

Group Session 26.03.2021

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Plan for today:

- Brief description of assignment III
- Short explanation of Dynamics
- Finding the **lagrangian** term this week, Euler-Lagrange will arrive next group session.
- Lagrangian with the spherical robot and 2-link planar arm

Mandatory Assignment 3

- Deadline: 19.04.2021, **3-weeks**
- Probably the most difficult assignment(Start early and do proper attempt! Trust me, you do not «**want second attempt**» on this assignment especially if you have other difficult courses.)
- Walkthrough of the mandatory assignment III
- Expressions might be long, use Python or Matlab to solve the equations to verify your calculated expressions.

Mandatory Assignment 3 cont.

- Try to do «**Task I**» by hand for 2-DOF robot, very relevant for exams as you will be finding the dynamic equations for 2-DOF

Strongly advise using programming to verify your answers.

And

- Try to do «**Task II**» by programming as you are finding the dynamic equations for 3-DOF. (I **STRONGLY DISCOURAGE** solving task II by hand, you will hate yourself beyond recognition)

Dynamics

From forward-kinematics, invers-kinematics and jacobian we are describing the motion of the robot without the **forces** that produces the motion.

- How much force do each of the respective joints need in order to move?
- How much force is needed to keep the respective joints in place?

For this, we need to first

- use «**Virtual displacements subject to holonomic constraints**»,
- use «**Principle of virtual work**»,
- use «**D’alemberts Principle to derive Euler-Lagrange Equations of motion**»

Euler-Lagrange

$$\tau_k = \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}$$

Modeling the robot for generalized coordinates

The Dynamic motion of the robotic manipulator

$$\tau = D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q)$$

Dynamics cont.

The Dynamic motion of the robotic manipulator

$$\tau = D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q)$$

Example: Given 2 DOF-robotic manipulator and 3-DOF manipulator

2-DOF

$$\tau = \underbrace{\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}}_{D(q)} \ddot{q} + \underbrace{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}}_{C(q, \dot{q})} \dot{q} + \underbrace{\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}}_{g(q)}$$

$D(q) \rightarrow$ Inertia
 $C(q, \dot{q}) \rightarrow$ Centrifugal and Coriolis
 $g(q) \rightarrow$ Gravity

3-DOF

$$\tau = \underbrace{\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}}_{D(q)} \ddot{q} + \underbrace{\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}}_{C(q, \dot{q})} \dot{q} + \underbrace{\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}}_{g(q)}$$

Dynamics cont.

- Very useful for Controller-systems
- Simulation and animation of robot motion

$$\tau_k = \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}$$

$$\tau = D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q)$$

What is \dot{q} and \ddot{q}

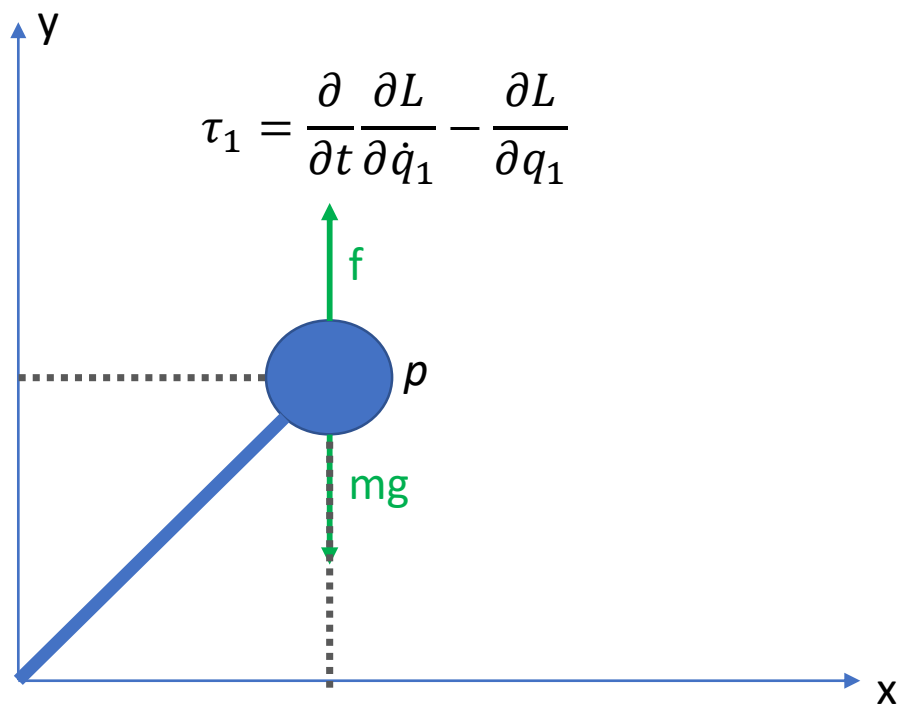
\dot{q} is velocity of the joint variable

\ddot{q} is acceleration of the joint variable

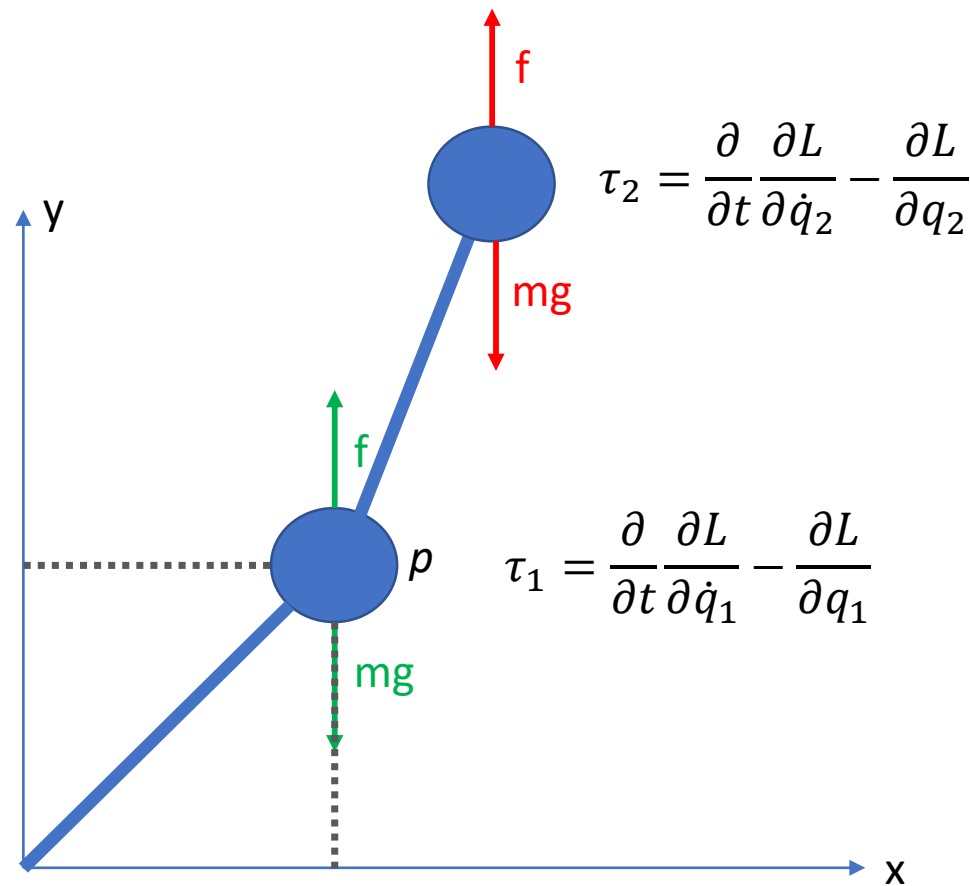
Lagrangian

- Difference between Kinetic energy and Potential energy
- With Kinetic and Potential energy, we can define a force-term known as torque using Euler-Lagrange.

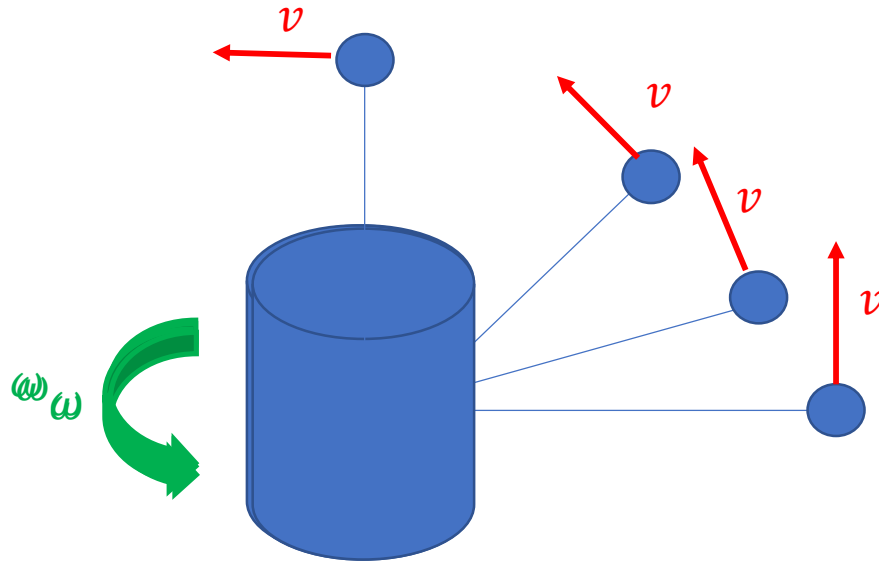
1-DOF Manipulator



2-DOF Manipulator

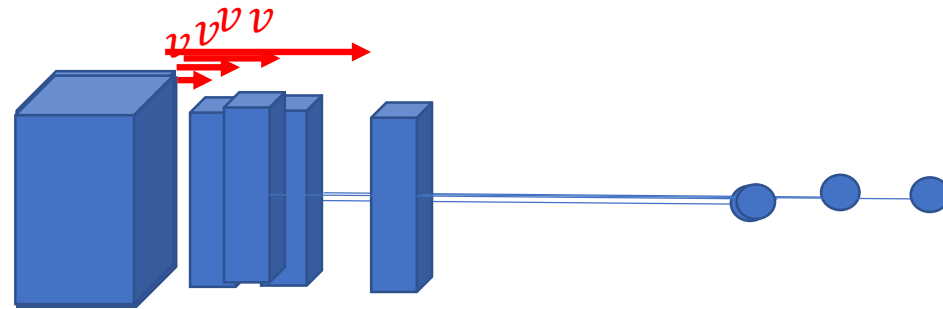


Revolute Joints



Angular, ω
-and
linear movement, v

Prismatic Joints



Only linear movement, v

What do we need to find the Lagrangian?

Kinetic Energy

$$K = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T I \omega$$

For one-link

$$K = \frac{1}{2} \dot{q}^T \left[\sum_{i=1}^n m_i J_{vi}(q)^T J_{vi}(q) + J_{\omega i}(q)^T R_i(q) I_i R_i(q)^T J_{\omega i}(q) \right] \dot{q}$$

For n-DOF links

Potential Energy

$$P = mgh$$

For one-link

$$P = \sum_{i=1}^n m_i g^T r_{ci}$$

For n-DOF links

$D(q)$

This is important for Euler-Lagrange, do not worry about this yet

Lagrangian

$$L = K - P$$

What do we need to find the Lagrangian?

Kinetic Energy:

$$v_i = J_v(q)\dot{q} \quad \text{Linear velocity}$$

$$\omega_i = J_\omega(q)\dot{q} \quad \text{Angular velocity}$$

$$I_i = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \quad \text{Inertia tensor}$$

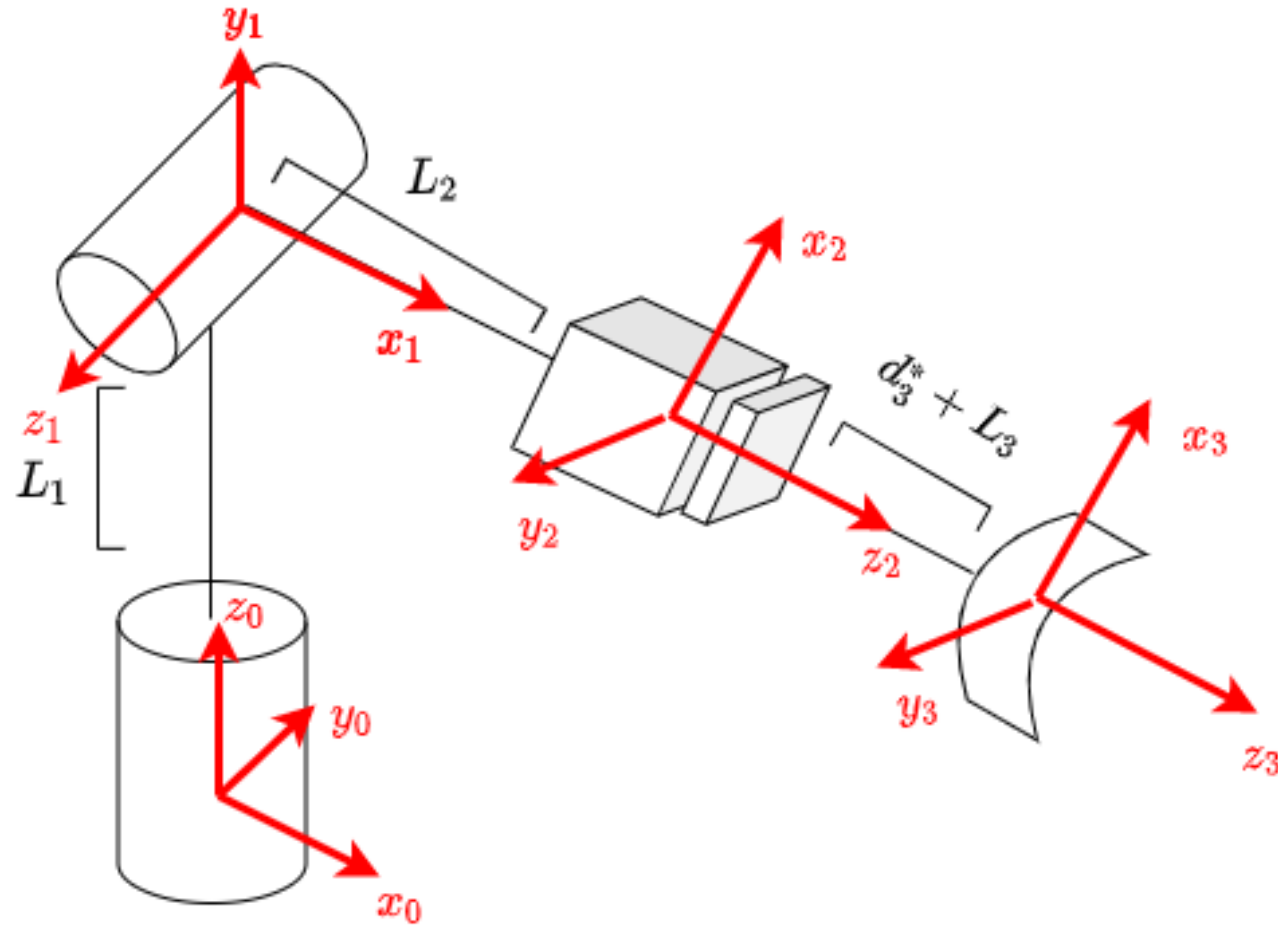
$$R_i = \begin{bmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{bmatrix} \quad \text{Rotation Matrices from Forward-kinematics}$$

Potential Energy:

- Assumption of mass-distribution for each link

$$h = r_{ci} \quad \text{height}$$

The spherical manipulator



Jacobian for the spherical manipulator

$$T_1^0 = A_1 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^0 = A_1 A_2 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_{2+\frac{\pi}{2}} & 0 & s_{2+\frac{\pi}{2}} & 0 \\ s_{2+\frac{\pi}{2}} & 0 & -c_{2+\frac{\pi}{2}} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -c_1 s_2 & s_1 & c_1 c_2 & 0 \\ -s_1 s_2 & -c_1 & c_2 s_1 & 0 \\ c_2 & 0 & s_2 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = A_1 A_2 A_3 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_{2+\frac{\pi}{2}} & 0 & s_{2+\frac{\pi}{2}} & 0 \\ s_{2+\frac{\pi}{2}} & 0 & -c_{2+\frac{\pi}{2}} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3^* + L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -c_1 s_2 & s_1 & c_1 c_2 & c_1 c_2 (d_3^* + L_2) \\ -s_1 s_2 & -c_1 & c_2 s_1 & c_2 s_1 (d_3^* + L_2) \\ c_2 & 0 & s_2 & L_1 + s_2 (d_3^* + L_2) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Helpful variables

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad z_1 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix} \quad z_2 = \begin{bmatrix} c_1 c_2 \\ c_2 s_1 \\ s_2 \end{bmatrix}$$

$$J_{v_i} = z_{i-1} \times (o_n - o_{i-1})$$

$$J_{v_i} = z_{i-1}$$

$$J_{\omega_i} = z_{i-1}$$

$$J_{\omega_i} = 0$$

$$(o_3 - o_0) = \begin{bmatrix} c_1 c_2 (d_3^* + L_2) \\ c_2 s_1 (d_3^* + L_2) \\ L_1 + s_2 (d_3^* + L_2) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 c_2 (d_3^* + L_2) \\ c_2 s_1 (d_3^* + L_2) \\ L_1 + s_2 (d_3^* + L_2) \end{bmatrix} \quad (o_3 - o_1) = \begin{bmatrix} c_1 c_2 (d_3^* + L_2) \\ c_2 s_1 (d_3^* + L_2) \\ L_1 + s_2 (d_3^* + L_2) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} = \begin{bmatrix} c_1 c_2 (d_3^* + L_2) \\ c_2 s_1 (d_3^* + L_2) \\ s_2 (d_3^* + L_2) \end{bmatrix}$$

$$J_{v_1} = z_{1-1} \times (o_3 - o_{1-1}) = z_0 \times (o_3 - o_0) = \begin{bmatrix} -c_2 s_1 (d_3^* + L_2) \\ c_1 c_2 (d_3^* + L_2) \\ 0 \end{bmatrix}$$

$$J_{v_2} = z_{2-1} \times (o_3 - o_{2-1}) = z_1 \times (o_3 - o_1) = \begin{bmatrix} -c_1 s_2 (d_3^* + L_2) \\ -s_1 s_2 (d_3^* + L_2) \\ c_2 (d_3^* + L_2) \end{bmatrix}$$

$$J_{v_3} = z_{3-1} = z_2 = \begin{bmatrix} c_1 c_2 \\ c_2 s_1 \\ s_2 \end{bmatrix}$$

$$J_{\omega_1} = z_{1-1} = z_0$$


$$J_{\omega_2} = z_{2-1} = z_1$$

$$J_{\omega_3} = 0$$

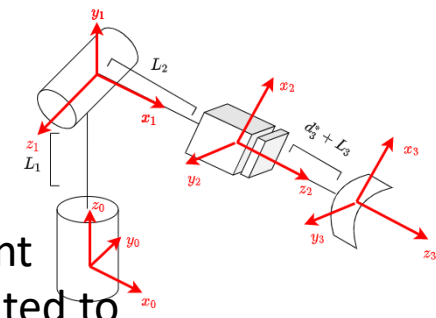
Jacobian

Putting all of it together yields the following Jacobian matrix

$$J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} = \begin{bmatrix} J_{v_1} & J_{v_2} & J_{v_3} \\ J_{\omega_1} & J_{\omega_2} & J_{\omega_3} \end{bmatrix} = \begin{bmatrix} -c_2 s_1 (d_3^* + L_2) & -c_2 s_1 (d_3^* + L_2) & c_1 c_2 \\ c_1 c_2 (d_3^* + L_2) & -s_1 s_2 (d_3^* + L_2) & c_2 s_1 \\ 0 & c_2 (d_3^* + L_2) & s_2 \\ 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

 J_v
 J_w

Finding K_1 for the spherical manipulator



- Setting d_3 and $L_2 = 0$, because they're related to third joint
- Setting second and third column = 0, because they're related to the second and third joint

$$J_v = \begin{bmatrix} -c_2 s_1 (d_3^* + L_2) & -c_2 s_1 (d_3^* + L_2) & c_1 c_2 \\ c_1 c_2 (d_3^* + L_2) & -s_1 s_2 (d_3^* + L_2) & c_2 s_1 \\ 0 & c_2 (d_3^* + L_2) & s_2 \end{bmatrix}$$

$$\longrightarrow J_{v_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$J_\omega = \begin{bmatrix} 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow J_{\omega_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{d}_3 \end{bmatrix}$$

$$I_1 = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} = 0$$

$$v_1 = J_{v_1}(q)\dot{q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{d}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\omega_1 = J_{\omega_1}(q)\dot{q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{d}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix}$$

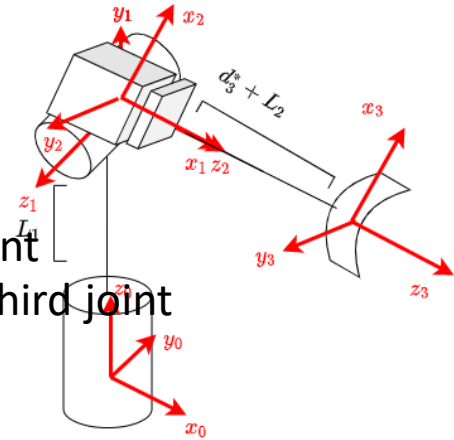
$$K_1 = \frac{1}{2} m_1 v_1^T v_1 + \frac{1}{2} \omega_1^T I_1 \omega_1$$

Plotting all of it together and receive the following

$$\begin{aligned} K_1 &= \frac{1}{2} m_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix}^T \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix} \\ &= 0 \end{aligned}$$

Finding K_2 for the spherical manipulator

- Setting d_3 and $L_2 = 0$, because they're related to third joint
- Setting third column = 0, because they're related to the third joint



$$J_v = \begin{bmatrix} -c_2 s_1 (d_3^* + L_2) & -c_2 s_1 (d_3^* + L_2) & c_1 c_2 \\ c_1 c_2 (d_3^* + L_2) & -s_1 s_2 (d_3^* + L_2) & c_2 s_1 \\ 0 & c_2 (d_3^* + L_2) & s_2 \end{bmatrix} \longrightarrow J_{v_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$J_\omega = \begin{bmatrix} 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \longrightarrow J_{\omega_2} = \begin{bmatrix} 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$I_2 = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} = 0$$

$$\dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{d}_3 \end{bmatrix}$$

$$v_2 = J_{v_2}(q) \dot{q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{d}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

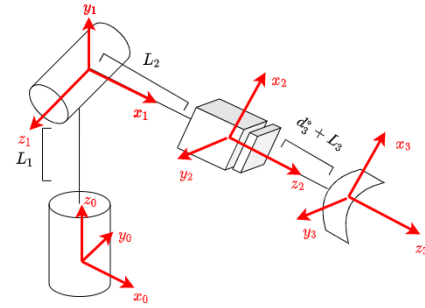
$$\omega_2 = J_{\omega_2}(q) \dot{q} = \begin{bmatrix} 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{d}_3 \end{bmatrix} = \begin{bmatrix} \dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix}$$

$$K_2 = \frac{1}{2} m_2 v_2^T v_2 + \frac{1}{2} \omega_2^T I_2 \omega_2$$

Plotting all of it together and receive the following

$$\begin{aligned} K_2 &= \frac{1}{2} m_2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix}^T \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix} \\ &= 0 \end{aligned}$$

Finding K_3 for the spherical manipulator



$$J_v = \begin{bmatrix} -c_2 s_1 (d_3^* + L_2) & -c_2 s_1 (d_3^* + L_2) & c_1 c_2 \\ c_1 c_2 (d_3^* + L_2) & -s_1 s_2 (d_3^* + L_2) & c_2 s_1 \\ 0 & c_2 (d_3^* + L_2) & s_2 \end{bmatrix} \xrightarrow{J_{v_3}} \begin{bmatrix} -c_2 s_1 (d_3^* + L_2) & -c_2 s_1 (d_3^* + L_2) & c_1 c_2 \\ c_1 c_2 (d_3^* + L_2) & -s_1 s_2 (d_3^* + L_2) & c_2 s_1 \\ 0 & c_2 (d_3^* + L_2) & s_2 \end{bmatrix}$$

$$J_\omega = \begin{bmatrix} 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{\quad} J_{\omega_3} = \begin{bmatrix} 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{d}_3 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} = 0$$

$$v_3 = J_{v_3}(q)\dot{q} = \begin{bmatrix} -c_2 s_1 (d_3^* + L_2) & -c_2 s_1 (d_3^* + L_2) & c_1 c_2 \\ c_1 c_2 (d_3^* + L_2) & -s_1 s_2 (d_3^* + L_2) & c_2 s_1 \\ 0 & c_2 (d_3^* + L_2) & s_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{d}_3 \end{bmatrix} = \begin{bmatrix} \dot{d}_3 c_1 c_2 - \dot{q}_1 c_2 s_1 (d_3^* + L_2) - \dot{q}_2 c_1 s_2 (d_3^* + L_2) \\ \dot{d}_3 s_1 c_2 + \dot{q}_1 c_2 c_1 (d_3^* + L_2) - \dot{q}_2 s_1 s_2 (d_3^* + L_2) \\ \dot{d}_3 s_2 + \dot{q}_2 c_2 (d_3^* + L_2) \end{bmatrix}$$

$$\omega_3 = J_{\omega_3}(q)\dot{q} = \begin{bmatrix} 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{d}_3 \end{bmatrix} = \begin{bmatrix} \dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix}$$

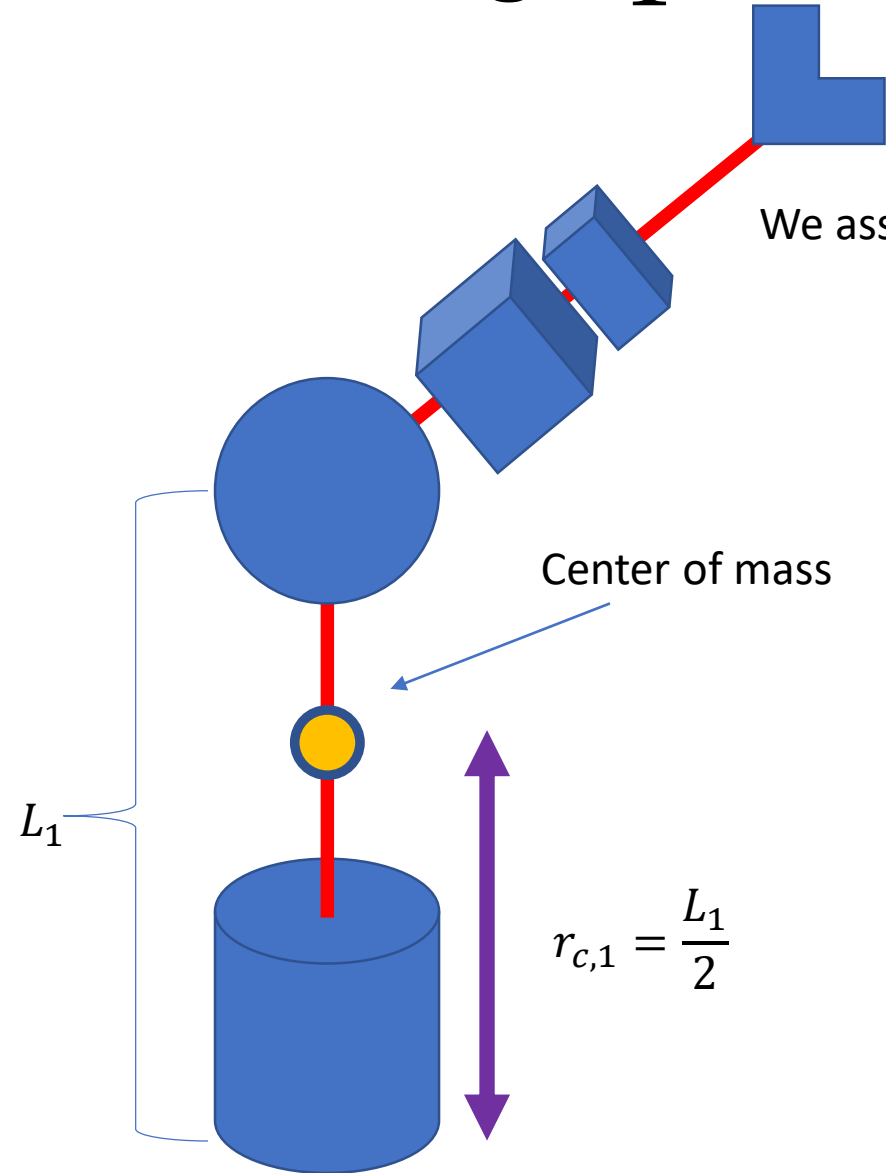
$$K_3 = \frac{1}{2} m_3 v_3^T v_3 + \frac{1}{2} \omega_3^T I_3 \omega_3$$

Plotting all of it together and receive the following

$$K_3 = \frac{1}{2} m_3 \begin{bmatrix} \dot{d}_3 c_1 c_2 - \dot{q}_1 c_2 s_1 (d_3^* + L_2) - \dot{q}_2 c_1 s_2 (d_3^* + L_2) \\ \dot{d}_3 s_1 c_2 + \dot{q}_1 c_2 c_1 (d_3^* + L_2) - \dot{q}_2 s_1 s_2 (d_3^* + L_2) \\ \dot{d}_3 s_2 + \dot{q}_2 c_2 (d_3^* + L_2) \end{bmatrix}^T \begin{bmatrix} \dot{d}_3 c_1 c_2 - \dot{q}_1 c_2 s_1 (d_3^* + L_2) - \dot{q}_2 c_1 s_2 (d_3^* + L_2) \\ \dot{d}_3 s_1 c_2 + \dot{q}_1 c_2 c_1 (d_3^* + L_2) - \dot{q}_2 s_1 s_2 (d_3^* + L_2) \\ \dot{d}_3 s_2 + \dot{q}_2 c_2 (d_3^* + L_2) \end{bmatrix} \\ + \frac{1}{2} \begin{bmatrix} \dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix}^T \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix}$$

$$= \frac{m_3 ((\dot{d}_3 s_2 + \dot{q}_2 c_2 (d_3^* + L_2))^2 + \dot{q}_1 c_2 s_1 (d_3^* + L_2) - \dot{d}_3 c_1 c_2 + (\dot{q}_2 c_1 s_2 (d_3^* + L_2))^2 + \dot{d}_3 s_1 c_2 + \dot{q}_1 c_2 c_1 (d_3^* + L_2) - \dot{q}_2 s_1 s_2 (d_3^* + L_2))}{2}$$

Finding P_1 for the spherical manipulator

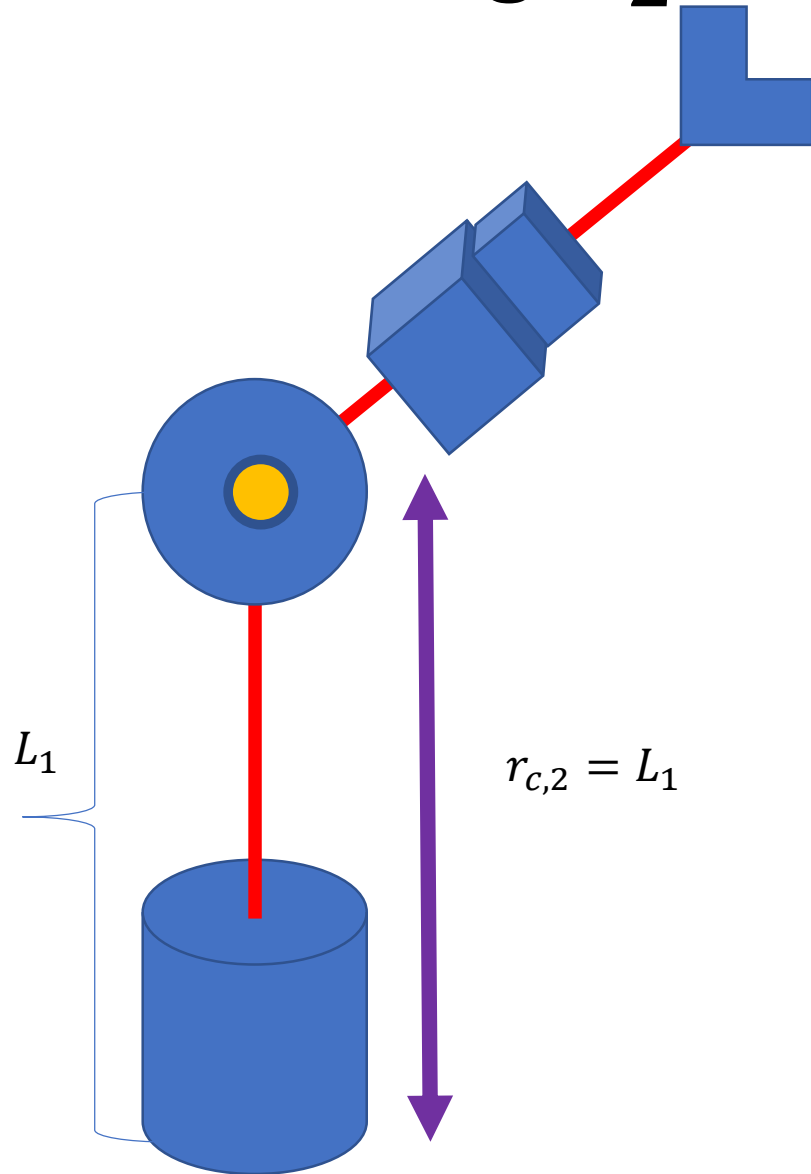


We assume the link is uniformly distributed, hence the mass center is centered

$$P_1 = m_1 g h_1 = m_1 g r_{c1}$$

$$P_1 = m_1 g h_1 = m_1 g \frac{L_1}{2}$$

Finding P_2 for the spherical manipulator

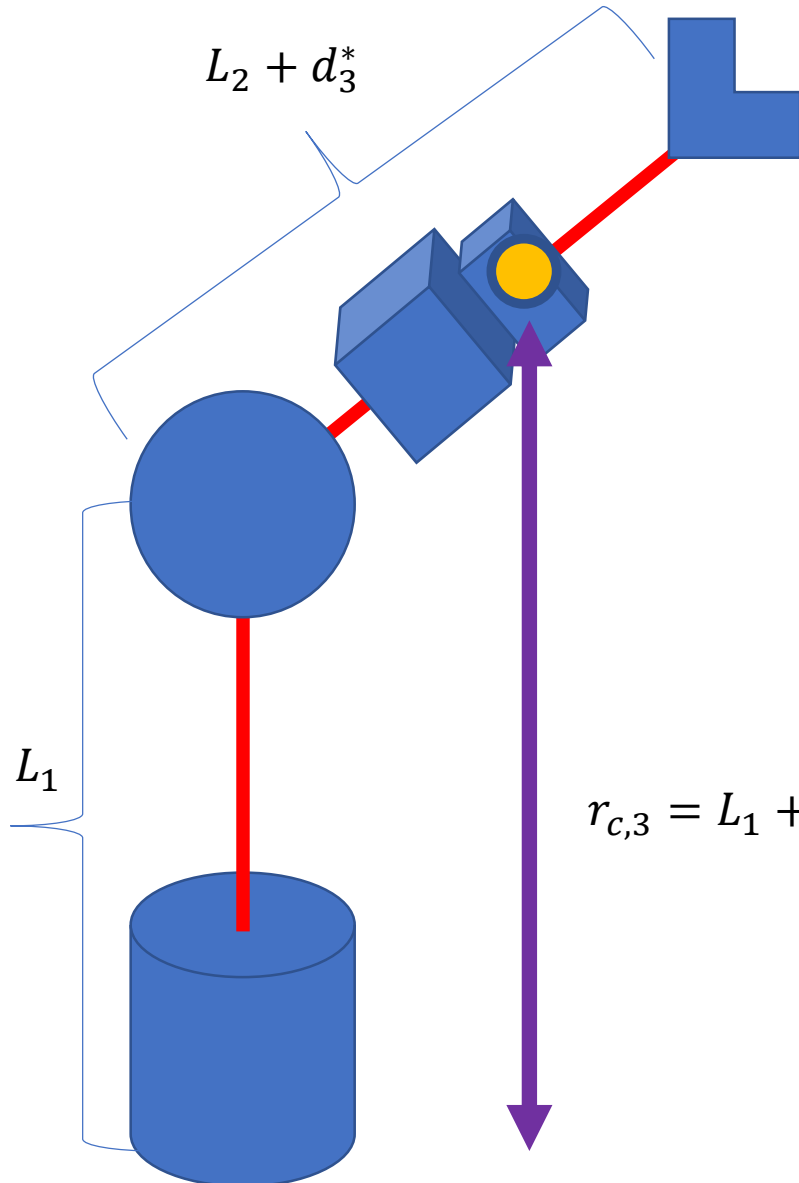


We assume the link is uniformly distributed, hence the mass center is centered

$$P_2 = m_2gh_2 = m_2gr_{c2}$$

$$P_2 = m_2gh_2 = m_2gL_1$$

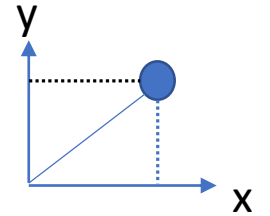
Finding P_3 for the spherical manipulator



We assume the link is uniformly distributed, hence the mass center is centered

$$P_3 = m_3gh_3 = m_3gr_{c3}$$

$$P_3 = m_3gh_3 = m_3gL_1 + \frac{s_2(L_2+d_3^*)}{2}$$



$$y = r * \sin(\theta)$$

$$r_{c,3} = L_1 + \sin(\theta_2) \frac{(L_2+d_3^*)}{2} = L_1 + \frac{s_2(L_2+d_3^*)}{2}$$

Lagrangian for the Spherical manipulator

$$K_1 = 0$$

$$K_2 = 0$$

$$K_3 = \frac{m_3((\dot{d}_3 s_2 + \dot{q}_2 c_2 (d_3^* + L_2))^2 + \dot{q}_1 c_2 s_1 (d_3^* + L_2) - \dot{d}_3 c_1 c_2 + (\dot{q}_2 c_1 s_2 (d_3^* + L_2))^2 + \dot{d}_3 s_1 c_2 + \dot{q}_1 c_2 c_1 (d_3^* + L_2) - \dot{q}_2 s_1 s_2 (d_3^* + L_2))}{2}$$

$$P_1 = m_1 g h_1 = m_1 g \frac{L_1}{2}$$

$$P_2 = m_2 g h_2 = m_2 g L_1$$

$$P_3 = m_3 g h_3 = m_3 g L_1 + \frac{s_2 (L_2 + d_3^*)}{2}$$

Lagrangian term

$$L = K - P$$

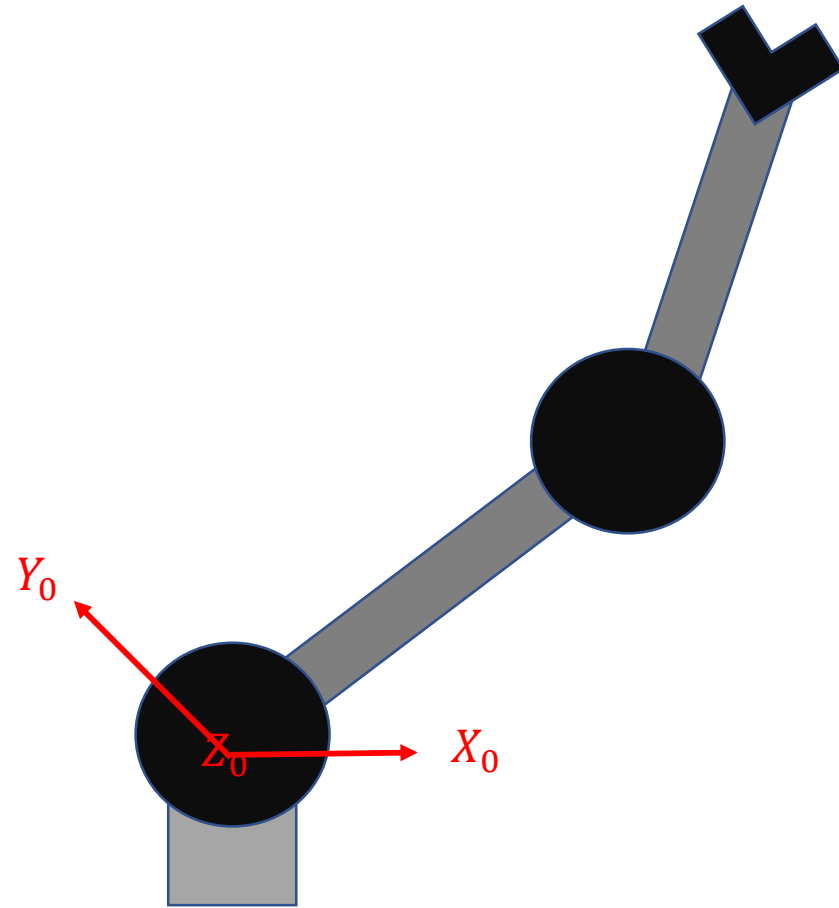
$$L = (K_1 + K_2 + K_3) - (P_1 + P_2 + P_3)$$

L

$$= \frac{m_3((\dot{d}_3 s_2 + \dot{q}_2 c_2 (d_3^* + L_2))^2 + \dot{q}_1 c_2 s_1 (d_3^* + L_2) - \dot{d}_3 c_1 c_2 + (\dot{q}_2 c_1 s_2 (d_3^* + L_2))^2 + \dot{d}_3 s_1 c_2 + \dot{q}_1 c_2 c_1 (d_3^* + L_2) - \dot{q}_2 s_1 s_2 (d_3^* + L_2))}{2} - \left(m_1 g \frac{L_1}{2} + m_2 g L_1 + m_3 g L_1 + \frac{s_2 (L_2 + d_3^*)}{2} \right)$$

5 minutes break!

2-Link Planar Arm



Forward-kinematics for the 2-Link planar Robot

DH-Table

Rot_{z,θ_i}	$Trans_{z,d_i}$	$Trans_{x,a_i}$	Rot_{x,α_i}
θ_1^*	0	L_1	0
θ_2^*	0	L_2	0

Forward-kinematics

$$T_1^0 = A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & L_1 c_1 \\ s_1 & c_1 & 0 & L_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_i = Rot_{z,\theta_i} Trans_{z,d_i} Trans_{x,a_i} Rot_{x,\alpha_i}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & L_1 c_1 \\ s_1 & c_1 & 0 & L_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_2^0 = A_1 A_2 = \begin{bmatrix} c_1 & -s_1 & 0 & L_1 c_1 \\ s_1 & c_1 & 0 & L_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 & 0 & L_2 c_2 \\ s_2 & c_2 & 0 & L_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{12} & -s_{12} & 0 & L_2 c_{12} + L_1 c_1 \\ s_{12} & c_{12} & 0 & L_2 s_{12} + L_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & L_2 c_2 \\ s_2 & c_2 & 0 & L_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Jacobian for the 2-Link planar Robot

$$T_1^0 = \begin{bmatrix} c_1 & -s_1 & 0 & L_1 c_1 \\ s_1 & c_1 & 0 & L_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^0 = \begin{bmatrix} c_1 & -s_1 & 0 & L_1 c_1 \\ s_1 & c_1 & 0 & L_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 & 0 & L_2 c_2 \\ s_2 & c_2 & 0 & L_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{12} & -s_{12} & 0 & L_2 c_{12} + L_1 c_1 \\ s_{12} & c_{12} & 0 & L_2 s_{12} + L_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$z_0 = [0 \ 0 \ 1] \quad O_0 = [0 \ 0 \ 0]$$

$$z_1 = [0 \ 0 \ 1] \quad O_1 = [L_1 c_1; \ L_1 s_1; \ 0;]$$

$$z_2 = [0 \ 0 \ 1] \quad O_2 = [L_2 c_{12} + L_1 c_1; \ L_2 s_{12} + L_1 s_1; \ 0;]$$

Jacobian for the 2-Link planar Robot cont.

The following robotic manipulator is (RR) so this gives us the following jacobian matrix:

$$J = \begin{bmatrix} J_{v1} & J_{v2} \\ J_{\omega1} & J_{\omega2} \end{bmatrix} = \begin{bmatrix} z_0 \times (O_2 - O_0) & z_1 \times (O_2 - O_1) \\ z_0 & z_1 \end{bmatrix}$$

$$J_{v1} = z_0 \times (O_2 - O_0) = [0 \ 0 \ 1] \times ([L_2 c_{12} + L_1 c_1; \ L_2 s_{12} + L_1 s_1; \ 0;] - [0 \ 0 \ 0])$$

$$J_{v2} = z_1 \times (O_2 - O_1) = [0 \ 0 \ 1] \times ([L_2 c_{12} + L_1 c_1; \ L_2 s_{12} + L_1 s_1; \ 0;] - [L_1 c_1; \ L_1 s_1; \ 0;])$$

$$J_{\omega1} = z_0 = [0 \ 0 \ 1]$$

$$J_{\omega2} = z_1 = [0 \ 0 \ 1]$$

Jacobian for the 2-Link planar Robot cont.

$$J = \begin{bmatrix} J_{v1} & J_{v2} \\ J_{\omega1} & J_{\omega2} \end{bmatrix} = \begin{bmatrix} z_0 \times (O_2 - O_0) & z_1 \times (O_2 - O_1) \\ z_0 & z_1 \end{bmatrix} = \begin{bmatrix} -L_2 s_{12} - L_1 s_1 & -L_2 s_{12} \\ L_2 c_{12} + L_1 s_1 & L_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Finding K_1 for 2-Link planar manipulator

Setting L_2 to 0 because they're related to joint 2

Setting second column to zero because they're related to joint 2

$$\begin{bmatrix} -L_2 s_{12} - L_1 s_1 & -L_2 s_{12} \\ L_2 c_{12} + L_1 c_1 & L_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$



$$\begin{bmatrix} -L_1 s_1 & 0 \\ L_1 c_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

$$J_{v_1} = \begin{bmatrix} -L_1 s_1 & 0 \\ L_1 c_1 & 0 \\ 0 & 0 \end{bmatrix} \quad J_{\omega_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$v_1 = J_{v_1}(q)\dot{q} = \begin{bmatrix} -L_1 s_1 & 0 \\ L_1 c_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -\dot{q}_1 L_1 s_1 \\ \dot{q}_1 L_1 c_1 \\ 0 \end{bmatrix}$$

$$I_1 = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} = 0$$

$$\omega_1 = J_{\omega_1}(q)\dot{q} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix}$$

$$K_1 = \frac{1}{2} m_1 v_1^T v_1 + \frac{1}{2} \omega_1^T I_1 \omega_1$$

Plotting all of it together and receive the following

$$K_1 = \frac{1}{2} m_1 \begin{bmatrix} -\dot{q}_1 L_1 s_1 \\ \dot{q}_1 L_1 c_1 \\ 0 \end{bmatrix}^T \begin{bmatrix} -\dot{q}_1 L_1 s_1 \\ \dot{q}_1 L_1 c_1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix}^T \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix}$$

$$K_1 = \frac{L_1^2 m_1 \dot{q}_1^2}{2}$$

Finding K_2 for 2-Link planar manipulator

$$\begin{bmatrix} -L_2 s_{12} - L_1 s_1 & -L_2 s_{12} \\ L_2 c_{12} + L_1 s_1 & L_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \xrightarrow{\text{Using the whole jacobian}} \begin{bmatrix} -L_2 s_{12} - L_1 s_1 & -L_2 s_{12} \\ L_2 c_{12} + L_1 s_1 & L_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$J_{v_2} = \begin{bmatrix} -L_2 s_{12} - L_1 s_1 & -L_2 s_{12} \\ L_2 c_{12} + L_1 s_1 & L_2 c_{12} \\ 0 & 0 \end{bmatrix}$$

$$J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

$$v_2 = J_{v_2}(q)\dot{q} = \begin{bmatrix} -L_2 s_{12} - L_1 s_1 & -L_2 s_{12} \\ L_2 c_{12} + L_1 s_1 & L_2 c_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} \dot{q}_1(-L_2 s_{12} - L_1 s_1) - \dot{q}_2 L_2 s_{12} \\ \dot{q}_1(L_2 c_{12} + L_1 s_1) + \dot{q}_2 L_2 c_{12} \\ 0 \end{bmatrix}$$

$$\omega_2 = J_{\omega_2}(q)\dot{q} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{bmatrix}$$

$$I_2 = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} = 0$$

$$K_2 = \frac{1}{2} m_2 v_2^T v_2 + \frac{1}{2} \omega_2^T I_2 \omega_2$$

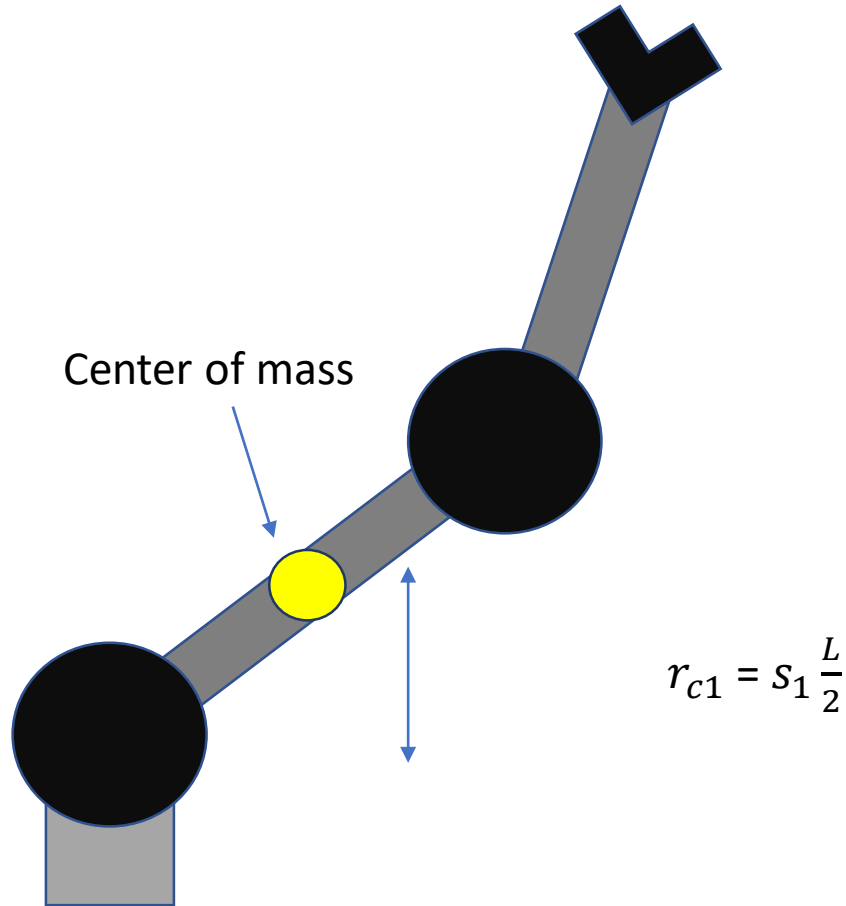
Plotting all of it together and receive the following

$$K_2 = \frac{1}{2} m_2 \begin{bmatrix} \dot{q}_1(-L_2 s_{12} - L_1 s_1) - \dot{q}_2 L_2 s_{12} \\ \dot{q}_1(L_2 c_{12} + L_1 s_1) + \dot{q}_2 L_2 c_{12} \\ 0 \end{bmatrix}^T \begin{bmatrix} \dot{q}_1(-L_2 s_{12} - L_1 s_1) - \dot{q}_2 L_2 s_{12} \\ \dot{q}_1(L_2 c_{12} + L_1 s_1) + \dot{q}_2 L_2 c_{12} \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{bmatrix}^T \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{bmatrix}$$

$$K_2 = \frac{m_2(L_1^2 \dot{q}_1^2 + 2c_2 L_1 L_2 \dot{q}_1^2 + 2c_2 L_1 L_2 \dot{q}_1 \dot{q}_2 + L_2^2 \dot{q}_1^2 + 2L_2^2 \dot{q}_1 \dot{q}_2 + L_2^2 \dot{q}_2^2)}{2}$$

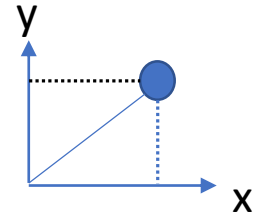
Finding P_1 for 2-link planar manipulator

We assume the link is uniformly distributed, hence the mass center is centered



$$P_1 = m_1 g h_1 = m_1 g r_{c1}$$

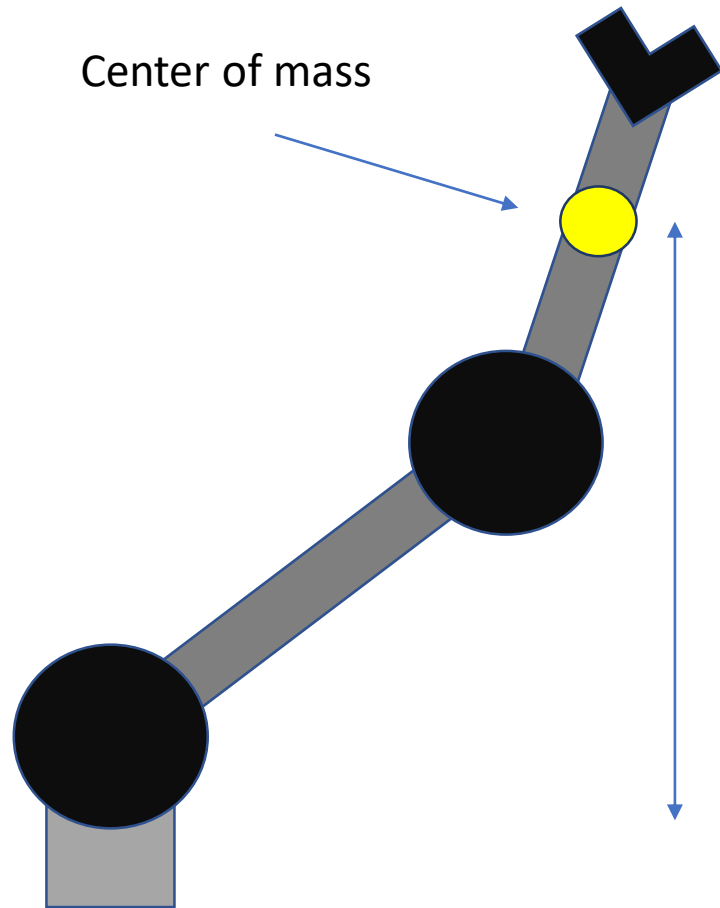
$$P_1 = m_1 g h_1 = m_1 g s_1 \frac{L}{2}$$



$$y = r * \sin(x)$$

Finding P_2 for 2-link planar manipulator

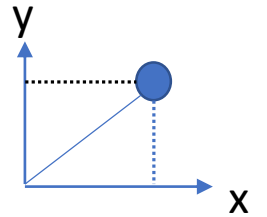
We assume the link is uniformly distributed, hence the mass center is centered



$$P_2 = m_2gh_2 = m_2gr_{c2}$$

$$P_2 = m_2gh_2 = m_2g(s_1L_1 + s_{12}\frac{L_2}{2})$$

$$r_{c,2} = s_1L_1 + s_{12}\frac{L_2}{2}$$



$$y = r * \sin(x)$$

Lagrangian for the 2-link Planar robot

$$K_1 = \frac{L_1^2 m_1 \dot{q}_1^2}{2}$$

$$K_2 = \frac{m_2(L_1^2 \dot{q}_1^2 + 2c_2 L_1 L_2 \dot{q}_1^2 + 2c_2 L_1 L_2 \dot{q}_1 \dot{q}_2 + L_2^2 \dot{q}_1^2 + 2L_2^2 \dot{q}_1 \dot{q}_2 + L_2^2 \dot{q}_2^2)}{2}$$

$$P_1 = m_1 g h_1 = m_1 g s_1 \frac{L_1}{2}$$

$$P_2 = m_2 g h_2 = m_2 g (s_1 L_1 + s_{12} \frac{L_2}{2})$$

Lagrangian

$$L = K - P$$

$$L = (K_1 + K_2) - (P_1 + P_2)$$

$$L = \left(\frac{L_1^2 m_1 \dot{q}_1^2}{2} + \frac{m_2(L_1^2 \dot{q}_1^2 + 2c_2 L_1 L_2 \dot{q}_1^2 + 2c_2 L_1 L_2 \dot{q}_1 \dot{q}_2 + L_2^2 \dot{q}_1^2 + 2L_2^2 \dot{q}_1 \dot{q}_2 + L_2^2 \dot{q}_2^2)}{2} \right) - \left(m_1 g s_1 \frac{L_1}{2} + m_2 g (s_1 L_1 + s_{12} \frac{L_2}{2}) \right)$$

Thats it for now!

Next group session, we will focus on deriving the Euler-Lagrange with our Lagrangian term!

$$\tau_k = \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}$$

$$\tau = D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q)$$

We will focus especially on time-differentiation

(Tidsderivasjon med hensyn på q-variablene)

$$\tau_k = \boxed{\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}_k}} - \frac{\partial L}{\partial q_k}$$

The jacobian way – 2 DOF planar robot

- It is also possible to find the relevant, J_v , velocities for the different kinetic energy components by finding the jacobian for J_1^0 and J_2^0 and then extracting the $J_{v,1}$ and $J_{v,2}$ for the respective K_1 and K_2
- Doing this approach may pose problem for shapes in matrices when programming as the shapes vary slightly in size as we increase n-DOF.

Inertia tensor

- Describes the mass density of a joint.
- how much force and energy is needed to rotate it.