Ch. 3: Inverse Kinematics
Ch. 4: Velocity Kinematics

Recap: kinematic decoupling

- Appropriate for systems that have an arm a wrist
  - Such that the wrist joint axes are aligned at a point
- For such systems, we can split the inverse kinematics problem into two parts:
  1. Inverse position kinematics: position of the wrist center
  2. Inverse orientation kinematics: orientation of the wrist
- First, assume 6DOF, the last three intersecting at o_c

$$R_6^0(q_1,\ldots,q_6) = R$$
$$o_6^0(q_1,\ldots,q_6) = o$$

- Use the position of the wrist center to determine the first three joint angles…

Recap: kinematic decoupling

- **Now, origin of tool frame, o_6, is a distance d_6 translated along z_5 (since z_5 and z_6 are collinear)**
  - **Thus, the third column of *R* is the direction of z_6 (w/ respect to the base frame) and we can write:**

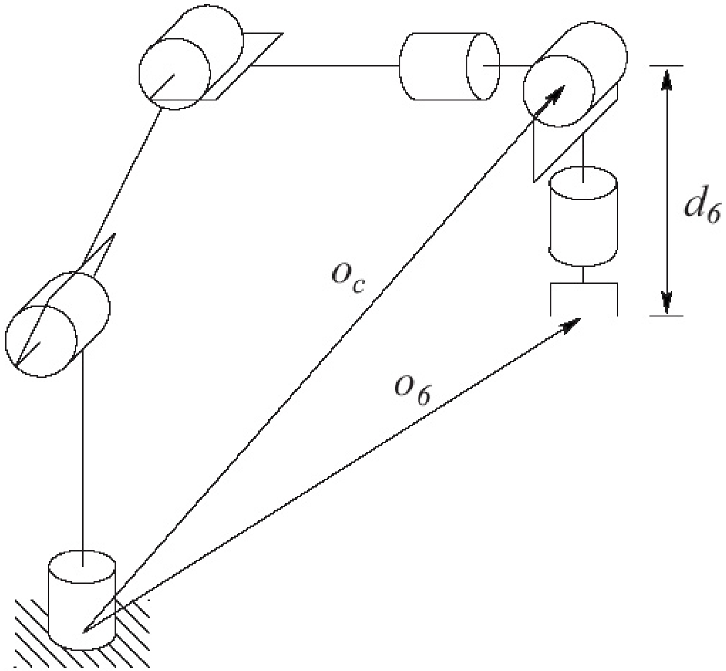
$$o = o_6^0 = o_c^o + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

  - **Rearranging:**

$$o_c^o = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

  - **Calling $o = [o_x\ o_y\ o_z]^T$, $o_c^0 = [x_c\ y_c\ z_c]^T$**

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$



Recap: kinematic decoupling

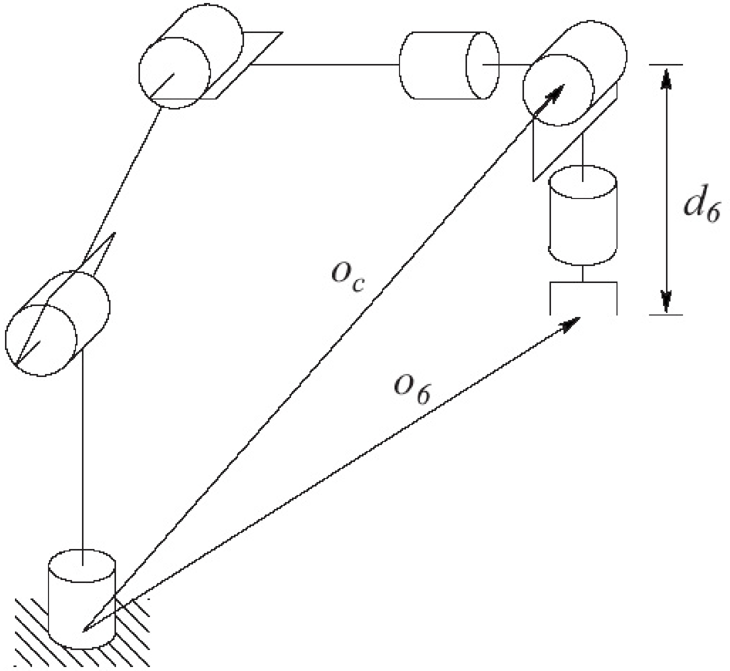
- **Since $[x_c\ y_c\ z_c]^T$ are determined from the first three joint angles, our forward kinematics expression now allows us to solve for the first three joint angles decoupled from the final three.**
  - **Thus we now have R_3^0**
  - **Note that:**

$$R = R_3^0 R_6^3$$

  - **To solve for the final three joint angles:**

$$R_6^3 = \left(R_3^0\right)^{-1} R = \left(R_3^0\right)^T R$$

  - **Since the last three joints for a spherical wrist, we can use a set of Euler angles to solve for them**



Recap: Inverse position kinematics

- **Now that we have $[x_c\ y_c\ z_c]^T$ we need to find q_1, q_2, q_3**
  - **Solve for q_i by projecting onto the x_{i-1}, y_{i-1} plane, solve trig problem**
  - **Two examples**
    - **elbow (RRR) manipulator: 4 solutions (left-arm elbow-up, left-arm elbow-down, right-arm elbow-up, right-arm elbow-down)**
    - **spherical (RRP) manipulator: 2 solutions (left-arm, right-arm)**

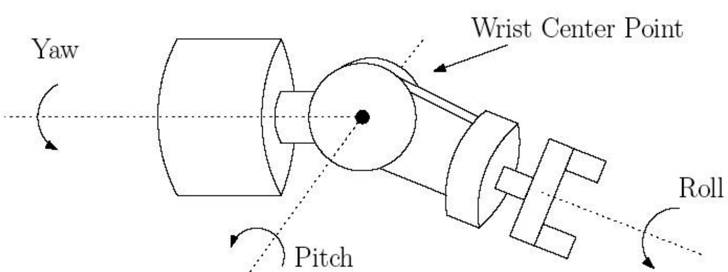
Inverse orientation kinematics

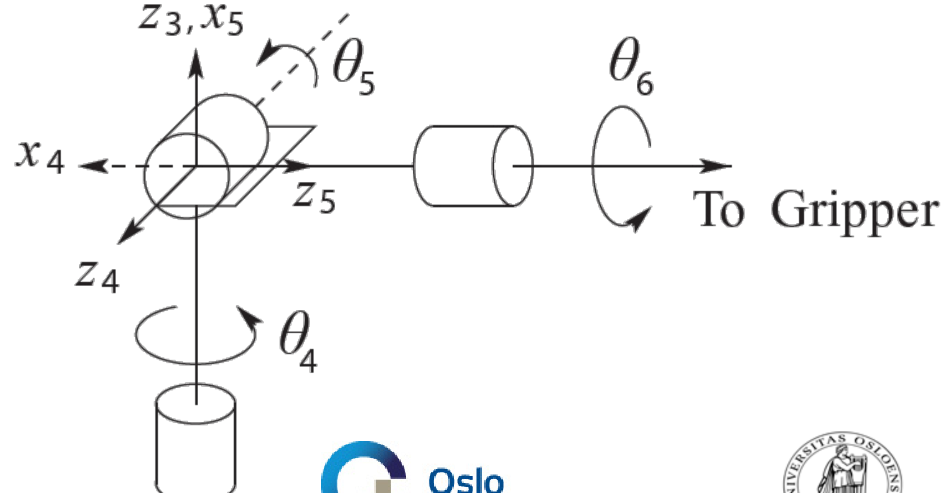
- **Now that we can solve for the position of the wrist center (given kinematic decoupling), we can use the desired orientation of the end effector to solve for the last three joint angles**
  - **Finding a set of Euler angles corresponding to a desired rotation matrix R**
  - **We want the final three joint angles that give the orientation of the tool frame with respect to o_3 (i.e. R_6^3)**

Inverse orientation: spherical wrist

- **Previously, we said that the forward kinematics of the spherical wrist were identical to a *ZYZ* Euler angle transformation:**

$$T_6^3 = A_4 A_5 A_6 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5c_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$





Inverse orientation: spherical wrist

- **The inverse orientation problem reduces to finding a set of Euler angles (θ_4, θ_5, θ_6) that satisfy:**

$$R_6^3 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & s_5c_6 & c_5 \end{bmatrix}$$

- **to solve this, take two cases:**
  1. **Both r_{13} and r_{23} are not zero (i.e. $\theta_5 \neq 0$)… nonsingular**
  2. **$\theta_5 = 0$, thus $r_{13} = r_{23} = 0$… singular**
- **Nonsingular case**
  - **If $\theta_5 \neq 0$, then $r_{33} \neq \pm 1$ and:**

$$c_5 = r_{33},\ s_5 = \pm\sqrt{1 - r_{33}^{\ 2}}$$

$$\theta_5 = \mathbf{atan2}\left(r_{33}, \pm\sqrt{1 - r_{33}^{\ 2}} \right)$$

Inverse orientation: spherical wrist

- **Thus there are two values for θ_5. Using the first ($s_5 > 0$):**

$$\theta_4 = \mathbf{atan}2(r_{13}, r_{23})$$
$$\theta_6 = \mathbf{atan}2(-r_{31}, r_{32})$$

- **Using the second value for θ_5 ($s_5 < 0$):**

$$\theta_4 = \mathbf{atan}2(-r_{13}, -r_{23})$$
$$\theta_6 = \mathbf{atan}2(r_{31}, -r_{32})$$

- **Thus for the nonsingular case, there are two solutions for the inverse orientation kinematics**

Inverse orientation: spherical wrist

- **In the singular case, θ_5 = 0 thus s_5 = 0 and $r_{13} = r_{23} = r_{31} = r_{32} = 0$**
- **Therefore, R_6^3 has the form:**

$$R_6^3 = \begin{bmatrix} c_4c_6 - s_4s_6 & -c_4s_6 - s_4c_6 & 0 \\ s_4c_6 + c_4s_6 & -s_4s_6 + c_4c_6 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{46} & -s_{46} & 0 \\ s_{46} & c_{46} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **So we can find the sum $\theta_4 + \theta_6$ as follows:**

$$\theta_4 + \theta_6 = \mathbf{atan}2(r_{11}, r_{21}) = \mathbf{atan}2(r_{11}, -r_{12})$$

- **Since we can only find the sum, there is an infinite number of solutions (singular configuration)**

Inverse Kinematics: general procedure

1. **Find q_1, q_2, q_3 such that the position of the wrist center is:**

$$o_c^o = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2. **Using q_1, q_2, q_3, determine R_3^0**

} inverse position kinematics

3. **Find Euler angles corresponding to the rotation matrix:**

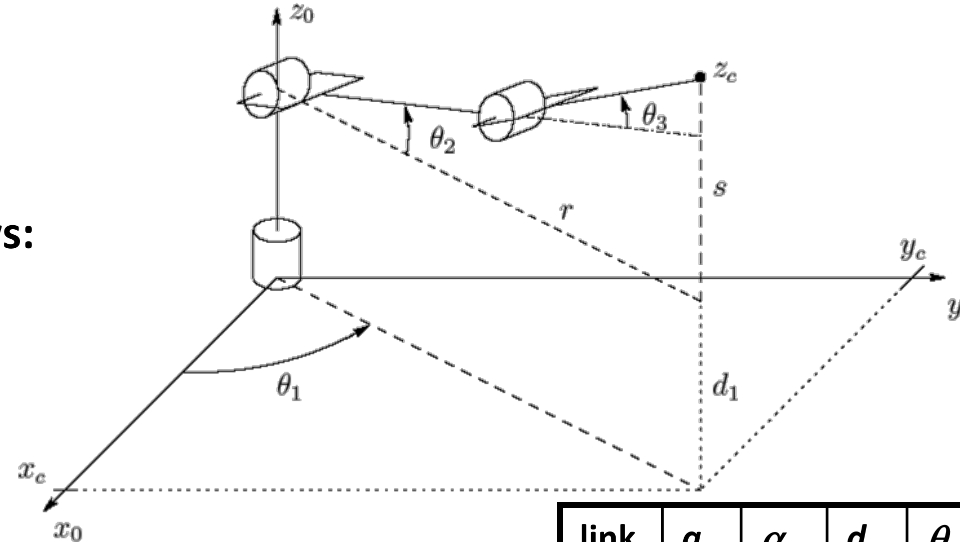
$$R_6^3 = \left(R_3^0\right)^{-1} R = \left(R_3^0\right)^T R$$

} inverse orientation kinematics

Example: RRR arm with spherical wrist

- **For the DH parameters below, we can derive R_3^0 from the forward kinematics:**

$$R_3^0 = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 \\ s_1c_{23} & -s_1s_{23} & -c_1 \\ s_{23} & c_{23} & 0 \end{bmatrix}$$



- **We know that R_6^3 is given as follows:**

$$R_6^3 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & s_5c_6 & c_5 \end{bmatrix}$$

- **To solve the inverse orientation kinematics:**

$$R_6^3 = \left(R_3^0\right)^T R$$

  - **For a given desired *R***

link	a_i	α_i	d_i	θ_i
1	0	90	d_1	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

Example: RRR arm with spherical wrist

- **Euler angle solutions can be applied. Taking the third column of $(R_3^0)^T R$**

$$c_4 s_5 = c_1 c_{23} r_{13} + s_1 c_{23} r_{23} + s_{23} r_{33}$$
$$s_4 s_5 = -c_1 s_{23} r_{13} - s_1 s_{23} r_{23} + c_{23} r_{33}$$
$$c_5 = s_1 r_{13} - c_1 r_{23}$$

- **Again, if $\theta_5 \neq 0$, we can solve for θ_5:**

$$\theta_5 = \mathbf{atan2}\left(s_1 r_{13} - c_1 r_{23}, \pm\sqrt{1-(s_1 r_{13} - c_1 r_{23})^2}\right)$$

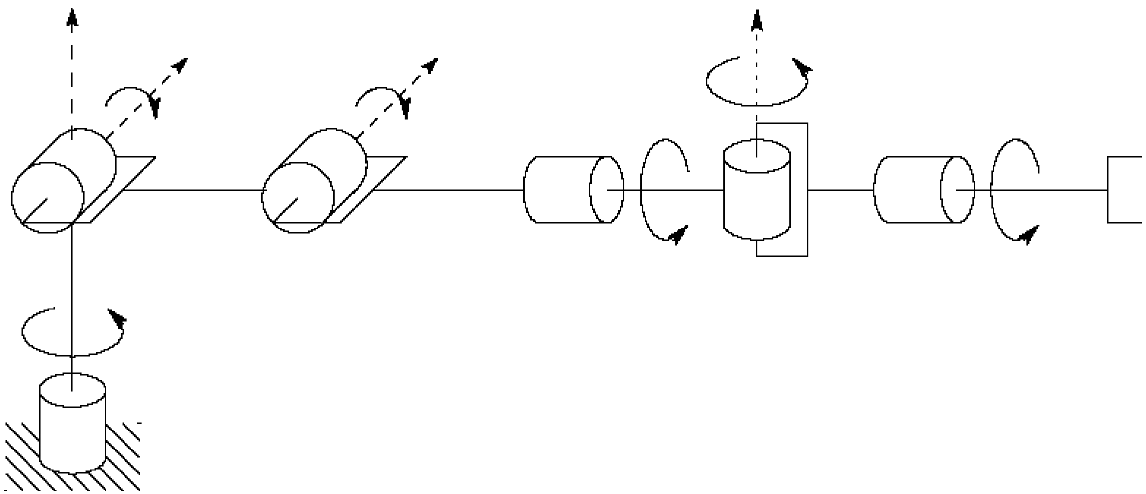
- **Finally, we can solve for the two remaining angles as follows:**

$$\theta_4 = \mathbf{atan2}(c_1 c_{23} r_{13} + s_1 c_{23} r_{23} + s_{23} r_{33}, -c_1 s_{23} r_{13} - s_1 s_{23} r_{23} + c_{23} r_{33})$$
$$\theta_6 = \mathbf{atan2}(-s_1 r_{11} + c_1 r_{21}, s_1 r_{12} - c_1 r_{22})$$

- **For the singular configuration ($\theta_5 = 0$), we can only find $\theta_4 + \theta_6$ thus it is common to arbitrarily set θ_4 and solve for θ_6**

Example: elbow manipulator with spherical wrist

- **Derive complete inverse kinematics solution**



link	a_i	α_i	d_i	θ_i
1	0	90	d_1	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3
4	0	-90	0	θ_4
5	0	0	0	θ_5
6	0	0	d_6	θ_3

- **we are given $H = T_6^0$ such that:**

$$o = \begin{bmatrix} o_x \\ o_y \\ o_z \end{bmatrix}, R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Example: elbow manipulator with spherical wrist

- **First, we find the wrist center:**

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$

- **Inverse position kinematics:**

$$\theta_1 = \mathbf{atan2}(x_c, y_c)$$

$$\theta_2 = \mathbf{atan2}\left(\sqrt{x_c{}^2 + y_c{}^2 - d^2}, z_c - d_1\right) - \mathbf{atan2}(a_2 + a_3 c_3, a_3 s_3)$$

$$\theta_3 = \mathbf{atan2}\left(D, \pm\sqrt{1 - D^2}\right)$$

- **Where *d* is the shoulder offset (if any) and *D* is given by:**

$$D = \frac{x_c{}^2 + y_c{}^2 - d^2 + (z_c - d_1)^2 - a_2{}^2 - a_3{}^2}{2a_2 a_3}$$

Example: elbow manipulator with spherical wrist

- **Inverse orientation kinematics:**
  - **Now that we know θ_1, θ_2, θ_3, we know R_3^0. need to find R_3^6:**

$$R_6^3 = \left(R_3^0\right)^T R$$

- **Solve for θ_4, θ_5, θ_6, Euler angles:**

$$\theta_4 = \mathbf{atan}2\left(c_1 c_{23} r_{13} + s_1 c_{23} r_{23} + s_{23} r_{33}, -c_1 s_{23} r_{13} - s_1 s_{23} r_{23} + c_{23} r_{33}\right)$$

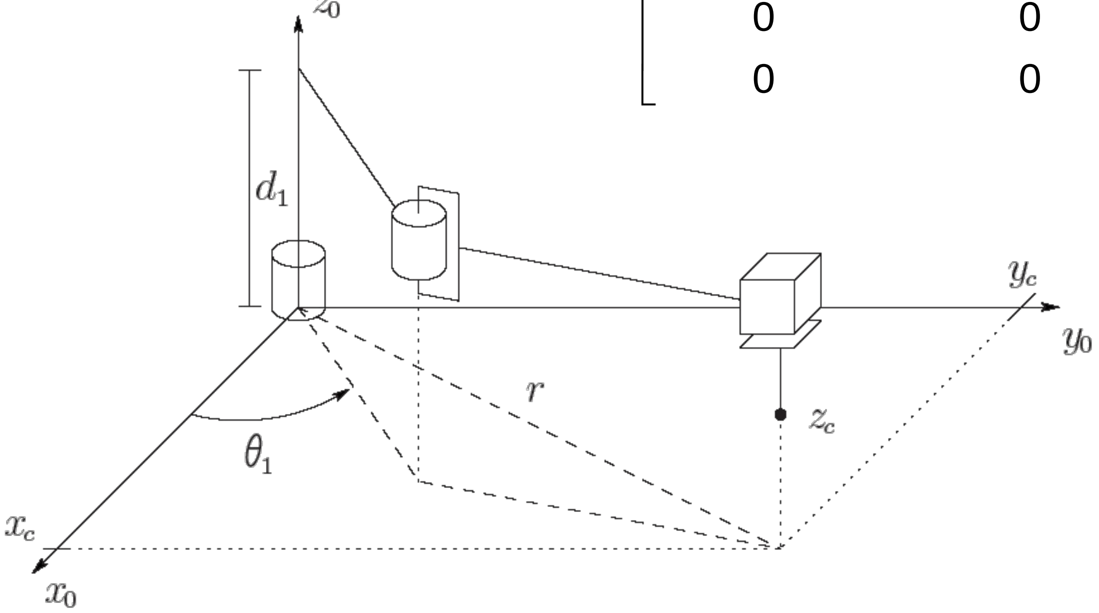
$$\theta_5 = \mathbf{atan}2\left(s_1 r_{13} - c_1 r_{23}, \pm\sqrt{1 - \left(s_1 r_{13} - c_1 r_{23}\right)^2}\right)$$

$$\theta_6 = \mathbf{atan}2\left(-s_1 r_{11} + c_1 r_{21}, s_1 r_{12} - c_1 r_{22}\right)$$

Example: inverse kinematics of SCARA manipulator

- **We are given T_4^0:** $T_4^0 = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & s_{12}c_4 - c_{12}s_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -c_{12}c_4 - s_{12}s_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1
2	a_2	180	0	θ_2
3	0	0	d_3	0
4	0	0	d_4	θ_4

Example: inverse kinematics of SCARA manipulator

- **Thus, given the form of T_4^0, *R* must have the following form:**

$$R = \begin{bmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **Where α is defined as:** $\alpha = \theta_1 + \theta_2 - \theta_4 = \mathbf{atan}2(r_{11}, r_{12})$
- **To solve for θ_1 and θ_2 we project the manipulator onto the x_0-y_0 plane:**

$$c_2 = \frac{o_x^{\ 2} + o_y^{\ 2} - a_1^{\ 2} - a_2^{\ 2}}{2a_1a_2}$$

- **This gives two solutions for θ_2:** $\theta_2 = \mathbf{atan}2\left(c_2, \pm\sqrt{1 - c_2^{\ 2}}\right)$
- **Once θ_2 is known, we can solve for θ_1:**

$$\theta_1 = \mathbf{atan}2(o_x, o_y)\text{-}\ \mathbf{atan}2(a_1 + a_2c_2, a_2s_2)$$

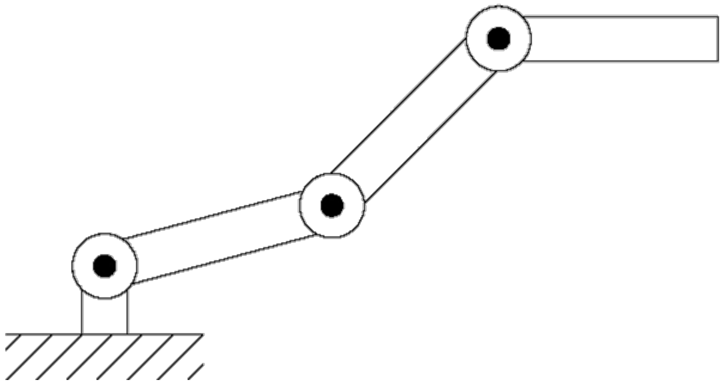
- **θ_4 is now give as:** $\theta_4 = \theta_1 + \theta_2 - \mathbf{atan}2(r_{11}, r_{12})$
- **Finally, it is trivial to see that $d_3 = o_z + d_4$**

Example: number of solutions

- **How many solutions to the inverse position kinematics of a planar 3-link arm?**
  - **given a desired $d=[d_x\ d_y]^T$, the forward kinematics can be written as:**

$$d_x = a_1c_1 + a_2c_{12} + a_3c_{123}$$
$$d_y = a_1s_1 + a_2s_{12} + a_3s_{123}$$

  - **Therefore the inverse kinematics problem is under-constrained (two equations and three unknowns)**



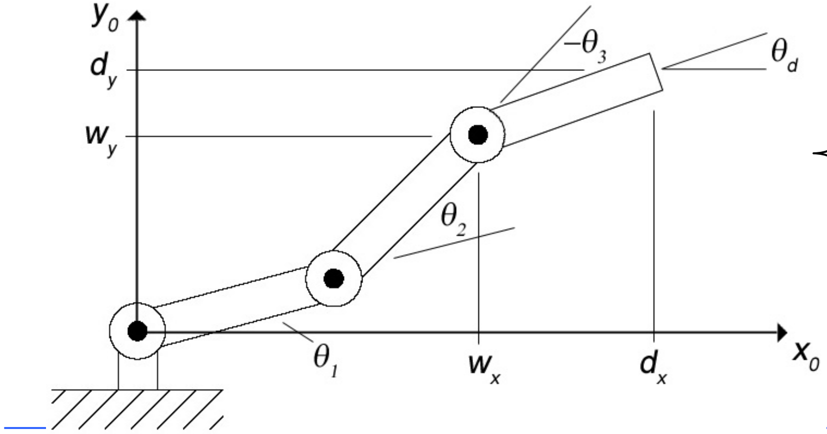
∞ solutions is *d* is inside the workspace
1 solution if *d* is on the workspace boundary
0 solutions else

Example: number of solutions

- **What if now we describe the desired position and orientation of the end effector?**
  - **given a desired $d=[d_x\ d_y]^T$, we can now call the position of o_2 the 'wrist center'. This position is given as:**

$$w_x = d_x - a_3 \cos(\theta_d)$$
$$w_y = d_y - a_3 \sin(\theta_d)$$

  - **Now we have reduced the problem to finding the joint angles that will give the desired position of the wrist center (we have done this for a 2D planar manipulator).**
  - **Finally, θ_3 is given as:** $\theta_3 = \theta_d - (\theta_1 + \theta_2)$



∞ solutions if the wrist center is on the origin
2 solutions if wrist center is inside the 2-link workspace
1 solution if wrist center is on the 2-link workspace boundary
0 solutions else

Velocity Kinematics

- **Now we know how to relate the end-effector position and orientation to the joint variables**
- **Now we want to relate end-effector linear and angular velocities with the joint velocities**
- **First we will discuss angular velocities about a fixed axis**
- **Second we discuss angular velocities about arbitrary (moving) axes**
- **We will then introduce the Jacobian**
  - **Instantaneous transformation between a vector in R^n representing joint velocities to a vector in R^6 representing the linear and angular velocities of the end-effector**
- **Finally, we use the Jacobian to discuss numerous aspects of manipulators:**
  - **Singular configurations**
  - **Dynamics**
  - **Joint/end-effector forces and torques**

Angular velocity: fixed axis

- **When a rigid body rotates about a fixed axis, every point moves in a circle**
  - **Let *k* represent the fixed axis of rotation, then the angular velocity is:**

$$\omega = \dot{\theta}\hat{k}$$

  - **The velocity of any point on a rigid body due to this angular velocity is:**

$$v = \omega \times r$$

  - **Where *r* is the vector from the axis of rotation to the point**
- **When a rigid body translates, all points attached to the body have the same velocity**

Next class…

- **Derivation of the Jacobian**