Addition of angular velocities

- **How do we determine the angular velocity of the tool frame due to the combination of multiple rotations of the joints?**
- **Angular velocities can be added once they are projected into the same coordinate frame.**
- **This can be extended to calculate the angular velocity for an *n*-link manipulator:**
  - **Suppose we have an *n*-link manipulator whose coordinate frames are related as follows:** $R_n^0 = R_1^0 R_2^1 \cdots R_n^{n-1}$
  - **Now we want to find the rotation of the n^{th} frame in the inertial frame:**

$$\dot{R}_n^0 = S(\omega_{0,n}^0) R_n^0$$

  - **We can define the angular velocity of the tool frame in the inertial frame:**

$$\begin{aligned}\omega_{0,n}^0 &= \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1 + R_2^0 \omega_{2,3}^2 + R_3^0 \omega_{3,4}^3 + \ldots + R_{n-1}^0 \omega_{n-1,n}^{n-1} \\ &= \omega_{0,1}^0 + \omega_{1,2}^0 + \omega_{2,3}^0 + \omega_{3,4}^0 + \ldots + \omega_{n-1,n}^0\end{aligned}$$

The Jacobian

- **Now we are ready to describe the relationship between the joint velocities and the end effector velocities.**
- **Assume that we have an *n*-link manipulator with joint variables $q_1, q_2, \ldots, q_n$**
  - **Our homogeneous transformation matrix that defines the position and orientation of the end effector in the inertial frame is:**

$$T_n^o = \begin{bmatrix} R_n^0(q) & o_n^0(q) \\ 0 & 1 \end{bmatrix}$$

  - **We can call the angular velocity of the tool frame $\omega_{0,n}{}^0$ and:**

$$S(\omega_{0,n}^0) = \dot{R}_n^0 (R_n^0)^T$$

  - **Call the linear velocity of the end effector:**

$$v_n^0 = \dot{o}_n^0$$

The Jacobian

- **Therefore, we want to come up with the following mappings:**

$$v_n^0 = J_v \dot{q}$$
$$\omega_n^0 = J_\omega \dot{q}$$

  - **Thus J_v and J_ω are 3xn matrices**
- **we can combine these into the following:** $\xi = J\dot{q}$
  - **where:**

$$\xi = \begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix}$$
$$J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}$$

- ***J* is called the Jacobian**
  - **6x*n* where *n* is the number of joints**

Deriving J_ω

- Remember that each joint *i* rotates around the axis z_{i-1}
- Thus we can represent the angular velocity of each frame with respect to the previous frame
  - If the i^{th} joint is revolute, this is:

$$\omega_{i-1,i}^{i-1} = \dot{q}_i z_{i-1}^{i-1} = \dot{q}_i \hat{k}$$

  - If the i^{th} joint is prismatic, the angular velocity of frame *i* relative to frame *i-1* is zero
  - Thus, based upon our rules of forming the equivalent angular velocity of the tool frame with respect to the base frame:

$$\begin{aligned}\omega_{0,n}^{0} &= \omega_{0,1}^{0} + R_1^0 \omega_{1,2}^{1} + R_2^0 \omega_{2,3}^{2} + R_3^0 \omega_{3,4}^{3} + \ldots + R_{n-1}^0 \omega_{n-1,n}^{n-1} \\ &= \rho_1 \dot{q}_1 \hat{k} + \rho_2 \dot{q}_2 R_1^0 \hat{k} + \rho_3 \dot{q}_3 R_2^0 \hat{k} + \rho_4 \dot{q}_4 R_3^0 \hat{k} + \ldots + \rho_n \dot{q}_n R_{n-1}^0 \hat{k} \\ &= \sum_{i=1}^{n} \rho_i \dot{q}_i z_{i-1}^0\end{aligned}$$

  - Where the term ρ_i determines if joint *i* is revolute (ρ_i =1) or prismatic (ρ_i =0)

Deriving J_v

- **Linear velocity of the end effector:**

$$\dot{o}_n^0 = \sum_{i=1}^{n} \frac{\partial o_n^0}{\partial q_i} \dot{q}_i$$

- **Therefore we can simply write the i^{th} column of J_v as:**

$$J_{v_i} = \frac{\partial o_n^0}{\partial q_i}$$

- **However, the linear velocity of the end effector can be due to the motion of revolute and/or prismatic joints**
- **Thus the end effector velocity is a linear combination of the velocity due to the motion of each joint**
  - **w/o L.O.G. we can assume all joint velocities are zero other than the i^{th} joint**
  - **This allows us to examine the end effector velocity due to the motion of either a revolute or prismatic joint**

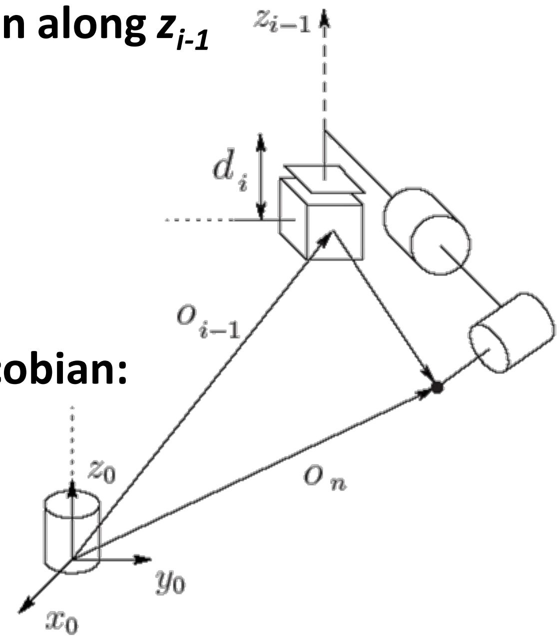
Deriving J_v

- **End effector velocity due to prismatic joints**
  - **Assume all joints are fixed other than the prismatic joint d_i**
  - **The motion of the end effector is pure translation along z_{i-1}**

$$\dot{o}_n^0 = \dot{d}_i R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \dot{d}_i z_{i-1}^0$$

  - **Therefore, we can write the i^{th} column of the Jacobian:**

$$J_{v_i} = z_{i-1}^0$$



Deriving J_v

- **End effector velocity due to revolute joints**
  - **Assume all joints are fixed other than the revolute joint θ_i**
  - **The motion of the end-effector is given by:**

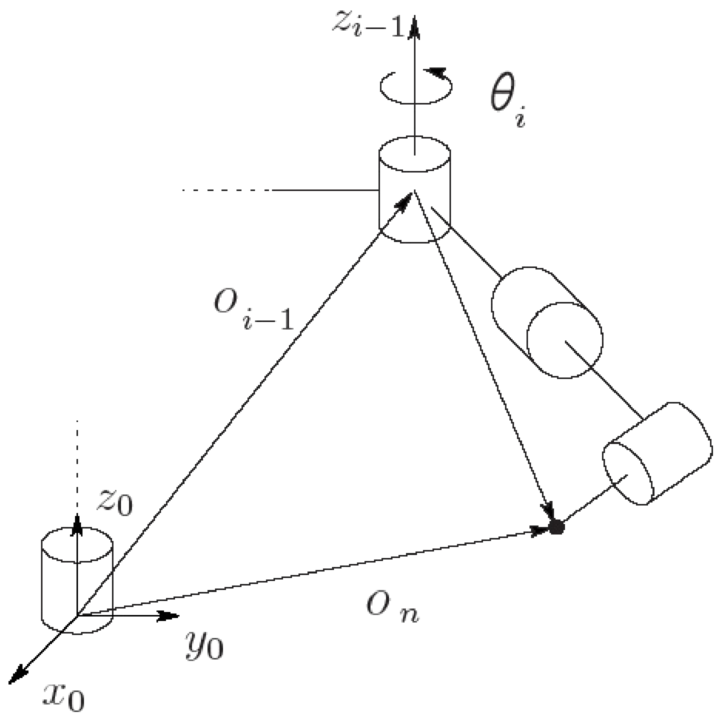
$$\dot{o}_n^0 = \omega_{i-1,i}^0 \times r = \dot{\theta}_i z_{i-1}^0 \times r$$

  - **Where the term *r* is the distance from the tool frame o_n to the frame o_{i-1}**

$$\dot{o}_n^0 = \dot{\theta}_i z_{i-1}^0 \times (o_n - o_{i-1})$$

  - **Thus we can write the i^{th} column of J_v:**

$$J_{v_i} = z_{i-1}^0 \times (o_n - o_{i-1})$$



The complete Jacobian

- **The *i*th column of J_v is given by:**

$$J_{v_i} = \begin{cases} z_{i-1} \times (o_n - o_{i-1}) & \text{for } i \text{ revolute} \\ z_{i-1} & \text{for } i \text{ prismatic} \end{cases}$$

- **The *i*th column of J_ω is given by:**

$$J_{\omega_i} = \begin{cases} z_{i-1} & \text{for } i \text{ revolute} \\ 0 & \text{for } i \text{ prismatic} \end{cases}$$

Singularities

- **We can now derive the Jacobian as a mapping given by the following:**

$$\xi = J(q)\dot{q}$$

- **This means that the columns of *J* form a basis for the space of possible end effector velocities**
- **Thus, for the end effector to be able to achieve any arbitrary body velocity ξ, *J* must have rank 6**
- **We know that *J* is 6x*n* and that:**

$$\text{for } A \in \mathbf{R}^{m \times n},\ \operatorname{rank}(A) \le \min(m,n)$$

- **Thus,** $\operatorname{rank}(J) \le \min(6,n)$
- **For example, for the two link planar manipulator,** $\operatorname{rank}(J) \le 2$
- **For example, for the Stanford manipulator,** $\operatorname{rank}(J) \le 6$
- **Note that the columns the Jacobian of a kinematically redundant manipulator are never linearly independent**

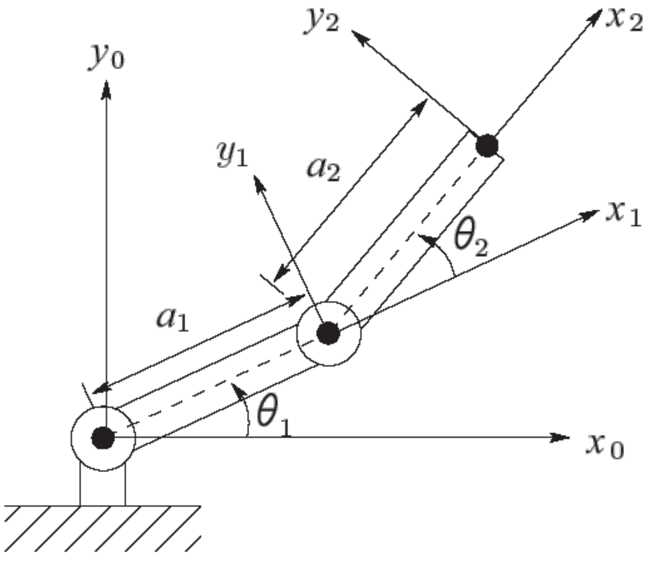
Singularities

- **But the rank of the Jacobian is not necessarily constant… it will of course depend upon the configuration**
- **Definition: we say that any configuration in which the rank of *J* is less than its maximum is a singular configuration**
  - **i.e. any configuration that causes *J* to lose rank is a singular configuration**
- **Characteristics of singularities:**
  - **At a singularity, motion in some directions will not be possible**
  - **At and near singularities, bounded end effector velocities would require unbounded joint velocities**
  - **At and near singularities, bounded joint torques may produce unbounded end effector forces and torques**
  - **Singularities often occur along the workspace boundary (i.e. when the arm is fully extended)**

Singularities

- **How do we determine singularities?**
  - **Simple: construct the Jacobian and observe when it will lose rank**
- **EX: two link manipulator**
  - **Previously, we found *J* to be:**

$$J(q)=\begin{bmatrix} -a_1s_1-a_2s_{12} & -a_2s_{12} \\ a_1c_1+a_2c_{12} & a_2c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$



  - **This loses rank if we can find some α such that:**

$$J_1=\alpha J_2 \quad \text{for } \alpha\in\mathbf{R}$$

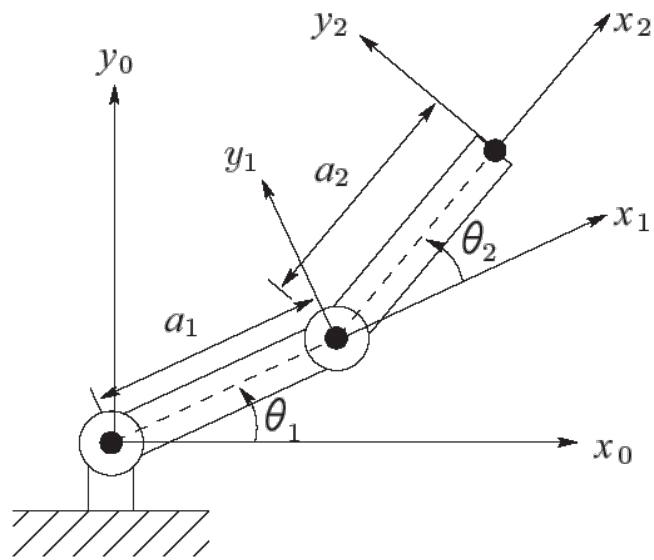
Singularities

- **This is equivalent to the following:**

$$a_1 s_1 + a_2 s_{12} = \alpha(a_2 s_{12})$$
$$a_1 c_1 + a_2 c_{12} = \alpha(a_2 c_{12})$$

- **Thus if $s_{12} = s_1$, we can always find an α that will reduce the rank of J**
- **Thus $\theta_2 = 0, \pi$ are two singularities**

$$\alpha = \frac{a_1 + a_2}{a_2} \qquad \alpha = \frac{a_2 - a_1}{a_2}$$



Determining Singular Configurations

- **In general, all we need to do is observe how the rank of the Jacobian changes as the configuration changes**
- **But it is not always as easy as the last example to observe how the rank changes**
- **There are some shortcuts for common manipulators: decoupling singularities**
  - **Analogous to kinematic decoupling**
  - **Assume that we have a 6DOF manipulator and that we can break the Jacobian into a block form**
  - **Then we can separate singularities into *arm singularities* and *wrist singularities***

Decoupling of Singularities

- **Assume that we have a 6DOF manipulator that has a 3-axis arm and a spherical wrist**
  - **thus the Jacobian is 6x6 and the maximum rank *J* can have is 6**
  - **Now we can say that the manipulator is in a singular configuration iff det(*J*(*q*)) = 0**
- **For the case of a kinematically decoupled manipulator, we can break up the Jacobian as follows:**
  - **Where J_p and J_o are represent the position and orientation portions of the Jacobian**
  - **J_o is given by the following:**

$$J = \begin{bmatrix} J_p & J_o \end{bmatrix}$$

$$J_o = \begin{bmatrix} z_3 \times (o_6 - o_3) & z_4 \times (o_6 - o_4) & z_5 \times (o_6 - o_5) \\ z_3 & z_4 & z_5 \end{bmatrix}$$

Decoupling of Singularities

- **Now, one further assumption: $o_3 = o_4 = o_5 = o_6 = o$**
  - **This allows us to note the form of Jo:**

$$J_o = \begin{bmatrix} 0 & 0 & 0 \\ z_3 & z_4 & z_5 \end{bmatrix}$$

  - **And we can split the total manipulator Jacobian as follows:**

$$J = \begin{bmatrix} J_{11} & 0 \\ J_{21} & J_{22} \end{bmatrix}$$

  - **Thus we can say:**

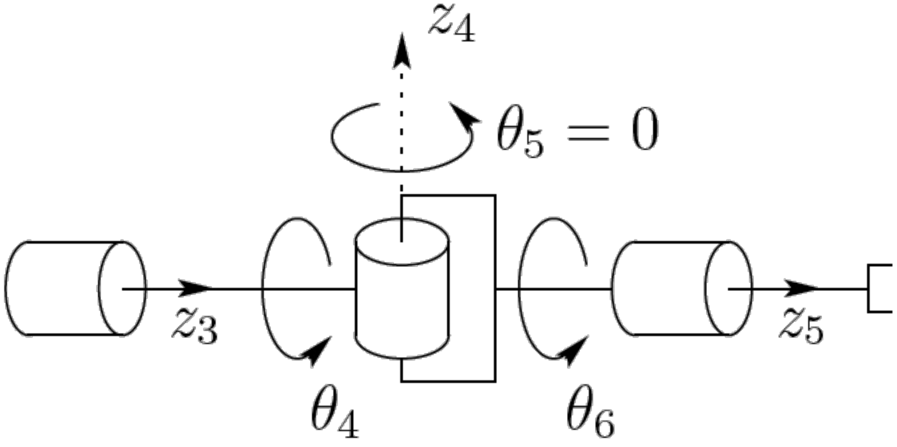
$$\det(J) = \det(J_{11})\det(J_{22})$$

Wrist singularities

- To determine the wrist singularities, we observe the determinant of J_{22}

$$J_{22} = [z_3 \quad z_4 \quad z_5]$$

- Thus the J_{22} has rank 3 when the three axes are linearly independent
  - This is always true, except when two of the axes are collinear
  - i.e. $\theta_5 = 0, \pi$ are the singularities for a spherical wrist



Arm singularities

- **To determine the arm singularities, we observe the determinant of J_{11}**
  - **First, if the i^{th} joint is revolute, the i^{th} column is J_{11} is given as follows:**

$$J_{11,i} = [z_{i-1} \times (o - o_{i-1})]$$

  - **First, if the i^{th} joint is prismatic, the i^{th} column is J_{11} is given as follows:**

$$J_{11,i} = [z_{i-1}]$$

- **We will now give examples for the common configurations we have been using: elbow, spherical, and SCARA manipulators**

Ex: elbow manipulator

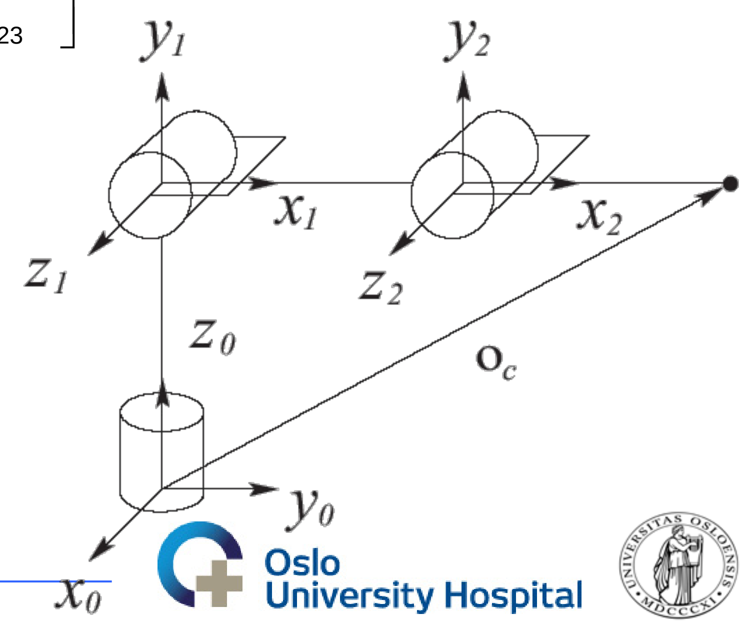
- **To determine the arm singularities, we observe the determinant of J_{11}**
  - **First, J_{11} is given as follows:**

$$J_{11} = [z_0 \times (o_c - o_0) \quad z_1 \times (o_c - o_1) \quad z_2 \times (o_c - o_2)]$$

$$= \begin{bmatrix} -a_2 s_1 c_2 - a_3 s_1 c_{23} & -a_2 s_2 c_1 - a_3 s_{23} c_1 & -a_3 c_1 s_{23} \\ a_2 c_1 c_2 + a_3 c_1 c_{23} & -a_2 s_1 s_2 - a_3 s_1 s_{23} & -a_3 s_1 s_{23} \\ 0 & a_2 c_2 + a_3 c_{23} & a_3 c_{23} \end{bmatrix}$$

  - **The determinant of J_{11} is:**

$$\det(J_{11}) = a_2 a_3 s_3 (a_2 c_2 + a_3 c_{23})$$

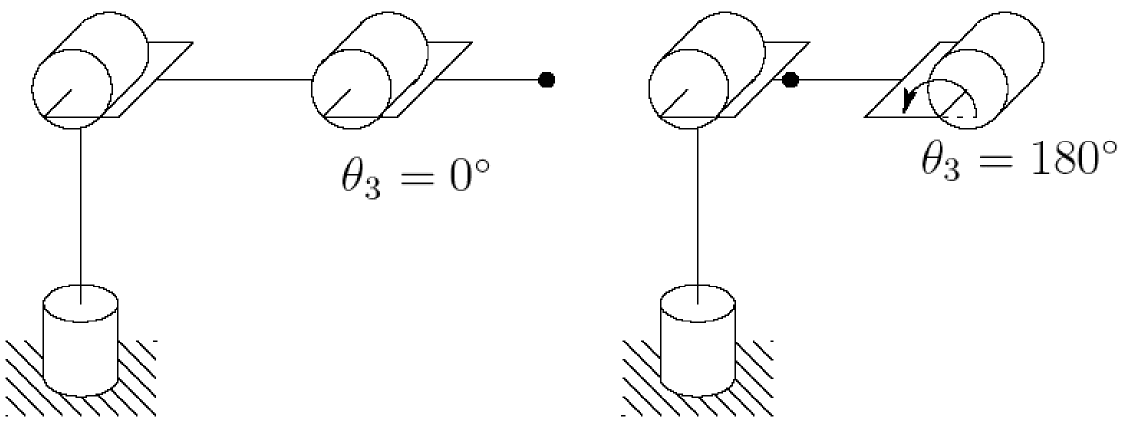


Ex: elbow manipulator

- **The determinant of J_{11} is:**

$$\det(J_{11}) = a_2 a_3 s_3 (a_2 c_2 + a_3 c_{23})$$

  - **Thus the arm is singular when $s_3 = 0$, i.e. $\theta_3 = 0, \pi$**
  - **This corresponds to the elbow being fully extended or fully retracted:**

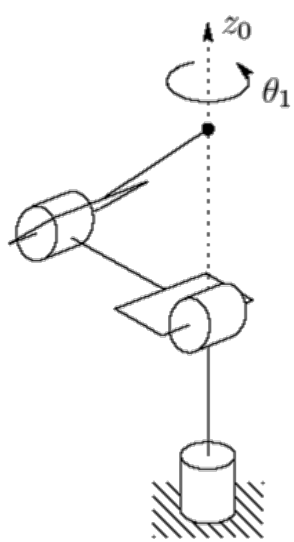


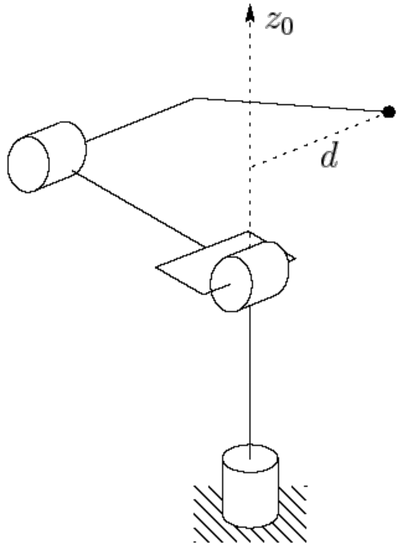
Ex: elbow manipulator

- **The determinant of J_{11} is:**

$$\det(J_{11}) = a_2 a_3 s_3 (a_2 c_2 + a_3 c_{23})$$

  - **Thus the arm is also singular when $a_2 c_2 + a_3 c_{23} = 0$**
  - **This corresponds to the wrist center intersecting the z_0 axis:**

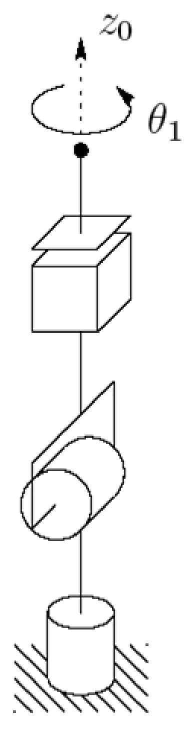




  - **But this is not possible if there is a shoulder offset:**

Ex: spherical manipulator

- **Since there is no 'elbow', the only singularity is when the wrist center intersects the base axis**



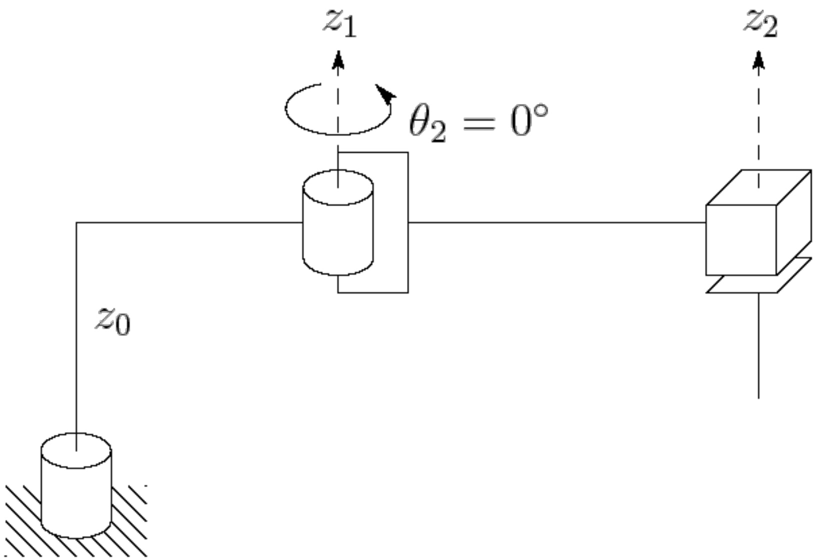
Ex: SCARA manipulator

- **First, we observe the construction of the Jacobian:**

$$J_{11} = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_1 s_{12} & 0 \\ a_1 c_1 + a_2 c_{12} & a_1 c_{12} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

- **The determinant is:**

$$\begin{aligned} \det(J_{11}) &= a_1^{\,2} c_1 s_{12} - a_1^{\,2} s_1 c_{12} \\ &= a_1^{\,2}(c_1 s_{12} - s_1 c_{12}) \\ &= a_1^{\,2}(c_1(s_1 c_2 + c_1 s_2) - s_1(c_1 c_2 - s_1 s_2)) \\ &= a_1^{\,2} s_2 \end{aligned}$$



- **Thus, the SCARA is singular**

for $s_2 = 0$, i.e. $\theta_2 = 0, \pi$

Force/torque relationships

- Similar to the relationship between the joint velocities and the end effector velocities, we are interested in expressing the relationship between the joint torques and the forces and moments at the end effector
  - Important for dynamics, force control, etc
- Let the vector of forces and moments at the end effector be represented as: $F = \begin{bmatrix} F_x & F_y & F_z & n_x & n_y & n_z \end{bmatrix}^T$
- Then we can express the joint torques, τ, as:

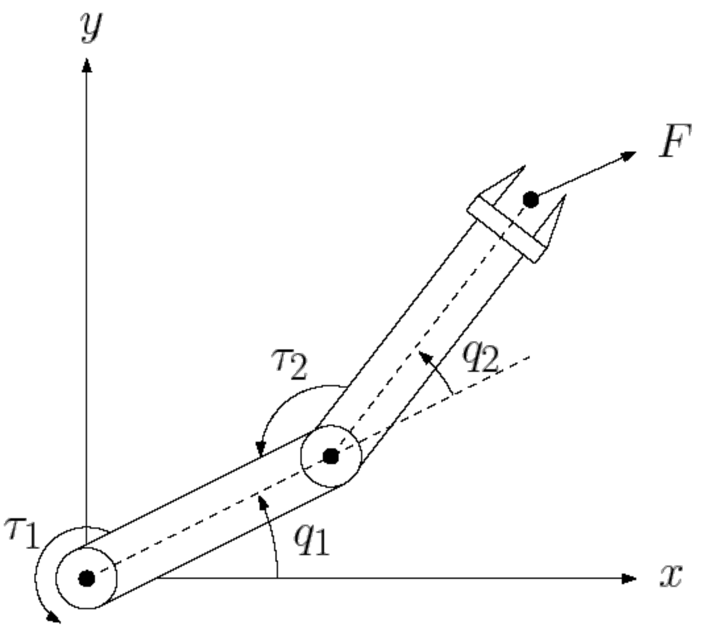
$$\tau = J^T(q)F$$

- We will derive this using the principal of virtual work when we discuss the dynamics of manipulators

Force/torque relationships

- **Example: for a force *F* applied to the end of a planar two-link manipulator, what are the resulting joint torques?**
  - **First, remember that the Jacobian is:**

$$J(q) = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

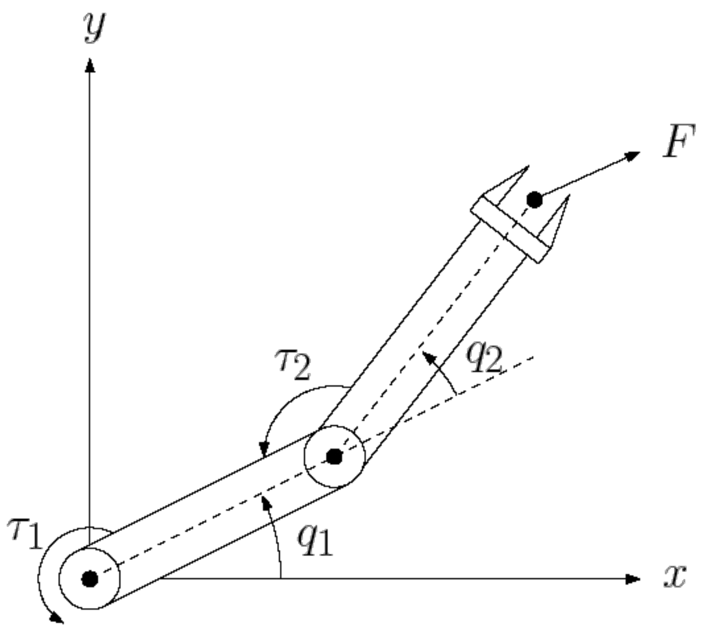


Force/torque relationships

- **Example: for a force *F* applied to the end of a planar two-link manipulator, what are the resulting joint torques?**
  - **Thus the joint torques are:**

$$\tau = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & a_1 c_1 + a_2 c_{12} & 0 & 0 & 0 & 1 \\ -a_2 s_{12} & a_2 c_{12} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \\ n_x \\ n_y \\ n_z \end{bmatrix}$$

$$= \begin{bmatrix} F_x(-a_1 s_1 - a_2 s_{12}) + F_y(a_1 c_1 + a_2 c_{12}) \\ F_x(-a_2 s_{12}) + F_y(a_2 c_{12}) \end{bmatrix}$$



Inverse velocity

- **We have developed the Jacobian as a mapping from joint velocities to end effector velocities:**

$$\xi = J\dot{q}$$

- **Now we want the inverse: what are the joint velocities for a specified end effector velocity?**
- **Simple case: if the Jacobian is square and nonsingular,**

$$\dot{q} = J^{-1}\xi$$

- **In all other cases, we need another method**
- **For systems that do not have exactly 6DOF, we cannot directly invert the Jacobian**
- **Thus there is only a solution to finding the joint velocities if ξ is in the range space of *J***

Inverse velocity

- Take the case of a manipulator with more than 6 joints
  - i.e. $n > 6$
- We can solve for the joint velocities using the right pseudo inverse
- For $J \in \mathbf{R}^{m \times n}$ with $m < n \Rightarrow \text{rank}(J) \leq m$
- If the manipulator is nonsingular, rank(J) = m and $(JJ^T)^{-1}$ exists
  - Thus we can write:

$$\begin{aligned} I &= \left(JJ^T\right)\left(JJ^T\right)^{-1} \\ &= J\left[J^T\left(JJ^T\right)^{-1}\right] \\ &= JJ^+ \end{aligned}$$

  - Where J^+ is the right pseudo inverse of J
  - Thus the solution for the joint velocities (with minimum norm) is:

$$\dot{q} = J^+ \xi$$

Inverse velocity

- **How do we construct J^+? Using SVD:**
  - **Generalization of methods that we would use for square matrices**
  - **We can write any *m* x *n* matrix *J* as a composition of three matrices:**

$$J = U\Sigma V^T$$

  - **Where the matrix *U* is *m* x *m* and contains the eigenvectors of JJ^T as its columns and Σ is a matrix that contains the singular values:**

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & & 0 \\ & \sigma_2 & & & & 0 \\ & & \cdot & & & 0 \\ & & & \cdot & & 0 \\ & & & & \sigma_m & 0 \end{bmatrix}$$

  - **And the singular values σ_i are the square roots of the eigenvalues of JJ^T:**

$$\sigma_i = \sqrt{\lambda_i}$$

Inverse velocity

- **Now *J⁺* is given by:**

$$J^{+} = V\Sigma^{+}U^{T}$$

  - **Where *Σ⁺* is:**

$$\Sigma = \begin{bmatrix} \sigma_1^{-1} & & & & 0 \\ & \sigma_2^{-1} & & & 0 \\ & & \cdot & & 0 \\ & & & \cdot & 0 \\ & & & \sigma_m^{-1} & 0 \end{bmatrix}^T$$

Next class...

- **Introduction to dynamics**