Group Session 09.04.2021

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Plans for today

- Time differentation
- Euler-Lagrange
- Dynamic Model of planar robot arm

Recap!

Next group session, we will focus on deriving the Euler-Lagrange with our Lagrangian term!

$$\tau_k = \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}$$

Before easter we found the lagrangian term for planar manipulator and spherical manipulator

$$\tau = D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q)$$

We will focus especially on time-differentation

(Tidsderivasjon med hensyn på q-variablene)

$$\tau_k = \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}$$

Tips and tricks for time-differentiation

- Divide between constants and generalized coordinates

- Generalized coordinates are $q_i, \dot{q}_i, \ddot{q}_i$

$$\frac{\partial}{\partial t} q = \dot{q}$$

$$\frac{\partial}{\partial t} \dot{q} = \ddot{q}$$

When differentiating wrt time, we often use these two following mathematical property:

- Product-rule(produktregelen)

- Chain-rule (Kjerneregelen)

Examples:

$f(g(x))$ $f(x)$

$m = mass$

$L_1 = Link\ length$

$\dot{q}_1 = Velocity\ for\ joint\ 1$

$\ddot{q}_1 = acceleration\ for\ joint\ 1$

Using product rule

$mc_1\dot{q}_1$

$u = m \qquad u' = 0$

$v = c_1\dot{q}_1 \qquad v' =$

Using chain-rule

$u = c_1 \qquad u' = -s_1\dot{q}_1$

$v = \dot{q}_1 \qquad v' = \ddot{q}_1$

yields

$u' * v + u * v' = -s_1\dot{q}_1\dot{q}_1 + c_1\ddot{q}_1 = -s_1\dot{q}_1^2 + c_1\ddot{q}_1$

$$u' * v + u * v' = 0 + m(-s_1\dot{q}_1^2 + c_1\ddot{q}_1) = \underline{\underline{m(-s_1\dot{q}_1^2 + c_1\ddot{q}_1)}}$$

Using product rule

$mL_1\dot{q}_1$

$u = mL_1 \qquad u' = 0$

$v = \dot{q}_1 \qquad v' = \ddot{q}_1$

$$u' * v + u * v' = 0 + mL_1\ddot{q}_1 = \underline{\underline{mL_1\ddot{q}_1}}$$

NB! Multiplication is commutative so you can choose how you use the product rule, try it!

Lagrangian term for planar manipulator

$$L = \left(\frac{L_1^2 m_1 \dot{q}_1^2}{2} + \frac{m_2(L_1^2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_1^2 + 2L_2^2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_2^2)}{2}\right) - (m_1 g\, s_1 \frac{L_1}{2} + m_2 g(s_1 L_1 + s_{12}\frac{L_2}{2}))$$

Euler-lagrange for planar manipulator

$$\tau_1 = \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1}$$

$$\tau_1 = \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1}$$

$$L = \left(\frac{L_1^2 m_1 \dot{q}_1^2}{2} + \frac{m_2(L_1^2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_1^2 + 2L_2^2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_2^2)}{2}\right) - (m_1 g\, s_1\frac{L_1}{2} + m_2 g(s_1L_1 + s_{12}\frac{L_2}{2}))$$

$$\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial\left(\frac{L_1^2 m_1 \dot{q}_1^2}{2} + \frac{m_2(L_1^2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_1^2 + 2L_2^2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_2^2)}{2}\right) - (m_1 g\, s_1\frac{L_1}{2} + m_2 g(s_1L_1 + s_{12}\frac{L_2}{2}))}{\partial \dot{q}_1}$$

$$\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial\left(\frac{L_1^2 m_1 \dot{q}_1^2}{2} + \frac{m_2(L_1^2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_1^2 + 2L_2^2\dot{q}_1\dot{q}_2 + \cancel{L_2^2\dot{q}_2^2})}{2}\right) - \cancel{(m_1 g\, s_1\frac{L_1}{2} + m_2 g(s_1L_1 + s_{12}\frac{L_2}{2}))}}{\partial \dot{q}_1}$$

$$\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial\left(\frac{L_1^2 m_1 \dot{q}_1^2}{2} + \frac{m_2(L_1^2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_1^2 + 2L_2^2\dot{q}_1\dot{q}_2)}{2}\right)}{\partial \dot{q}_1}$$

Differentiation of $\frac{\partial L}{\partial \dot{q}_1}$ wrt $\dot{q}_1$

$$\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial\left(\frac{L_1^2 m_1 \dot{q}_1^2}{2} + \frac{m_2(L_1^2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_1^2 + 2L_2^2\dot{q}_1\dot{q}_2)}{2}\right)}{\partial \dot{q}_1}$$

Calculating $\frac{\partial L}{\partial \dot{q}_1}$

$$\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial\left(\frac{L_1^2 m_1 \dot{q}_1^2}{2} + \frac{m_2(L_1^2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_1^2 + 2L_2^2\dot{q}_1\dot{q}_2)}{2}\right)}{\partial \dot{q}_1}$$

$$\frac{\partial L}{\partial \dot{q}_1} = m_2L_1^2\dot{q}_1 + m_2L_2^2\dot{q}_1 + m_2L_2^2\dot{q}_2 + m_2 2L_1L_2\dot{q}_1c_2 + m_2L_1L_2\dot{q}_2c_2 + L_1^2m_1\dot{q}_1$$

Differentiation of $\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_1}$ wrt t

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial}{\partial t}(m_2 L_1^2 \dot{q}_1 + m_2 L_2^2 \dot{q}_1 + m_2 L_2^2 \dot{q}_2 + m_2 2 L_1 L_2 \dot{q}_1 c_2 + m_2 L_1 L_2 \dot{q}_2 c_2 + L_1^2 m_1 \dot{q}_1)$$

We can divide each term and solve with respect to time

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial}{\partial t}(\overbrace{m_2 L_1^2 \dot{q}_1}^{\text{I}} + \overbrace{m_2 L_2^2 \dot{q}_1}^{\text{II}} + \overbrace{m_2 L_2^2 \dot{q}_2}^{\text{III}} + \overbrace{m_2 2 L_1 L_2 \dot{q}_1 c_2}^{\text{IV}} + \overbrace{m_2 L_1 L_2 \dot{q}_2 c_2}^{\text{V}} + \overbrace{L_1^2 m_1 \dot{q}_1}^{\text{VI}})$$

Let us solve all of these terms individually and thorough

Differentiation of $\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_1}$ wrt t

I: $m_2 L_1^2 \dot{q}_1$

$u = m_2 L_1^2 \quad u' = 0$

$v = \dot{q}_1 \quad v' = \ddot{q}_1$

$u' * v + u * v' = 0 + m_2 L_1^2 \ddot{q}_1 = m_2 L_1^2 \ddot{q}_1$

II: $m_2 L_2^2 \dot{q}_1$

$u = m_2 L_2^2 \quad u' = 0$

$v = \dot{q}_1 \quad v' = \ddot{q}_1$

$u' * v + u * v' = 0 + m_2 L_2^2 \ddot{q}_1 = m_2 L_2^2 \ddot{q}_1$

III: $m_2 L_2^2 \dot{q}_2$

$u = m_2 L_1^2 \quad u' = 0$

$v = \dot{q}_2 \quad v' = \ddot{q}_2$

$u' * v + u * v' = 0 + m_2 L_1^2 \ddot{q}_2 = m_2 L_1^2 \ddot{q}_2$

IV: $m_2 2 L_1 L_2 \dot{q}_1 c_2$

$u = m_2 2 L_1 L_2 \quad u' = 0$

$v = \dot{q}_1 c_2 \quad v' =$

Using chain-rule

$u = c_2 \quad u' = -s_2 \dot{q}_2$

$v = \dot{q}_1 \quad v' = \ddot{q}_1$

$u' * v + u * v' = -s_2 \dot{q}_2 \dot{q}_1 + c_2 \ddot{q}_1$

$$u' * v + u * v' = 0 + m_2 2 L_1 L_2 (-s_2 \dot{q}_2 \dot{q}_1 + c_2 \ddot{q}_1)$$
$$= m_2 2 L_1 L_2 (-s_2 \dot{q}_2 \dot{q}_1 + c_2 \ddot{q}_1)$$

Differentiation of $\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_1}$ wrt t , cont.

V: $m_2 L_1 L_2 \dot{q}_2 c_2$

$u = m_2 L_1 L_2 \quad u' = 0$

$v = \dot{q}_2 c_2 \quad v' =$

Using chain-rule

$u = c_2 \quad u' = -s_2 \dot{q}_2$

$v = \dot{q}_2 \quad v' = \ddot{q}_2$

$u' * v + u * v' = -s_2 \dot{q}_2 \dot{q}_2 + c_2 \ddot{q}_1 = -s_2 \dot{q}_2^2 + c_2 \ddot{q}_2$

$u' * v + u * v' = 0 + m_2 L_1 L_2(-s_2 \dot{q}_2^2 + c_2 \ddot{q}_2) = m_2 L_1 L_2(-s_2 \dot{q}_2^2 + c_2 \ddot{q}_2)$

VI: $L_1^2 m_1 \dot{q}_1$

$u = L_1^2 m_1 \quad u' = 0$

$v = \dot{q}_1 \quad v' = \ddot{q}_1$

$u' * v + u * v' = 0 + L_1^2 m_1 \ddot{q}_1 = L_1^2 m_1 \ddot{q}_1$

Putting it all together for $\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_1}$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial}{\partial t}(m_2 L_1^2 \dot{q}_1 + m_2 L_2^2 \dot{q}_1 + m_2 L_2^2 \dot{q}_2 + m_2 2 L_1 L_2 \dot{q}_1 c_2 + m_2 L_1 L_2 \dot{q}_2 c_2 + L_1^2 m_1 \dot{q}_1)$$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_1} = m_2 L_1^2 \ddot{q}_1 + m_2 L_1^2 \ddot{q}_2 + m_2 L_2^2 \ddot{q}_1 + m_2 2 L_1 L_2(-s_2 \dot{q}_2 \dot{q}_1 + c_2 \ddot{q}_1) + m_2 L_1 L_2(-s_2 \dot{q}_2^2 + c_2 \ddot{q}_2) + L_1^2 m_1 \ddot{q}_1$$

Differentiation of $\frac{\partial L}{\partial q_1}$ wrt q_1

$$\frac{\partial L}{\partial q_1} = \frac{\partial\left(\frac{L_1^2 m_1 \dot{q}_1^2}{2} + \frac{m_2(L_1^2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_1^2 + 2L_2^2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_2^2)}{2}\right) - (m_1g\, s_1\frac{L_1}{2} + m_2g(s_1L_1 + s_{12}\frac{L_2}{2}))}{\partial q_1}$$

$$\frac{\partial L}{\partial q_1} = \frac{\partial\left(\frac{L_1^2 m_1 \dot{q}_1^2}{2} + \frac{m_2(L_1^2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_1^2 + 2L_2^2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_2^2)}{2}\right) - (m_1g\, s_1\frac{L_1}{2} + m_2g(s_1L_1 + s_{12}\frac{L_2}{2}))}{\partial q_1}$$

$$\frac{\partial L}{\partial q_1} = \frac{\partial(m_1g\, s_1\frac{L_1}{2} + m_2g(s_1L_1 + s_{12}\frac{L_2}{2}))}{\partial q_1}$$

Differentiation of $\frac{\partial L}{\partial q_1}$ wrt q_1, cont.

$$\frac{\partial L}{\partial q_1} = \frac{\partial(m_1 g\, s_1 \frac{L_1}{2} + m_2 g(s_1 L_1 + s_{12}\frac{L_2}{2}))}{\partial q_1}$$

$$\frac{\partial L}{\partial q_1} = m_2 g c_{12}\frac{L_2}{2} + m_2 g c_1 L_1 + m_1 g c_1 \frac{L_1}{2}$$

Calculating τ_1

$$\tau_1 = \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1}$$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_1} = m_2 L_1^2 \ddot{q}_1 + m_2 L_1^2 \ddot{q}_2 + m_2 L_2^2 \ddot{q}_1 + m_2 2 L_1 L_2(-s_2 \dot{q}_2 \dot{q}_1 + c_2 \ddot{q}_1) + m_2 L_1 L_2(-s_2 \dot{q}_2^2 + c_2 \ddot{q}_2) + L_1^2 m_1 \ddot{q}_1$$

$$\frac{\partial L}{\partial q_1} = -m_2 g c_{12}\frac{L_2}{2} - m_2 g c_1 L_1 - m_1 g c_1 \frac{L_1}{2}$$

$$\tau_1 = m_2 L_1^2 \ddot{q}_1 + m_2 L_1^2 \ddot{q}_2 + m_2 L_2^2 \ddot{q}_1 + m_2 2 L_1 L_2(-s_2 \dot{q}_2 \dot{q}_1 + c_2 \ddot{q}_1) + m_2 L_1 L_2(-s_2 \dot{q}_2^2 + c_2 \ddot{q}_2) + L_1^2 m_1 \ddot{q}_1 - (-m_2 g c_{12}\frac{L_2}{2} - m_2 g c_1 L_1 - m_1 g c_1 \frac{L_1}{2})$$

$$= m_2 L_1^2 \ddot{q}_1 + m_2 L_1^2 \ddot{q}_2 + m_2 L_2^2 \ddot{q}_1 + m_2 2 L_1 L_2(-s_2 \dot{q}_2 \dot{q}_1 + c_2 \ddot{q}_1) + m_2 L_1 L_2(-s_2 \dot{q}_2^2 + c_2 \ddot{q}_2) + L_1^2 m_1 \ddot{q}_1 + m_2 g c_{12}\frac{L_2}{2} + m_2 g c_1 L_1 + m_1 g c_1 \frac{L_1}{2})$$

Differentiation of $\frac{\partial L}{\partial \dot{q}_2}$ wrt $\dot{q}_2$

$$\tau_2 = \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2}$$

$$\frac{\partial L}{\partial \dot{q}_2} = \frac{\partial\left(\frac{L_1^2 m_1 \dot{q}_1^2}{2} + \frac{m_2(L_1^2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_1^2 + 2L_2^2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_2^2)}{2}\right) - (m_1 g\, s_1 \frac{L_1}{2} + m_2 g(s_1 L_1 + s_{12}\frac{L_2}{2}))}{\partial \dot{q}_2}$$

$$\frac{\partial L}{\partial \dot{q}_2} = \frac{\partial\left(\frac{L_1^2 m_1 \dot{q}_1^2}{2} + \frac{m_2(L_1^2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_1^2 + 2L_2^2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_2^2)}{2}\right) - (m_1 g\, s_1 \frac{L_1}{2} + m_2 g(s_1 L_1 + s_{12}\frac{L_2}{2}))}{\partial \dot{q}_2}$$

$$\frac{\partial L}{\partial \dot{q}_2} = \frac{\partial\left(\frac{m_2(2c_2L_1L_2\dot{q}_1\dot{q}_2 + 2L_2^2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_2^2)}{2}\right)}{\partial \dot{q}_2}$$

$$\frac{\partial L}{\partial \dot{q}_2} = m_2c_2L_1L_2\dot{q}_1 + m_2L_2^2\dot{q}_1 + m_2L_2^2\dot{q}_2$$

Differentiation of $\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_2}$ wrt t

$$\frac{\partial L}{\partial \dot{q}_2} = m_2 c_2 L_1 L_2 \dot{q}_1 + m_2 L_2^2 \dot{q}_1 + m_2 L_2^2 \dot{q}_2$$

We can divide each term and solve with respect to time

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_2} = \frac{\partial}{\partial t}(\underbrace{m_2 c_2 L_1 L_2 \dot{q}_1}_{\text{I}} + \underbrace{m_2 L_2^2 \dot{q}_1}_{\text{II}} + \underbrace{m_2 L_2^2 \dot{q}_2}_{\text{III}})$$

Differentiation of $\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_2}$ wrt t, cont.

I: $m_2 c_2 L_1 L_2 \dot{q}_1$

$u = m_2 L_1 L_2$ $u' = 0$

$v = \dot{q}_1 c_2$ $v' =$

Using chain-rule

$u = c_2$ $u' = -s_2 \dot{q}_2$

$v = \dot{q}_1$ $v' = \ddot{q}_1$

$u' * v + u * v' = -s_2 \dot{q}_2 \dot{q}_1 + c_2 \ddot{q}_1$

$$u' * v + u * v' = 0 + m_2 L_1 L_2(-s_2 \dot{q}_2 \dot{q}_1 + c_2 \ddot{q}_1)$$
$$= m_2 L_1 L_2(-s_2 \dot{q}_2 \dot{q}_1 + c_2 \ddot{q}_1)$$

II: $m_2 L_2^2 \dot{q}_1$

$u = m_2 L_2^2$ $u' = 0$

$v = \dot{q}_1$ $v' = \ddot{q}_1$

$$u' * v + u * v' = 0 + m_2 L_2^2 \ddot{q}_1 = m_2 L_2^2 \ddot{q}_1$$

III: $m_2 L_2^2 \dot{q}_2$

$u = m_2 L_2^2$ $u' = 0$

$v = \dot{q}_2$ $v' = \ddot{q}_2$

$$u' * v + u * v' = 0 + m_2 L_2^2 \ddot{q}_2 = m_2 L_2^2 \ddot{q}_2$$

Putting it all together for $\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_2}$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_2} = \frac{\partial}{\partial t}(m_2 c_2 L_1 L_2 \dot{q}_1 + m_2 L_2^2 \dot{q}_1 + m_2 L_2^2 \dot{q}_2)$$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_2} = m_2 L_1 L_2(-s_2 \dot{q}_2 \dot{q}_1 + c_2 \ddot{q}_1) + m_2 L_2^2 \ddot{q}_1 + m_2 L_2^2 \ddot{q}_2$$

Differentiation of $\frac{\partial L}{\partial q_2}$ wrt q_2

$$\frac{\partial L}{\partial q_2} = \frac{\partial\left(\frac{L_1^2 m_1 \dot{q}_1^2}{2} + \frac{m_2(L_1^2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_1^2 + 2L_2^2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_2^2)}{2}\right) - (m_1 g\, s_1\frac{L_1}{2} + m_2 g(s_1L_1 + s_{12}\frac{L_2}{2}))}{\partial q_2}$$

$$\frac{\partial L}{\partial q_2} = \frac{\partial\left(\frac{L_1^2 m_1 \dot{q}_1^2}{2} + \frac{m_2(L_1^2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_1^2 + 2L_2^2\dot{q}_1\dot{q}_2 + L_2^2\dot{q}_2^2)}{2}\right) - (m_1 g\, s_1\frac{L_1}{2} + m_2 g(s_1L_1 + s_{12}\frac{L_2}{2}))}{\partial q_2}$$

Differentiation of $\frac{\partial L}{\partial q_2}$ wrt q_2, cont.

$$\frac{\partial L}{\partial q_2} = \frac{\partial\left(\frac{m_2(2c_2L_1L_2\dot{q}_1^2 + 2c_2L_1L_2\dot{q}_1\dot{q}_2)}{2}\right) - m_2g(s_{12}\frac{L_2}{2})}{\partial q_2}$$

$$\frac{\partial L}{\partial q_2} = \frac{\partial(m_2c_2L_1L_2\dot{q}_1^2 + m_2c_2L_1L_2\dot{q}_1\dot{q}_2 - m_2gs_{12}\frac{L_2}{2})}{\partial q_2}$$

$$\frac{\partial L}{\partial q_2} = -m_2L_1L_2s_2\dot{q}_1^2 - m_2L_1L_2\dot{q}_2s_2 - m_2gc_{12}\frac{L_2}{2}$$

Calculating τ_2

$$\tau_2 = \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2}$$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_2} = m_2 L_1 L_2(-s_2\dot{q}_2\dot{q}_1 + c_2\ddot{q}_1) + m_2 L_2^2\ddot{q}_1 + m_2 L_2^2\ddot{q}_2$$

$$\frac{\partial L}{\partial q_2} = -m_2 L_1 L_2 s_2 \dot{q}_1^2 - m_2 L_1 L_2 \dot{q}_2 s_2 - m_2 g c_{12}\frac{L_2}{2}$$

$$\tau_2 = m_2 L_1 L_2(-s_2\dot{q}_2\dot{q}_1 + c_2\ddot{q}_1) + m_2 L_2^2\ddot{q}_1 + m_2 L_2^2\ddot{q}_2 - (-m_2 L_1 L_2 s_2 \dot{q}_1^2 - m_2 L_1 L_2 \dot{q}_2 s_2 - m_2 g c_{12}\frac{L_2}{2})$$

$$\tau_2 = m_2 L_1 L_2(-s_2\dot{q}_2\dot{q}_1 + c_2\ddot{q}_1) + m_2 L_2^2\ddot{q}_1 + m_2 L_2^2\ddot{q}_2 + m_2 L_1 L_2 s_2 \dot{q}_1^2 + m_2 L_1 L_2 \dot{q}_2 s_2 + m_2 g c_{12}\frac{L_2}{2})$$

Dynamic model for the 2-DOF planar manipulator

$$\tau_1 = \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} =$$

$$= m_2 L_1^2 \ddot{q}_1 + m_2 L_1^2 \ddot{q}_2 + m_2 L_2^2 \ddot{q}_1 + m_2 2 L_1 L_2 (-s_2 \dot{q}_2 \dot{q}_1 + c_2 \ddot{q}_1) + m_2 L_1 L_2 (-s_2 \dot{q}_2^2 + c_2 \ddot{q}_2) + L_1^2 m_1 \ddot{q}_1 + m_2 g c_{12} \frac{L_2}{2}$$

$$+ m_2 g c_1 L_1 + m_1 g c_1 \frac{L_1}{2})$$

$$\tau_2 = \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} = m_2 L_1 L_2 (-s_2 \dot{q}_2 \dot{q}_1 + c_2 \ddot{q}_1) + m_2 L_2^2 \ddot{q}_1 + m_2 L_2^2 \ddot{q}_2 + m_2 L_1 L_2 s_2 \dot{q}_1^2 + m_2 L_1 L_2 \dot{q}_2 s_2 + m_2 g c_{12} \frac{L_2}{2})$$

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} m_2 L_1^2 \ddot{q}_1 + m_2 L_1^2 \ddot{q}_2 + m_2 L_2^2 \ddot{q}_1 + m_2 2 L_1 L_2 (-s_2 \dot{q}_2 \dot{q}_1 + c_2 \ddot{q}_1) + m_2 L_1 L_2 (-s_2 \dot{q}_2^2 + c_2 \ddot{q}_2) + L_1^2 m_1 \ddot{q}_1 + m_2 g c_{12} \frac{L_2}{2} + m_2 g c_1 L_1 + m_1 g c_1 \frac{L_1}{2}) \\ m_2 L_1 L_2 (-s_2 \dot{q}_2 \dot{q}_1 + c_2 \ddot{q}_1) + m_2 L_2^2 \ddot{q}_1 + m_2 L_2^2 \ddot{q}_2 + m_2 L_1 L_2 s_2 \dot{q}_1^2 + m_2 L_1 L_2 \dot{q}_2 s_2 + m_2 g c_{12} \frac{L_2}{2}) \end{bmatrix}$$

This is our equation of motion without inertia tensors.

$$\tau = D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q)$$

Two ways of finding $C(q,\dot{q})\dot{q}$

I) Given D(q)

Dynamic model for the 2-DOF planar manipulator

$$\tau = D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q)$$

$$\tau = \begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}D_{11} & D_{12}\\ D_{21} & D_{22}\end{bmatrix}\begin{bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{bmatrix} + \begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} + \begin{bmatrix}g_1\\ g_2\end{bmatrix}$$

$$D_{11} = m_2L_1^2 + m_2L_2^2 + 2L_1L_2m_2c_2$$

$$D_{12} = L_1L_2m_2c_2 + m_2L_2^2$$

$$D_{21} = m_2L_2^2 + L_1L_2m_2c_2$$

$$D_{22} = m_2L_2^2$$

$$C_{11} = -2L_1L_2m_2s_2\dot{q}_2$$

$$C_{12} = -2L_1L_2m_2s_2(\dot{q}_2 + \dot{q}_1)$$

$$C_{21} = 2L_1L_2m_2s_2\dot{q}_1$$

$$C_{22} = 0$$

$$g_1 = m_2gc_{12}\frac{L_2}{2} + m_2gc_1L_1 + m_1gc_1\frac{L_1}{2}$$

$$g_2 = m_2gc_{12}\frac{L_2}{2}$$

$$\tau = \begin{bmatrix}\tau_1 \\ \tau_2\end{bmatrix} = \begin{bmatrix} m_2L_1^2\ddot{q}_1 + m_2L_1^2\ddot{q}_2 + m_2L_2^2\ddot{q}_1 + m_2 2L_1L_2(-s_2\dot{q}_2\dot{q}_1 + c_2\ddot{q}_1) + m_2L_1L_2(-s_2\dot{q}_2^2 + c_2\ddot{q}_2) + L_1^2m_1\ddot{q}_1 + m_2gc_{12}\frac{L_2}{2} + m_2gc_1L_1 + m_1gc_1\frac{L_1}{2}) \\ m_2L_1L_2(-s_2\dot{q}_2\dot{q}_1 + c_2\ddot{q}_1) + m_2L_2^2\ddot{q}_1 + m_2L_2^2\ddot{q}_2 + m_2L_1L_2s_2\dot{q}_1^2 + m_2L_1L_2\dot{q}_2s_2 + m_2gc_{12}\frac{L_2}{2}) \end{bmatrix}$$

These two equations are the equal

$$\tau = \begin{bmatrix}\tau_1 \\ \tau_2\end{bmatrix} = \begin{bmatrix} m_2L_1^2 + m_2L_2^2 + 2L_1L_2m_2c_2 & -2L_1L_2m_2c_2 + m_2L_2^2 \\ m_2L_2^2 + L_1L_2m_2c_2 & m_2L_2^2 \end{bmatrix}\begin{bmatrix}\ddot{q}_1 \\ \ddot{q}_2\end{bmatrix} + \begin{bmatrix} -2L_1L_2m_2s_2\dot{q}_2 & -2L_1L_2m_2s_2(\dot{q}_2 + \dot{q}_1) \\ 2L_1L_2m_2s_2\dot{q}_1 & 0 \end{bmatrix}\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix} + \begin{bmatrix} m_2gc_{12}\frac{L_2}{2} + m_2gc_1L_1 + m_1gc_1\frac{L_1}{2} \\ m_2gc_{12}\frac{L_2}{2} \end{bmatrix}$$

Lagrangian term for Spherical manipulator

$$L = \frac{m_3((\dot{d}_3 s_2 + \dot{q}_2 c_2(d_3^* + L_2))^2 + \dot{q}_1 c_2 s_1(d_3^* + L_2) - \dot{d}_3 c_1 c_2 + \left(\dot{q}_2 c_1 s_2(d_3^* + L_2)\right)^2 + \dot{d}_3 s_1 c_2 + \dot{q}_1 c_2 c_1(d_3^* + L_2) - \dot{q}_2 s_1 s_2(d_3^* + L_2)}{2} - \left(m_1 g \frac{L_1}{2} + m_2 g L_1 + m_3 g L_1 + \frac{s_2(L_2 + d_3^*)}{2}\right)$$

Euler Lagrange for Spherical manipulator

$$\tau_1 = \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1}$$

$$\tau_2 = \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2}$$

$$\tau_3 = \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_3} - \frac{\partial L}{\partial q_3}$$

I will not do this by hand,

Dynamic model for the Spherical manipulator

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$$

$$\tau = D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q)$$

$$\downarrow$$

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

FINISH

Hint: for mandatory assignment 3

- The equation of motion or the dynamic model should be small for the simplified 2-DOF Crustcrawler robot in Task I.