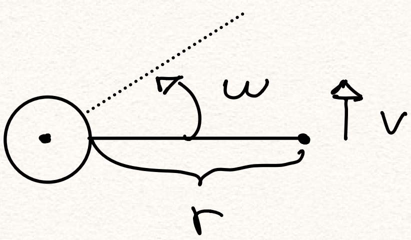
Plan:
- Velocities
- Jacobian
- Example: Spherical manipulator
- Singularities

- Want to find the velocity of the tool relative to the base.
- We can only measure the changes in the joint variables.
  --> So how?

Velocity:
- Rigid body rotates about a fixed axis:
  - k is the unit vector in the direction of the axis.



$$\omega = \dot{\theta} k$$
$$v = \omega \times r$$

- Translation for a rigid body:
  - All points attached to the body moves with the same velocity.

- End effector velocity:
- $\xi = J\dot{q}$ (q = joint variables)
- J, Jacobian: Transformation from joint variables to end-effector velocities.

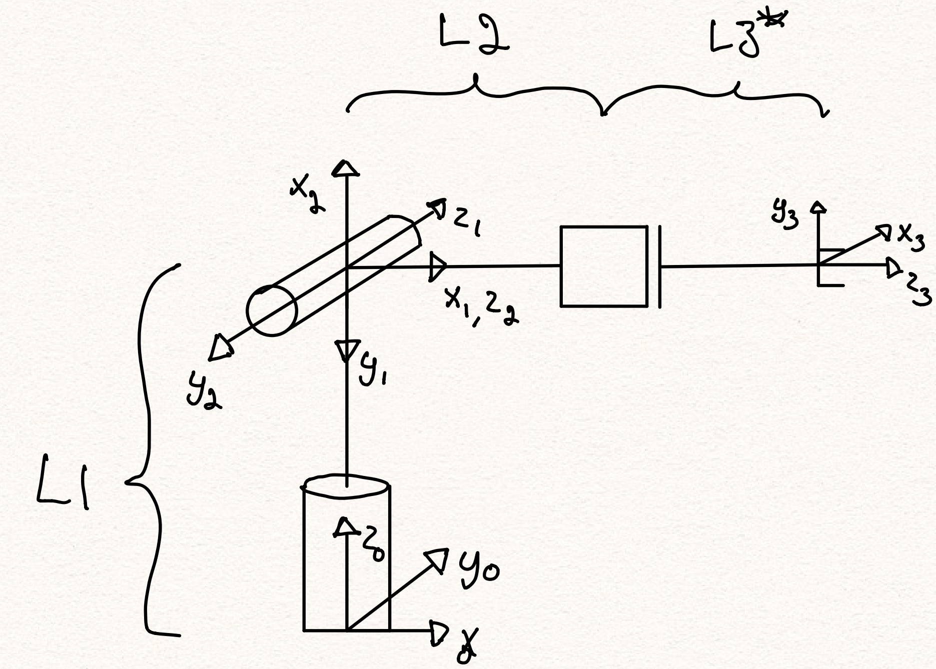
$$J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \qquad J_{v_i} \begin{cases} z_{i-1} \times (O_n - O_{i-1}) & \text{revolute} \\ z_{i-1} & \text{prismatic} \end{cases} \qquad J_{\omega_i} \begin{cases} z_{i-1} & \text{revolute} \\ 0 & \text{prismatic} \end{cases}$$

$$J = \begin{bmatrix} \frac{\partial F}{\partial x_1} & \cdots & \frac{\partial F}{\partial x_n} \end{bmatrix} \text{ (Alternatively)}$$

Jacobian:
- Useful for:
  - Calculation and execution of smooth movement.
  - Calculation of singularities.
  - Deriving the dynamical equations.
  - Translate joint variables to force and moment in the tool.

$$\left.\begin{aligned} v_i &= J_v \dot{q} \\ \omega_i &= J_\omega \dot{q} \end{aligned}\right\} \xi = \begin{bmatrix} v_i \\ \omega_i \end{bmatrix}$$

Example: Spherical manipulator



$\boxed{RRP}$

$$T_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & L1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_2^0 = \begin{bmatrix} c_1 s_2 & s_1 & c_1 c_2 & 0 \\ s_1 s_2 & -c_1 & s_1 c_2 & 0 \\ c_2 & 0 & -s_2 & L1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = \begin{bmatrix} -S_1 & C_1S_2 & C_1C_2 & C_1C_2(L2+L3^*) \\ C_1 & S_1S_2 & S_1C_2 & S_1C_2(L2+L3^*) \\ 0 & C_2 & -S_2 & L1 - S_2(L2+L3^*) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$J = \begin{bmatrix} z_0 \times (O_3 - O_0) & z_1 \times (O_3 - O_1) & z_2 \\ z_0 & z_1 & 0 \end{bmatrix}$$

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad z_1 = \begin{bmatrix} -S_1 \\ C_1 \\ 0 \end{bmatrix} \qquad z_2 = \begin{bmatrix} C_1C_2 \\ S_1C_2 \\ -S_2 \end{bmatrix}$$

$$O_3 - O_0 = O_3 = \begin{array}{cc} \begin{bmatrix} C_1C_2(L2+L3) \\ S_1C_2(L2+L3) \\ L1 - S_2(L2+L3) \end{bmatrix} & \begin{array}{c} a \\ b \\ c \end{array} \end{array}$$

$$O_3 - O_1 = \begin{array}{cc} \begin{bmatrix} C_1C_2(L2+L3) \\ S_1C_2(L2+L3) \\ -S_2(L2+L3) \end{bmatrix} & \begin{array}{c} d \\ e \\ f \end{array} \end{array}$$

$$z_0 \times (O_3 - O_0) = \begin{matrix} \cancel{0} & 0 & 1 & 0 & 0 & \cancel{\ } \\ \cancel{a} & b & c & a & b & \cancel{c} \end{matrix}$$

$$\begin{bmatrix} -b \\ a \\ 0 \end{bmatrix} = \begin{bmatrix} -s_1 c_2 (L2 + L3) \\ c_1 c_2 (L2 + L3) \\ 0 \end{bmatrix}$$

$$z_0 \times (O_3 - O_1) = \begin{matrix} \cancel{-s_1} & c_1 & 0 & -s_1 & c_1 & \cancel{0} \\ \cancel{d} & e & f & d & e & \cancel{f} \end{matrix}$$

$$\begin{bmatrix} c_1 f \\ s_1 f \\ -s_1 e - c_1 d \end{bmatrix} = \begin{bmatrix} -c_1 s_2 (L2 + L3) \\ -s_1 s_2 (L2 + L3) \\ -\underbrace{\cancel{(s_1^2 + c_1^2)}}_{1}(c_2 (L2 + L3)) \end{bmatrix}$$

$$J = \begin{bmatrix} -s_1 c_2 (L2 + L3) & -c_1 s_2 (L2 + L3) & c_1 c_2 \\ c_1 c_2 (L2 + L3) & -s_1 s_2 (L2 + L3) & s_1 c_2 \\ 0 & -c_2 (L2 + L3) & -s_2 \\ 0 & -s_1 & 0 \\ 0 & c_1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Singularities:
- Configuration where motion in some directions is impossible.
- Minor movement of the tool could require infinite force in the joints.
  - "Does not mean that joints will fly off into the stratosphere at superluminal speeds, but the joints might strain and break."
- Mathematically: rang J(q) < max
- Appears when the manipulator loses a degree of freedom.
- $|J| = 0$
- $|J_v| = 0$ (arm singularities)