Plan:
- Independent Joint Modeling
- Dynamic model recap
- Laplace transforms
- Transfer functions (short)
- Block diagrams
- P, PD, PID setpoint controllers

Independent Joint Modeling
- Controlling a whole manipulator is fairly difficult
  - We focus instead on controlling only one joint at a time
- All interaction between joints (dynamic coupling) will be classified as noise

Dynamic Model of a Robot
From the equations of motion from dynamics we have:
$J(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + g(q) = f$

$J(q)\ddot{q}$ - inertia
$C(q,\dot{q})\dot{q}$ - coriolis/centrifugal forces
$B\dot{q}$ - viscous friction (damping)
$g(q)$ - gravitational forces
f - torque/force from actuators

$$\left.\begin{array}{l}\text{Coriolis/centrifugal}\\ \text{Gravity}\\ \text{Coupling } (J(q)\ddot{q} \to J\ddot{q})\end{array}\right\} \to D \text{ - disturbance}$$

$J\ddot{q} + B\dot{q} + D = f$

- Not the same notations as in velocity kinematics: J is not Jacobian,
  D matrix is not the one from dynamics
- Inertia and inertial forces are not the same thing, they are loosely related, if at all
- Coriolis and centrifugal forces are often classified as inertial (fictitious) forces

Laplace Transform
  - Time -> frequency
  - Ordinary differential equation -> linear equation
  - Differentiation in time -> multiplication by s in Laplace
  - Integration in time -> division by s in Laplace

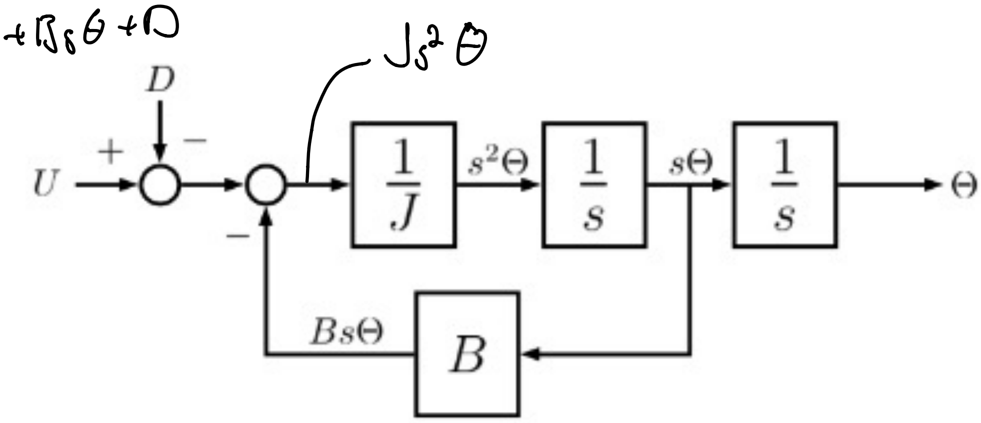
E.g: $J\ddot{q} + B\dot{q} + D = f \to Js^2\Theta + Bs\Theta + D = f$

Transfer functions
- Y(s) = H(s)X(s)
- Y(s) - system output
- X(s) - system input
- H(s) - transfer function (transforms input into output)
- Rearrange equations to find: H(s) = Y(s)/X(s)
- If numerator (Y) is set to 0, we can find the poles of the system by solving for s
- If denominator (X) is set to 0, we can find the zeros
  - Useful for determining stability of the system

Block Diagrams
- Helpful for visualizing equations and feedback/feedforward loops

Block diagram for the generic system:

$U = Js^2\Theta + Bs\Theta + D$



Block diagram with basic building blocks

- On the left side, we start with the full torque equation denoted as U (control effort)
- As the signal proceeds through the blocks, we "peel" terms off one by one, until we end up with only Θ
- The whole diagram represents a robotic system
  - Can be used as a building block for a larger diagram of controller blocks

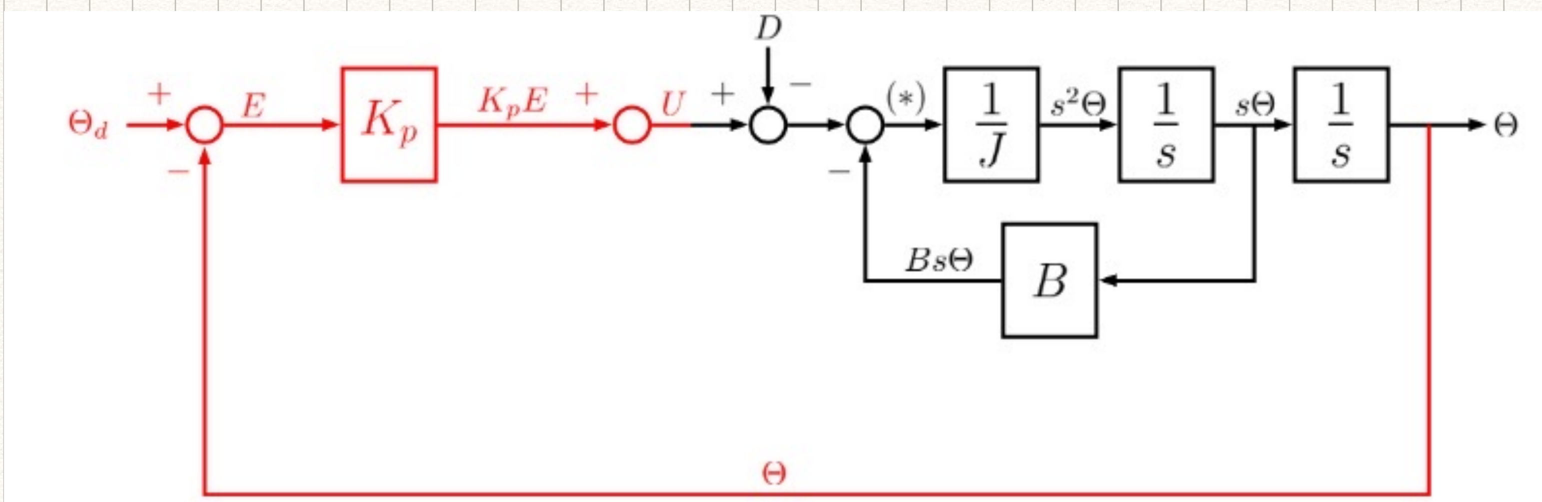
Setpoint controllers
- Controllers that drive a robot to a set point
- Current angle: Θ
- Desired angle: Θ_d
- Error: $e(t) = \Theta_d - \Theta$
- Controllers use the error term to calculate the control effort U (output torque/force)
- Controllers try to reduce the error to 0

- Types of setpoint controllers
  - Proportional (P)
  - Proportional Derivative (PD)
  - Proportional Integral Derivative (PID)
  (- Proportional Integral (PI))

P controller
- $U(t) = K_p e(t)$
- Control effort proportional to the controller
- Laplace: $U(s) = K_p E(s)$

$e = (\theta_d - \theta)$

- Block diagram:

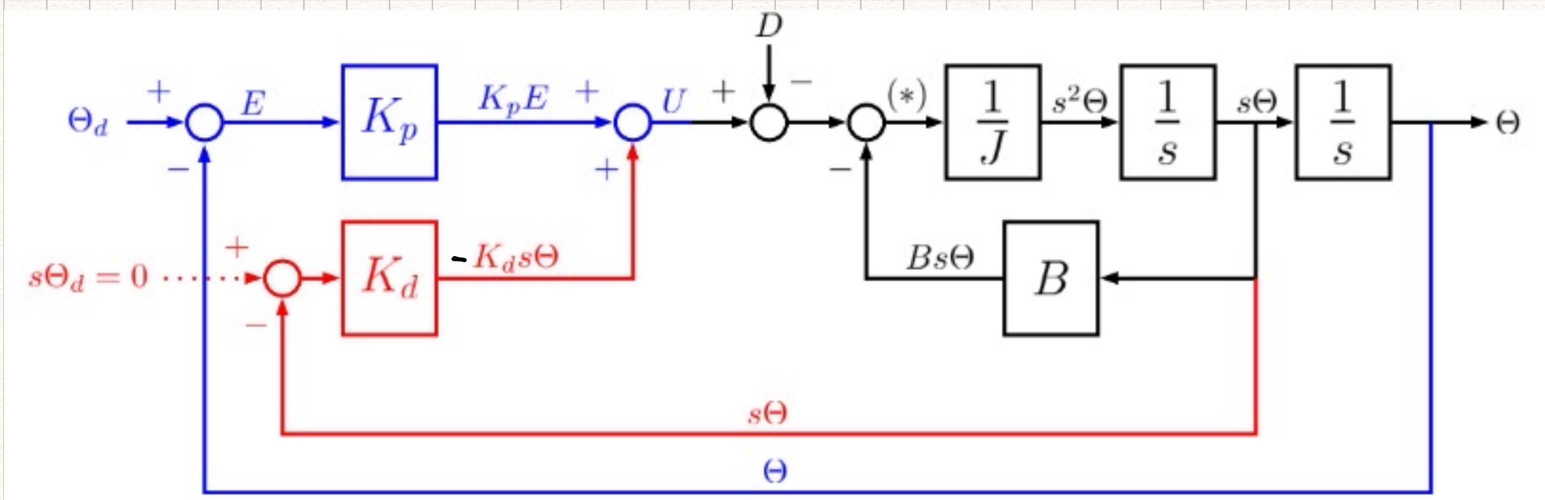


- Increased K_p gives:
  - Faster response
  - Decrease in steady state error
  - Increased oscillations
- Proportional term by itself will not eliminate the error

PD controller
- Builds on the P controller by adding a derivative term
- Derivative term represents how fast the error changes - used to "predict" future error
- $U(t) = K_p e(t) + K_d \dot{e}(t)$

$\dot{e} = \dot{\theta}_d - \dot{\theta}$ $\quad$ $\dot{\theta}_d = 0$ $\quad$ $K_d \dot{e} = -K_d \dot{\theta}$

- Laplace: $U(s) = K_p(\Theta_d - \Theta) - K_d s\Theta$
  - Write out the error terms, since the desired velocity for theta is 0 (no oscillations) we can simplify
- Block diagram:



- Increased K_d reduces oscillations, but can make the robot stop before reaching the desired point

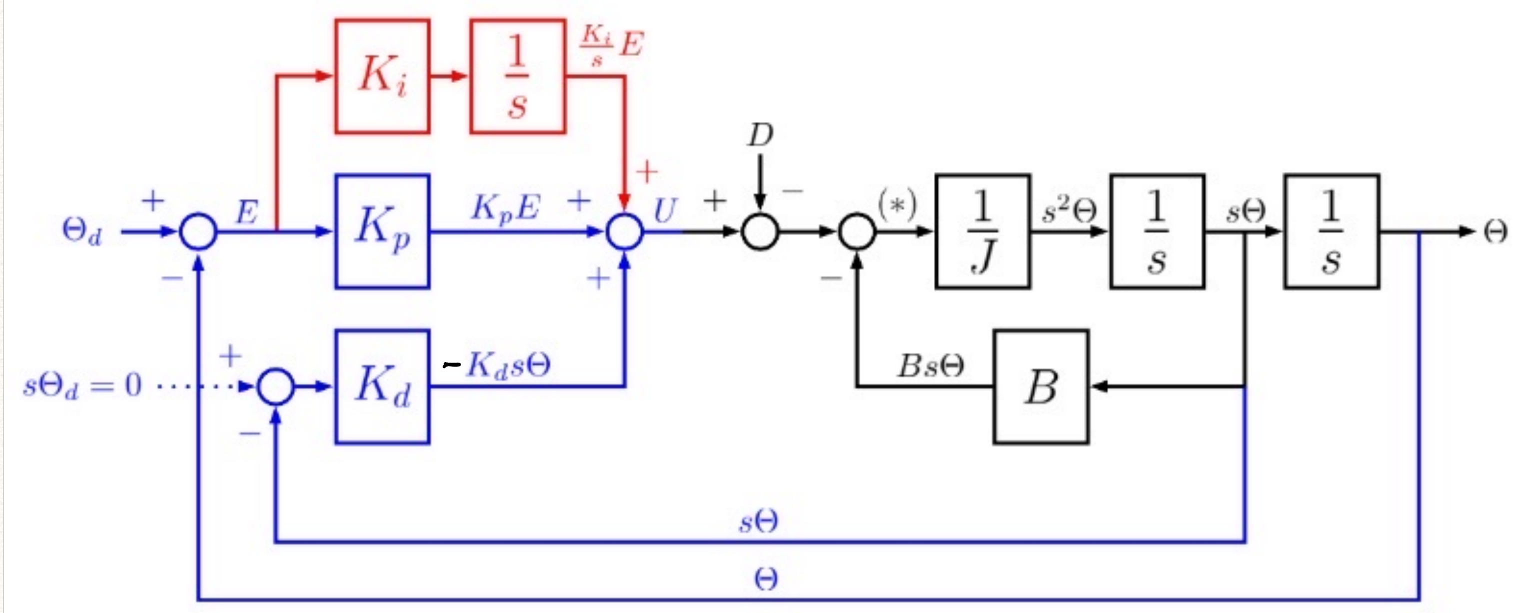
PID controller
- Builds on the PD controller by adding an integral term
- Integral term accumulates past errors over time
- $U(t) = K_p e(t) + K_d \dot{e}(t) + K_i \int e(t)dt$
- Laplace: $U(s) = (K_p + K_d s + \frac{K_i}{s})E(s)$

$e = \Theta_d - \Theta$

$\dot{e}(t) = \dot{\Theta}_d - \dot{\Theta}$

- Block diagram:

$-K_d s\Theta$ $\quad$ $\dot{e}(s) = s\Theta_d - s\Theta$



- Good choice of K_i gives the necessary push to prevent early stopping due to the derivative term, eliminating the error
- If K_i is too big, expect an overshoot, oscillations and instability

Session 2:

- Transfer functions (recap)
- Estimate controller constants
- Steady state error

Transfer functions:

- Solve for θ

P - controller:

$u(s) = K_p E(s) = K_p(\theta_d - \theta)$ controller

$u(s) = Js^2\theta + Bs\theta + D$ dynamic system

$\downarrow$

$Js^2\theta + Bs\theta + D = K_p\theta_d - K_p\theta$

$\theta(Js^2 + Bs + K_p) = K_p\theta_d - D$

$\theta = \dfrac{K_p\theta_d - D}{Js^2 + Bs + K_p}$ ~ Transfer function

Estimating controller constants

- Determined by the characteristic polynomial from the transfer function.

$\Omega(s) = Js^2 + Bs + K_p$ (P-controller)

- General damped second order system:

$s^2 + 2\zeta\omega s + \omega^2 = 0$

ζ: Damping ratio

- $\zeta = 1 \rightarrow$ critically damped
  - Customary in robotics
  - Produces the fastest non-oscillatory response.

ω: Closed loop natural frequency

$$\Omega(s) = 0 \rightarrow Js^2 + Bs + K_p = 0$$

$$s^2 + \frac{B}{J}s + \frac{K_p}{J} = 0 \qquad \text{(second order sys)}$$

Matching with the general system gives

$$\frac{K_p}{J} = \omega^2 \rightarrow \boxed{K_p = J\omega^2}$$

Steady state error:

- Final value theorem
  - $\lim_{t \to \infty} f(t) = \lim_{s \to 0} s F(s)$
  - Relates the frequency domain to the time domain as time approaches infinity
- Steady state error is the error when $t = \infty$

$$e_{ss} = \lim_{s \to 0} sE(s)$$

$$\Theta = \frac{K_p \Theta_d - D}{Js^2 + Bs + K_p} \quad , \quad \Omega = Js^2 + Bs + K_p$$

$$E(s) = \Theta_d - \frac{K_p \Theta_d - D}{\Omega}$$

$$= \Theta_d - \frac{K_p \Theta_d}{\Omega} + \frac{D}{\Omega}$$

$$= \frac{\Theta_d (Js^2 + Bs) + K_p \Theta_d - K_p \Theta_d}{\Omega} + \frac{D}{\Omega}$$

$$= \frac{\Theta_d (Js^2 + Bs)}{\Omega} + \frac{D}{\Omega}$$

step reference input + constant D:

$$\Theta_d(s) = \frac{\Omega^d}{s} \quad , \quad D(s) = \frac{D}{s}$$

$$\lim_{s \to 0} sE(s) = \frac{s \cdot \frac{\Omega^d}{s} (Js^2 + Bs)}{\Omega} + \frac{s \cdot \frac{D}{s}}{\Omega}$$

$$= \frac{\Omega^d (Js^2 + Bs)}{\Omega} + \frac{D}{\Omega}$$

$$s = 0 \rightarrow \underbrace{\frac{\Omega^d (J\overset{0}{\overset{\shortparallel}{s^2}} + B\overset{0}{\overset{\shortparallel}{s}})}{\Omega}}_{0} + \frac{D}{\underset{0}{\underset{\shortparallel}{Js^2}} + \underset{0}{\underset{\shortparallel}{Bs}} + K_p}$$

$$e_{ss} = \frac{D}{K_p}$$