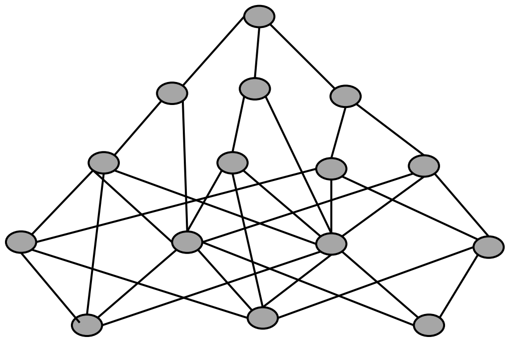
9*8*7*6 = 3024 nodes

9*8*7*6*5 = 15120 nodes

9*8*7*6*5* 4 = 60480 nodes

……

9*8*7*6*5*4*3*2*1 = 362 880 nodes



Example of a collapsed tree (a DAG)

As before:

1 node

9 nodes

72 **different** nodes

Fewer than before:

252 **different** nodes

756 **different** nodes

1260 **different** nodes

1680 **different** nodes

……

126 **different** nodes = $\binom{9}{4}$

$\binom{9}{4}$ = (9 8 7 6) / (1 2 3 4) = 126

This usually requires a lot of memory!

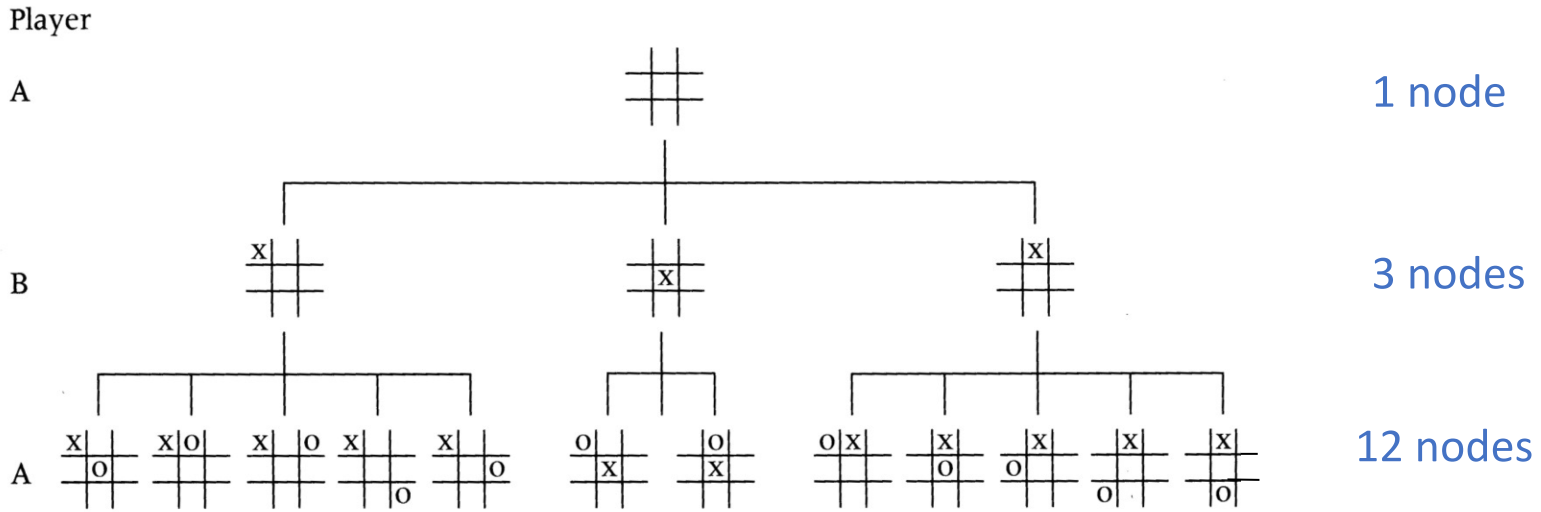
In some games, e.g. Tic-Tac-Toe, you can gain a lot by recognizing equal nodes, and not repeat the analysis for these

In Tic-Tac-Toe we then never need more than 1680 nodes during breath first search

In Chess this is very important!

Representing symmetric situations by the same node

- We can also gain a lot by looking at symmetries:
  - Two situations are symmetric if the rest of the game from these two situations will also be symmetric according to the rules of the game
  - Represent positions that are symmetries of each other by the same node
  - Tic-Tac-Toe: Symmetric solutions will always be at the same depth, but this is not generally the case!
  - In e.g. chess there are fewer symmetries to utilize
- Using this will often reduce the needs for memory/time further!



The "value" of a position, and zero-sum games

- During a game, we will always store:
  - A number (value) characterizing how good the situation is for player A
    - High values are good for A, and low values are bad
    - Thus all nodes of a game-tree have a value (seen from A)
  - If we want to see the game from B's point of view, we usually negate the values
- We want a "strategy for A"
  - That is: Some kind of rule telling A what to do in all possible "A-situations" (those where it is A's turn to make a move)
  - We will, for a given position, look for a strategy that will give A a win
    - **But note**: Such a strategy will often not exist!

Fully analyzable games

- “Fully analyzable games” means: The full tree can be traversed and analyzed
  - Then there will be three possible values for each A-situation S (usually represented as +1, -1 or 0)

1. A has a strategy so that A will win whatever B does, if A follows that strategy from S (score: +1 for A)
2. Whatever A does from S, B has a winning strategy from the new situation (score: -1 for A).
3. If A and B both play perfectly, it will end in a tie, or the game will go on for ever (score: 0 for both)
   - Situation 3 can only occur for some games. E.g.: The game Tic-Tac-Toe ends in a tie if both players play as good as possible.

Another example: The game Nim

- We start with two (or more?) piles of sticks
  - Number of sticks: *m* and *n*
- One player can take any number of sticks from one pile, but have to take at least 1
- The player taking the last stick has lost

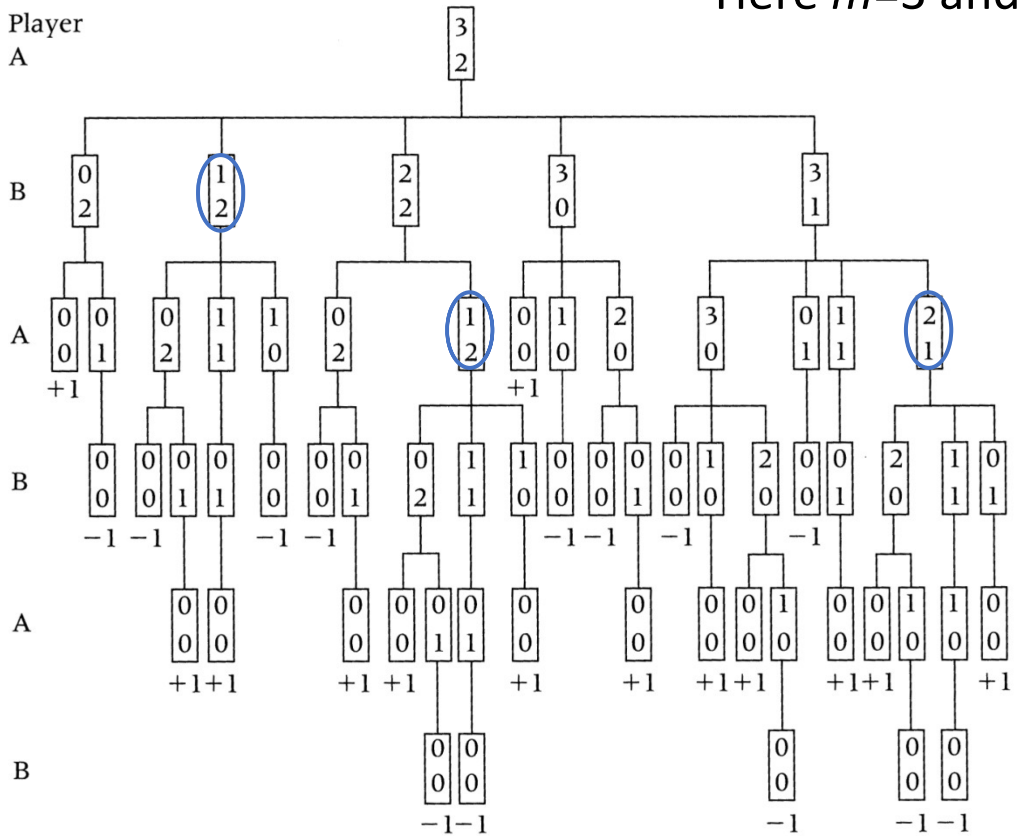
– Nim will never end in a tie.

– With *m*=3 and *n*=2, the full game tree is shown to the right.

– The value seen from A is indicated for the final situations (leaf nodes).

– **Next problem:** What is the value of the rest of the nodes?

Here *m*=3 and *n*=2



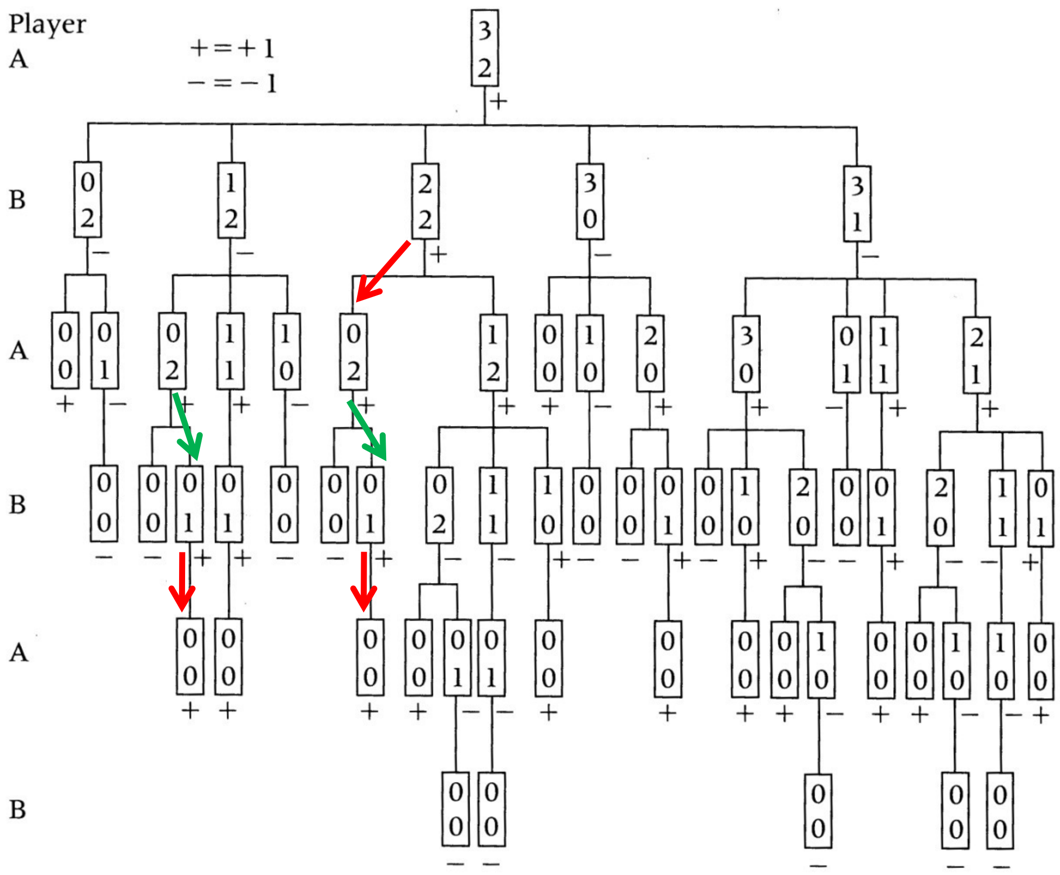
NB: We could reduce the number of separate nodes by recognizing symmetries and equivalent nodes (see e.g. **blue circles** above)

How can we find a strategy so that A wins? Or prove that no such strategy exists!

- A wants to find an optimal move from any given position
- We must assume that also B will do optimal moves seen from B's point of view
  - Thus B will move to the subnode with smallest value (since +1 and -1 are as seen from A)

Min-Max Strategy:

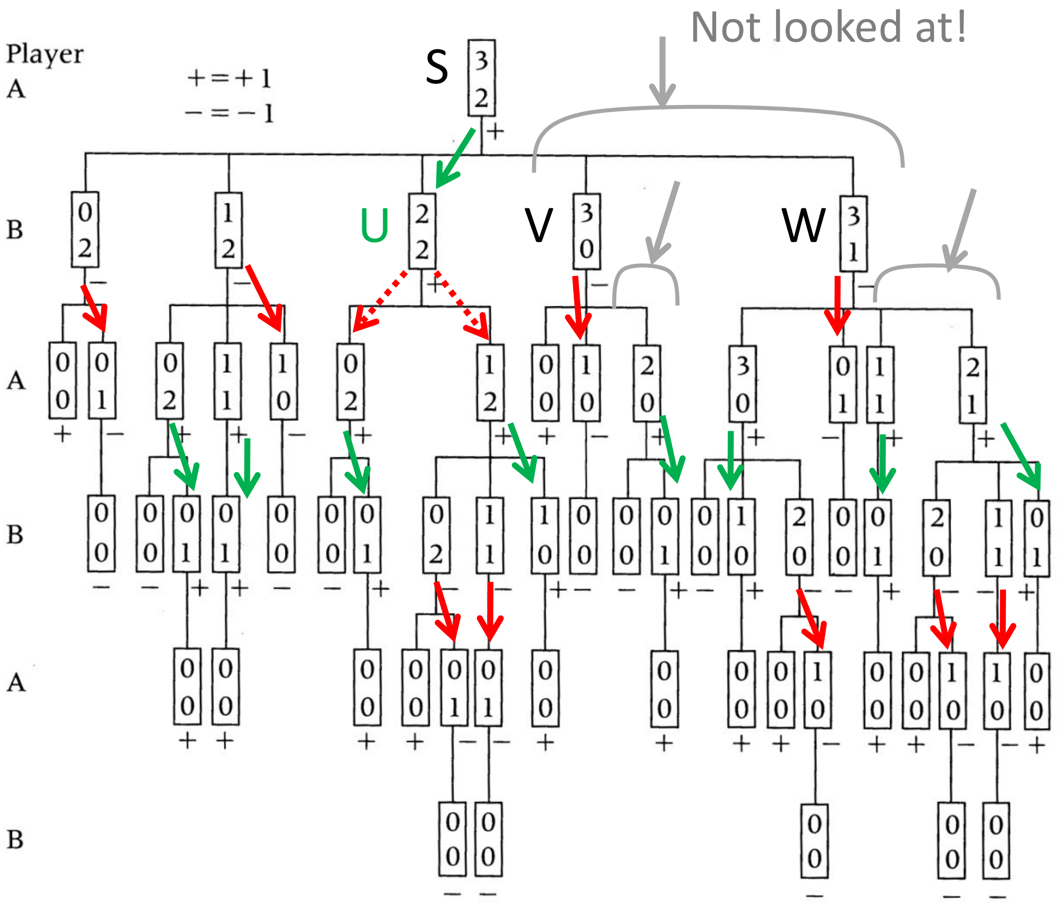
- To compute the value of a node, we have to know the values of all the subnodes
- This can be done by a depth first search, computing node values during the withdrawal (postfix)



Values for A-nodes: If possible, move to a node with value +1 (and mark current node with +1). Otherwise make a random move

Values for B-nodes: If possible, move to a node with value -1. Otherwise make a random move

The Min-Max-Algorithm in action, with simple alpha-beta cutoff



- Previous slide: The search is done by a depth first traversal of the game tree, computing values on withdrawal (postfix)
- The result of this is given in the figure to the left as + and -.

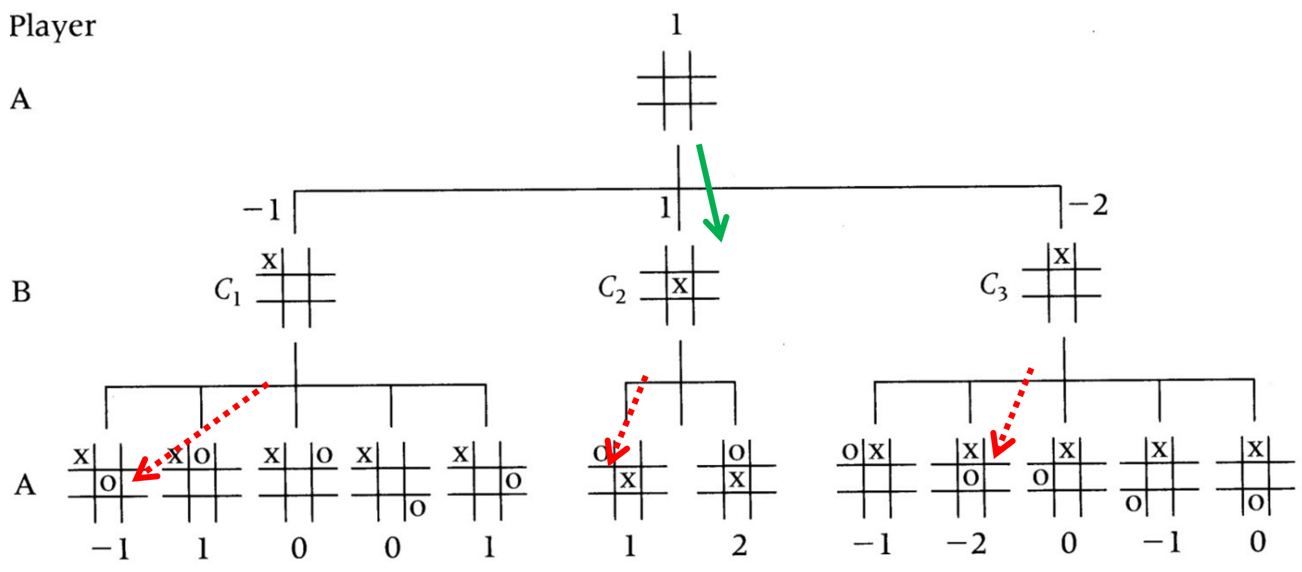
Possible optimalization:

- From the start-position S, assume that A has looked at three of its subtrees (from the left). A has then found a winning node U (marked +1). *Then the value of V and W does not matter.*
- This is a simple version of ***alpha-beta cutoff*** *(pruning)*

- **Green arrows:** Good moves for A from winning situations for A
- **Red arrows:** Good move for B from winning situations for B

What if the game tree is too large to traverse?

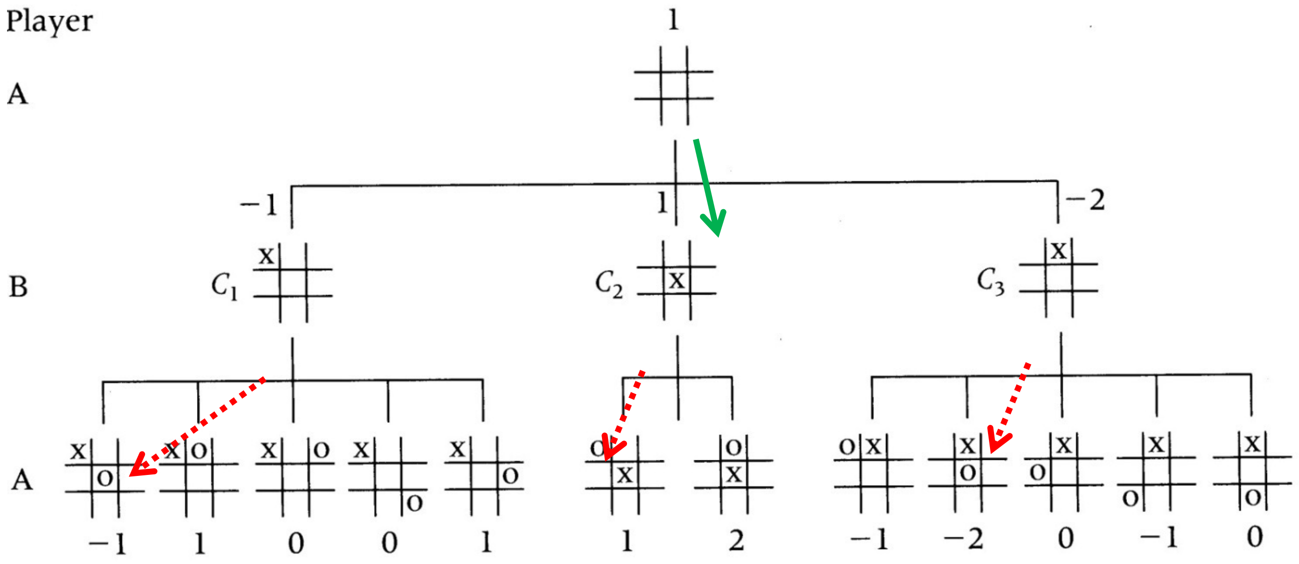
- Search to a certain depth, and then estimate (with some heuristic function) how good the situation is for A at the nodes at that depth. We then usually use other values than only: +1, -1 and 0
- In the figure above we go to depth 2
- The heuristic function above is: the number of “winning lines for A” minus the number of “winning lines for B” (this is given above for each leaf node)
  - A “winning line” for A is a column, row or diagonal where B has not filled any of the three positions (so that A can still hope to fill them all, and win)
- The best move for A from the start position is therefore (according to this heuristic) to go to C_2



What if the game tree is too large to traverse?

- However, this heuristic is not good later on in the game. It does not take into account that winning is better than any heuristic. We therefore, in addition, give winning nodes the value $+\infty$ (no such node here).
- This will give quite a good strategy. But, as said above: Tic-Tac-Toe will end in a tie if both players play perfectly.
- We have to add that the tie-situation (e.g. the one below) gets the value 0. Thus, if we fully analyze the game, the value of the root node will be 0.

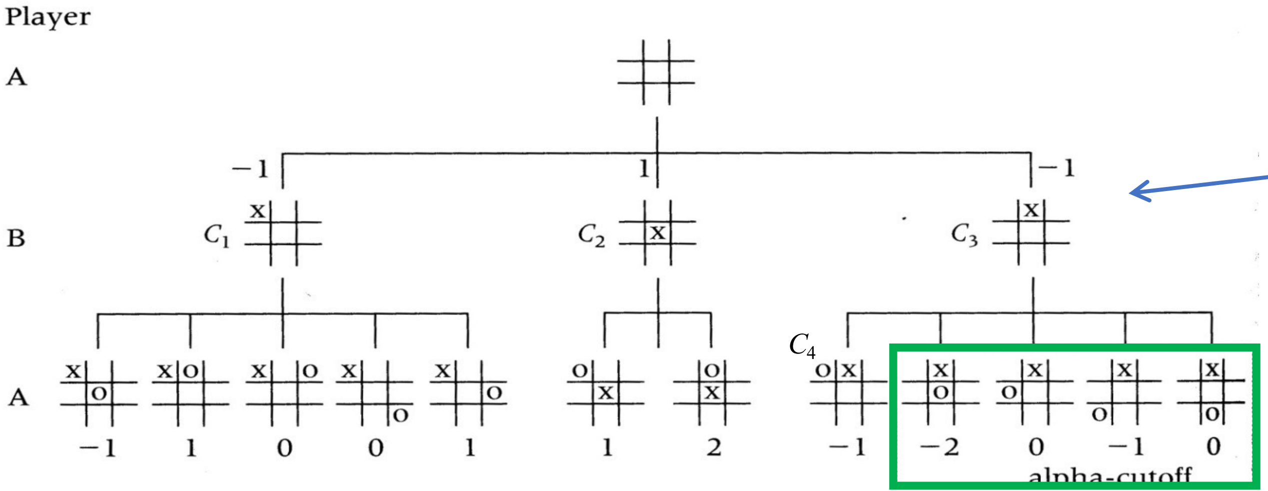
O	O	X
X	X	O
O	X	X



NOTE: The difficult choice for a game-programmer is between searching *very deep* or using a *good, but time consuming,* heuristic function!

Intuition: Alpha-beta cutoff (pruning) (Assuming it is A's move)

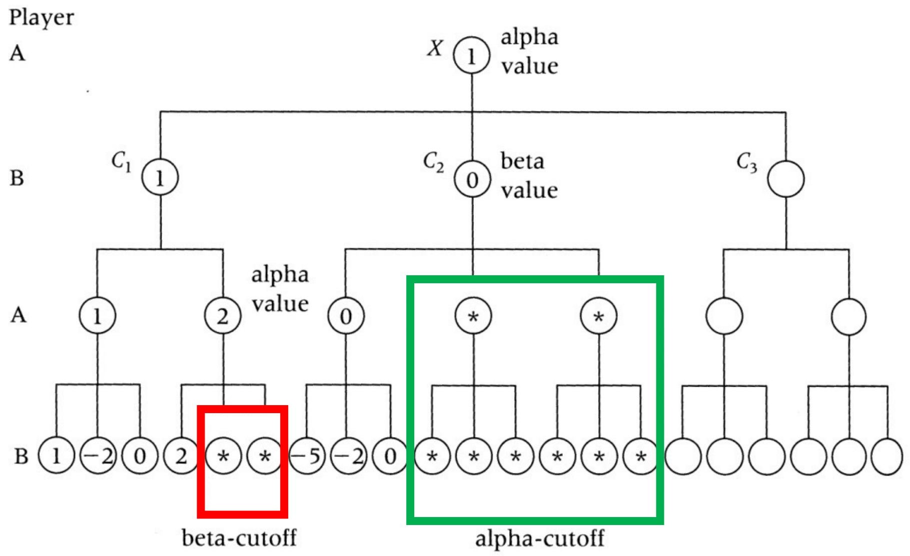
- A will consider all the possible moves from the current situation, one after the other...
- After a while, A has noted that the best move seen so far is a move in which A can obtain the value *u* (after C_1 and C_2, u = 1)
- A looks at the next potential move, which would lead to situation C_3 and then looks at the subnodes of C_3. A soon observes that *B has a very good move* (C_4) giving value v = -1 Thus the value of C_3 cannot be better (for A) than -1 as B will minimize at C_3. This is true independent of what value the other subtrees of C_3 gives
- As v < u, player A has no interest in looking for even better moves for B from situation C_3. A already knows that it has a better move than to C_3, which is C_2



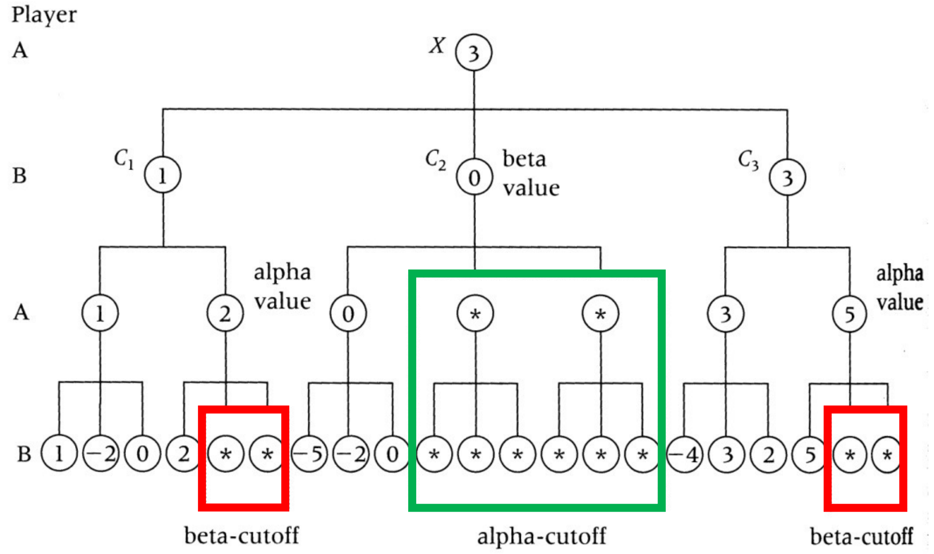
Should have become -2, but value -1 (after C4) is enough for A to conclude that a move to C_3 is not the best (to C_2 is better, with value 1)

Examples showing alpha-beta cutoff

- When A considers the next move:
    - Cutoffs from A-situations is called **alpha-cutoffs**
    - Corresponding cutoffs from B-situations are called **beta-cutoffs**
- The figures show alpha- and beta-cutoffs at different stages of a DF-search of a game tree
- When implementing alpha-beta-cutoffs during a DF-search, it is usual to switch viewpoints between the levels
    - Then we can always maximize the value
    - But we have to negate all values for each new level



(a)



(b)

Examples showing alpha-beta cutoff

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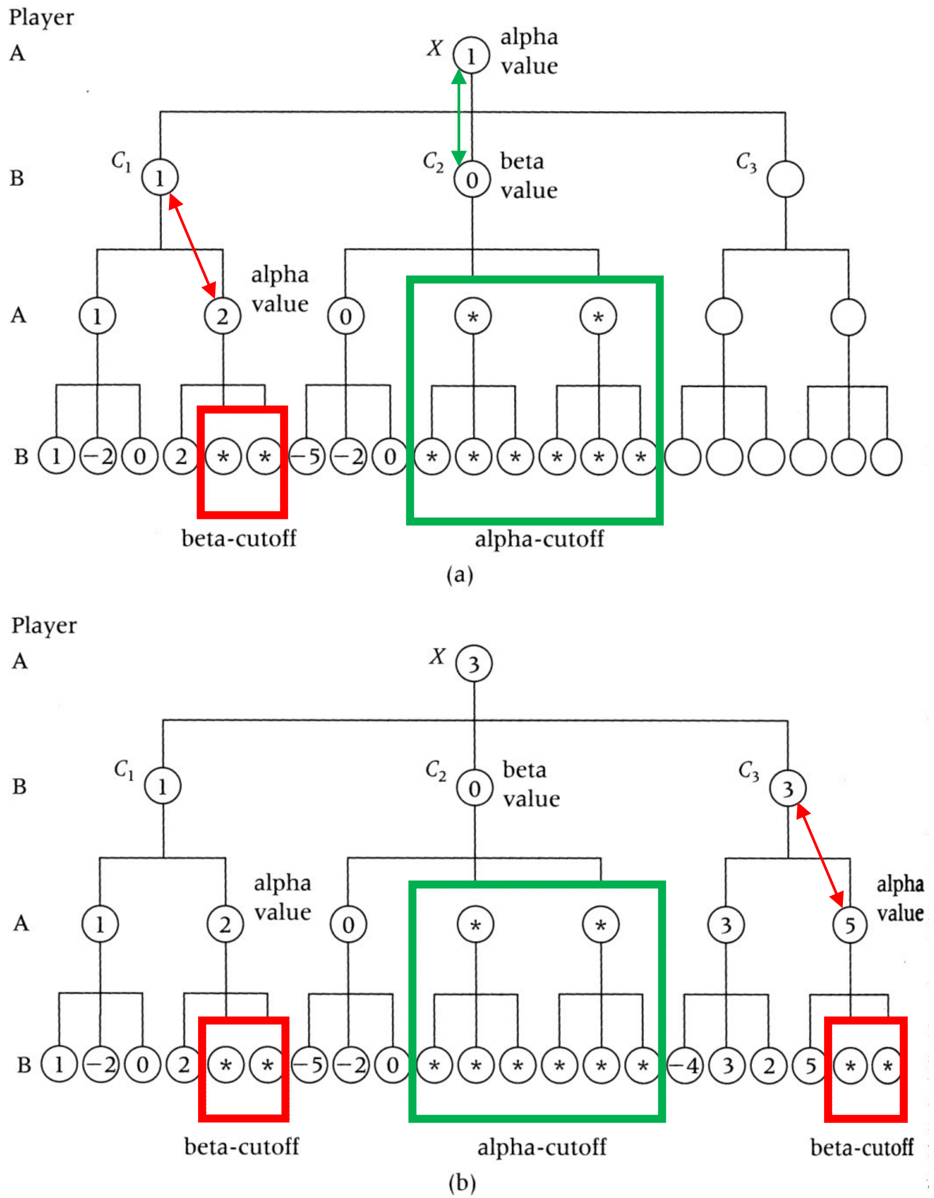

```
real function ABNodeValue (
    X,          // The node we compute alpha/beta value for. Children: C[1],C[2]… C[k]
    numLev,     // Number of levels left
    parentVal) // The alpha/beta-value from the parent node (-LB from the parent)
// Returned value: The final alpha/beta-value for the node X
{
    real LB;    // Will hold current Lower Bound for the alpha/beta value of node X

    if <X is a terminal node> or numLev = 0 then {
        return <An estimate of the quality of the situation (the heuristic)>;
    } else {
        LB :=  - ABNodeValue(C[1], NumLev-1, ∞); // Recursive call
        for i :=  2 to k do {
            if  LB >= parentValue  then {
                return LB;                          // Cutoff, no further calculation
            }
            else {
                LB := max(LB, - ABNodeValue(C[i], Numlev-1,  - LB) ); //Recursive call

            }
        }
    }
    return LB;
}
```

Start the recursive call to calculate value for the (current) rootnode (down to depth 10) by calling

```
ABNodeValue(rootnode, 10, -∞)                         // This ”-” is missing in the textbook
```

Misprints in the textbook (B&P)

- There are some simple misprints in the program at page 741 in the textbook (may be corrected in some editions):
  - ”AB” is missing in the name of the procedure in the recursive call.
  - A right parenthesis is missing at the end of the line where **max** is called.
  - A minus (“-”) is missing in the arguments of the initial call

- These errors are corrected on the previous slide!