UiO : Department of Technology Systems
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Lecture 11 - Control Theory

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Control of Manipulators and Mobile Robots

UNIK4490/TEK4030

Lecture overview

- General introduction to control theory
  - Motivation
    - Self regulating systems (pendulum at equilibrium)
    - Unstable systems (car speed)
  - Open and closed loop systems
    - Open loop (washing machine)
      - Feed forward (car speed – incline as disturbance model)
    - Closed loop
      - Feedback (cruise control)
  - Stability
    - Stable systems
    - Unstable systems

*

Definition input/output stability

- Asymptotically stable if:
  - $y \rightarrow 0$ when $t \rightarrow \infty$ and u has a finite duration and amplitude
- Marginally stable if:
  - $|y| < \infty$ for all $t \geq 0$ and u has a finite duration and amplitude
- Unstable otherwise

Robot Control

- We want to control the joint positions (configuration) of a robot
  - Therefore we need to figure out how we can determine the motor/actuator inputs so as to command the robot to a desired configuration
- In general, the input (voltage/current) does not create instantaneous motion to a desired configuration because of the robots dynamical properties
- There are also other real world elements that affect the robots motion like backlash (clearance between gear tooths - causes hysteresis) and the properties of the motor/actuator

Robot Control

- We shall focus on single input single output systems (SISO) meaning that we only look at each joint by itself

- We therefore assume that each joint is affected only by itself
- This means that the contributions from the other joints are treated as a disturbance

- Before we can start to control a joint, we need to model the joint and look at its properties.

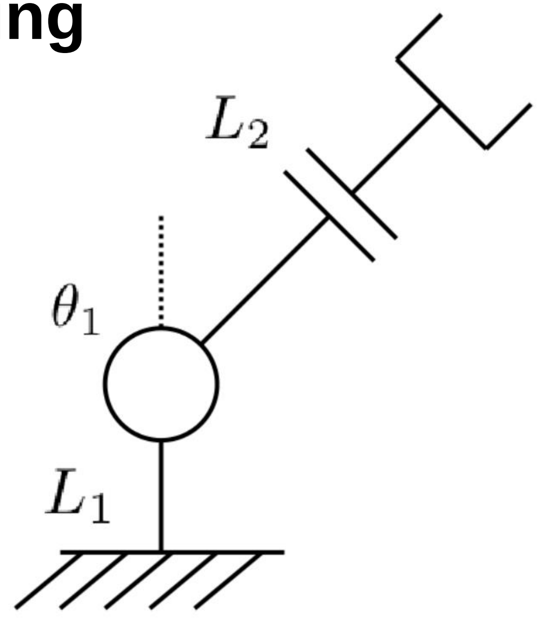
Robot control - Modelling

- From dynamics we have the ordinary differential equation (ODE) for the robot. We have *n* non-linear equations as a result of *n* joint variables, all expressed in the time domain.
- These *n* equations often depends on each other

- We want to transform these equations into *n* linear, independent equations

- We do this by treating the non-linear effects and the dependence of other joints as disturbance

Example – Robot modeling



The dynamic equations are given as

$$m L_2^2 \ddot{\theta}_1 + 2 m L_2 \dot{L}_2 \dot{\theta}_1 - m g s_1 L_2 = \tau_1$$
$$m \ddot{L}_2 - m L_2 \dot{\theta}_1^2 - m g c_1 = F_2$$

We shall now give a linearization of the first equation

Robot control - Modeling

- We now have a set of independent linear equations
- We want to analyse the properties of these systems
- Laplace transform can be used to transform the equations into the frequency domain
  - This transforms the equations from a ordinary differential equations into a linear equations, which easier to solve
  - We can analyse important system properties in the frequency domain, such as stability and step response
- Block diagrams can be used to visualize the equations for the system and possible feedback loops

Laplace transform

- The Laplace transform is defined as follows:

$$x(s) = \int_0^\infty e^{-st} x(t) dt$$

- For example, Laplace transform of a derivative:

$$L\{\dot{x}(t)\} = L\left\{\frac{dx(t)}{dt}\right\} = \int_0^\infty e^{-st} \frac{dx(t)}{dt} dt$$

  – Integrating by parts:

$$L\left\{\frac{dx(t)}{dt}\right\} = e^{-st} x(t)\Big|_0^\infty + s\int_0^\infty e^{-st} x(t) dt$$
$$= sx(s) - x(0)$$

Laplace transform

- Similarly, Laplace transform of a second derivative:

$$L\{\ddot{x}(t)\} = L\left\{\frac{d^2x(t)}{dt^2}\right\} = \int_0^\infty e^{-st}\frac{d^2x(t)}{dt^2}dt = s^2x(s) - sx(0) - \dot{x}(0)$$

- Thus, if we have a generic 2nd order system described by the following ODE:

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = F(t)$$

- And we want to get a transfer function representation of the system, take the Laplace transform of both sides:

$$mL\{\ddot{x}(t)\} + bL\{\dot{x}(t)\} + kL\{x(t)\} = L\{F(t)\}$$
$$m\left(s^2x(s) - sx(0) - \dot{x}(0)\right) + b\left(sx(s) - x(0)\right) + kx(s) = F(s)$$

Laplace transform

- Continuing:

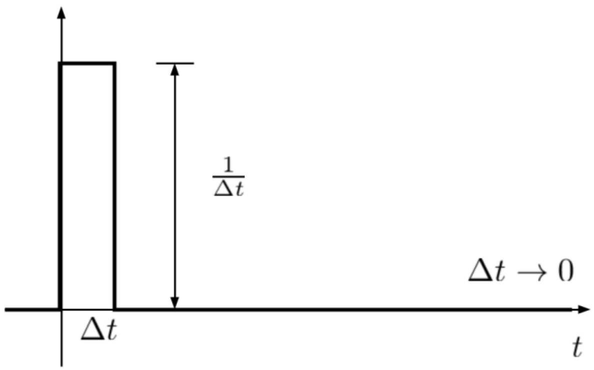
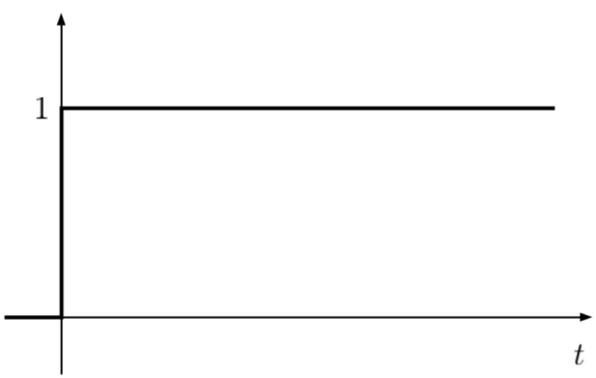
$$\left(ms^2 + bs + k\right)x(s) = F(s) + m\dot{x}(0) + (ms + c)x(0)$$

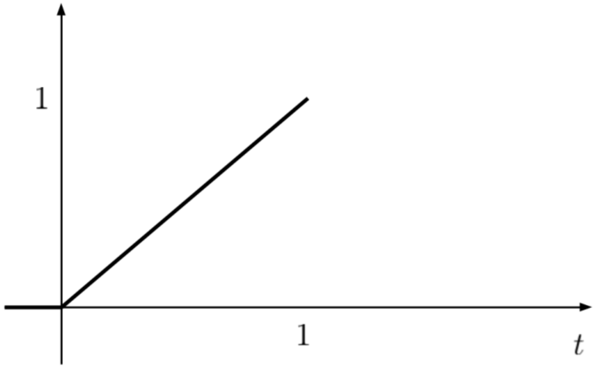
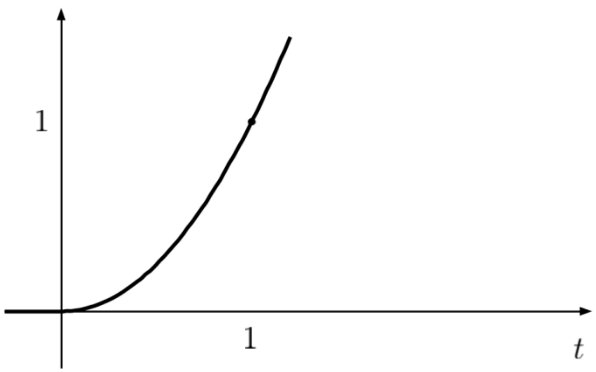
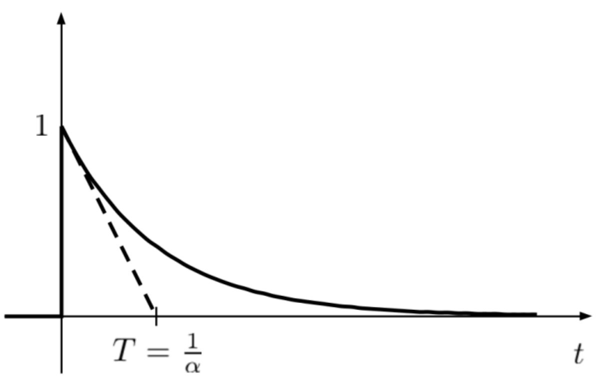
- The *transient response* is the solution of the above ODE if the *forcing function* $F(t) = 0$

- The steady state response is the solution of the above equation if the initial conditions are zero
- This yields the equation

$$\left(ms^2 + bs + k\right)x(s) = F(s)$$

Common Laplace functions

	$\delta(t) = \lim_{\Delta t \to 0} g(t, \Delta t)$ $g(t, \Delta t) = \begin{cases} \frac{1}{\Delta t} & , 0 < t < \Delta t \\ 0 & , 1 < 0, t > \Delta t \end{cases}$	1
	$f(t) = \begin{cases} 1 & , t \geq 0 \\ 0 & , t < 0 \end{cases}$	$\frac{1}{s}$

	$f(t) = \begin{cases} t & ,t \geq 0 \\ 0 & ,t < 0 \end{cases}$	$\frac{1}{s^2}$
	$f(t) = \begin{cases} t^2 & ,t \geq 0 \\ 0 & ,t < 0 \end{cases}$	$\frac{1}{s^3}$
	$f(t) = \begin{cases} e^{-\alpha t} & ,t \geq 0 \\ 0 & ,t < 0 \end{cases}$	$\frac{1}{s+\alpha} = \frac{T}{1+Ts}$

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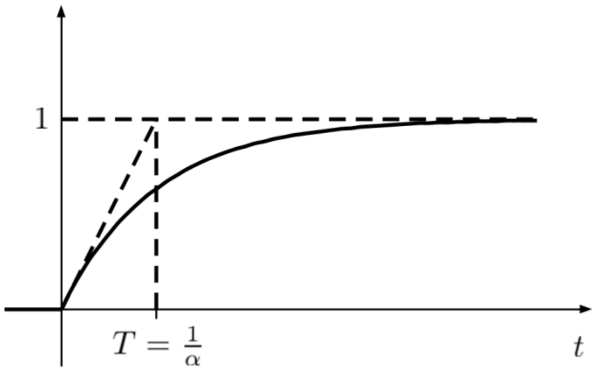
<td>
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<td>$$f(t) = \begin{cases} 1 - e^{-\alpha t} & , t \geq 0 \\ 0 & , t < 0 \end{cases}$$</td>
<td>$$\frac{\alpha}{s(s+\alpha)} = \frac{1}{s(1+Ts)}$$</td>
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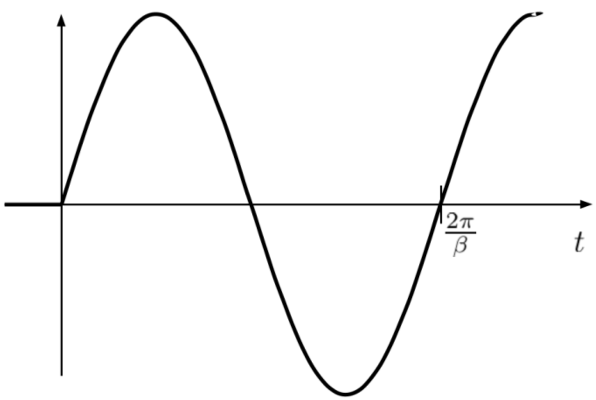
<td>
</td>
<td>$$f(t) = \begin{cases} \sin \beta t & , t \geq 0 \\ 0 & , t < 0 \end{cases}$$</td>
<td>$$\frac{\beta}{s^2+\beta^2} = \frac{\beta}{(s+j\beta)(s-j\beta)}$$</td>
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</table>

Review of the Laplace transform

- Properties of the Laplace transform
  - Takes an ODE to a algebraic equation
  - Differentiation in the time domain is multiplication by s in the Laplace domain
  - Integration in the time domain is multiplication by $1/s$ in the Laplace domain
  - Considers initial conditions
    - i.e. transient and steady-state response
  - The Laplace transform is a linear operator

Transfer functions

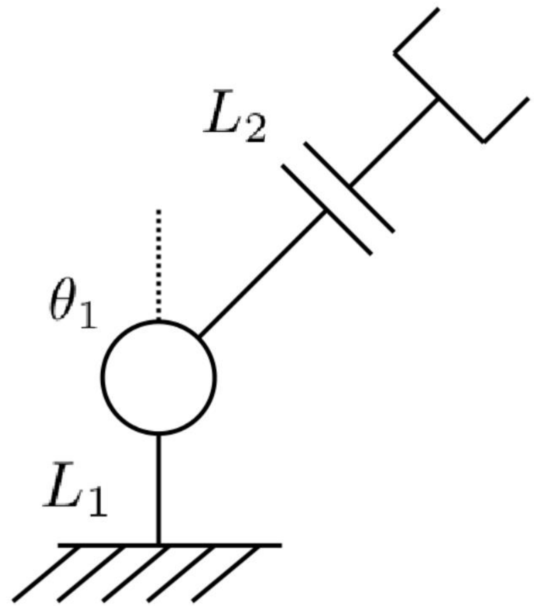
- When all initial conditions of a Laplace transform are zero, the response Y(s) of a linear system is given by its input X(s) and its transfer function H(s)

$$Y(s) = H(s)\,X(s)$$

$$\frac{Y(s)}{X(s)} = H(s)$$

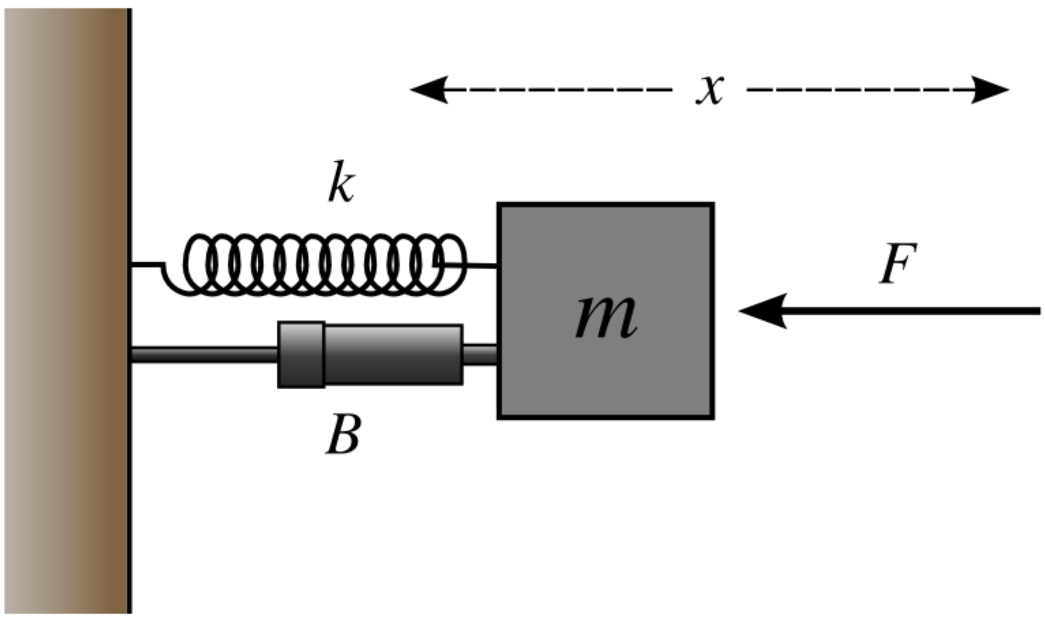
- The denominator (X(s)) of the transfer function is called the characteristic polynomial

Example cont. – Robot modeling



We can now derive the transfer functions for the robot in the frequency domain

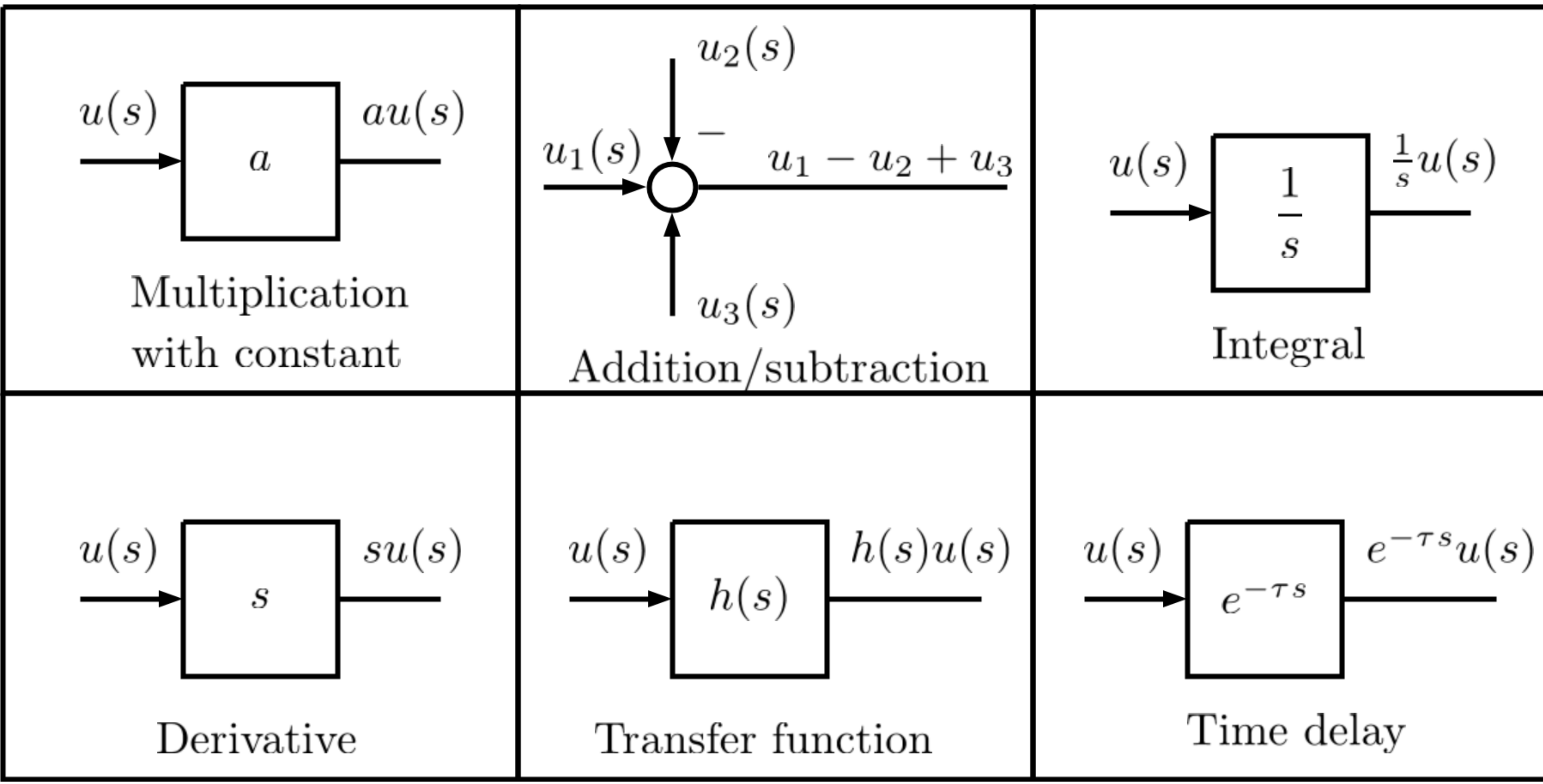
Example – Mass spring damper system



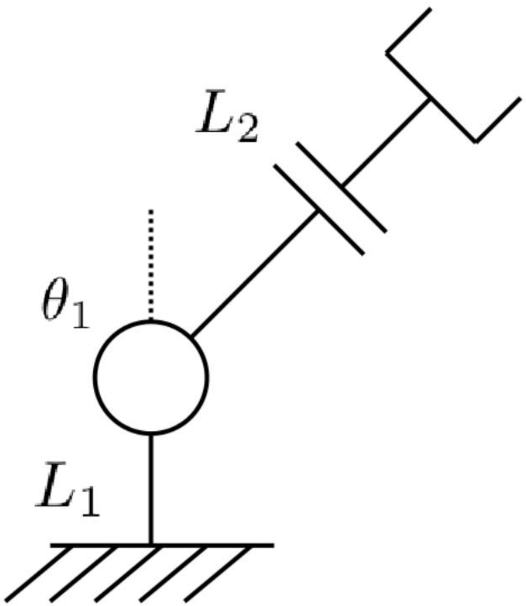
Block diagrams

- A block diagram visalizes one or more equations
- Makes it easier to see feedback and feed forward loops

Block diagram symbols



Example cont. – Robot modeling



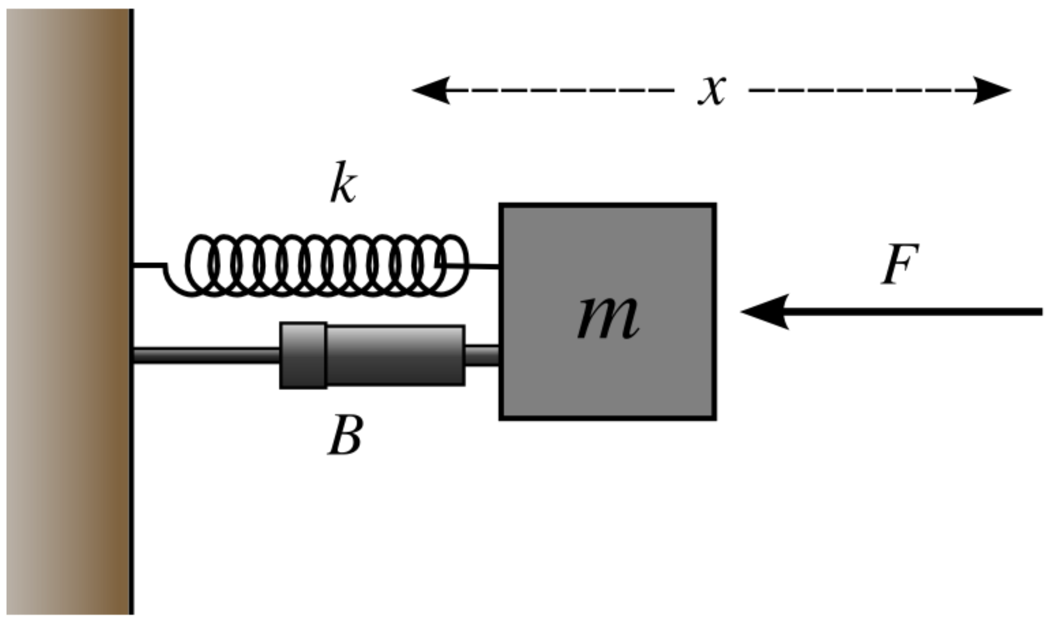
Drawing a block diagram

The transfer function as given as:

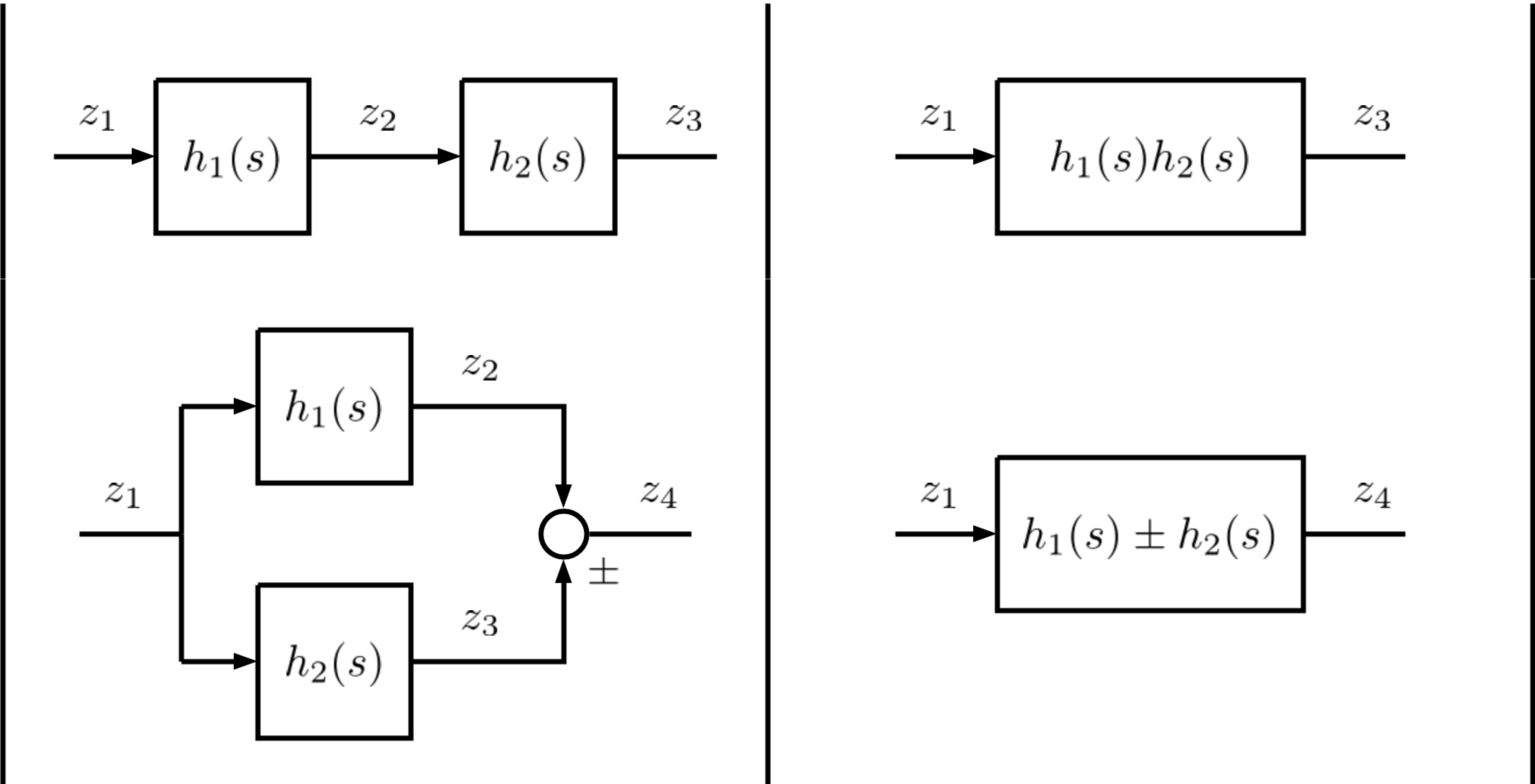
$$Js^2\theta_1 - G\theta_1 + D = \tau_1$$

Drawing a block diagram of a mass spring damper system

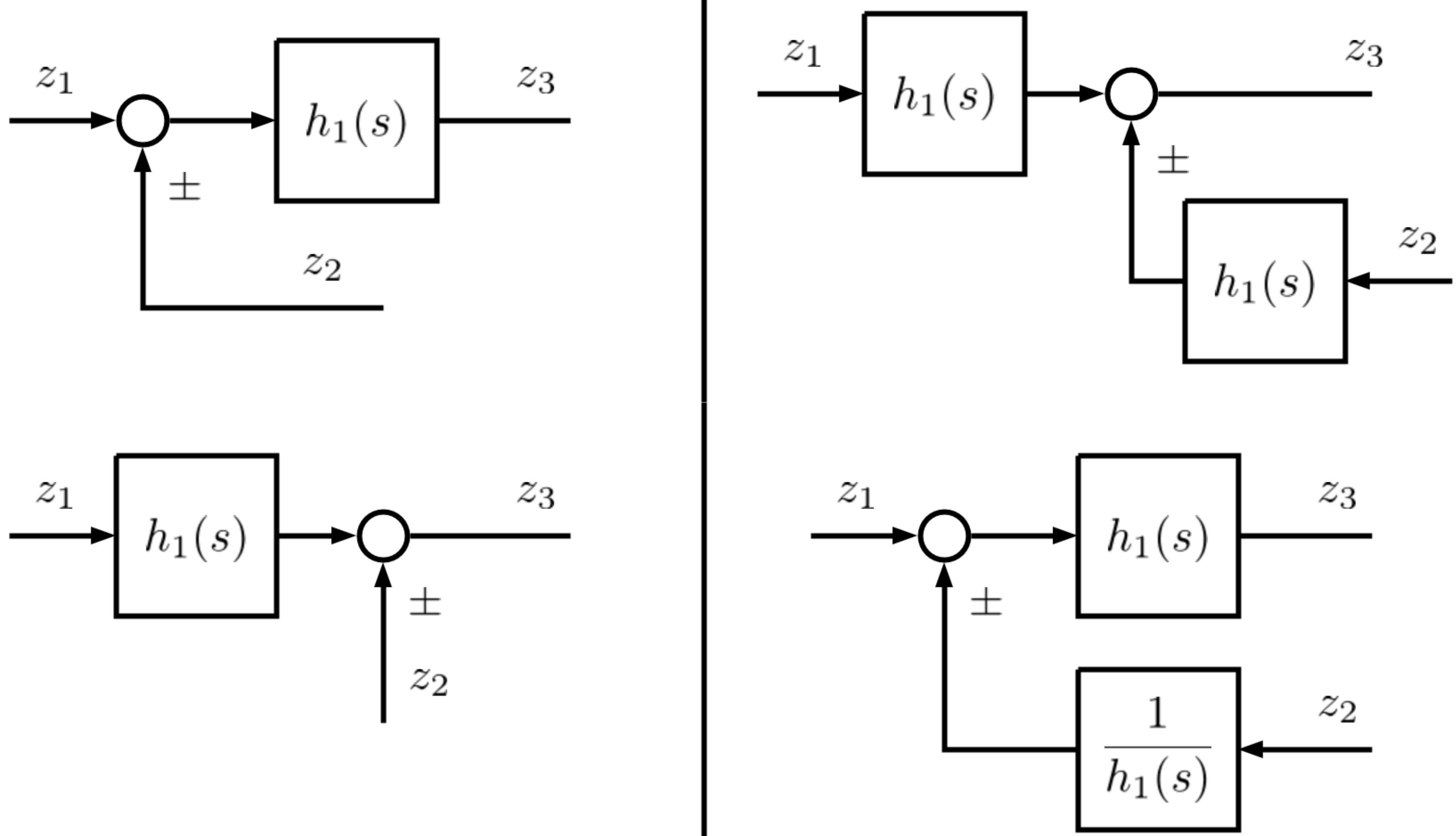
- $xs^2 = \frac{1}{m}(u - fxs - kx)$



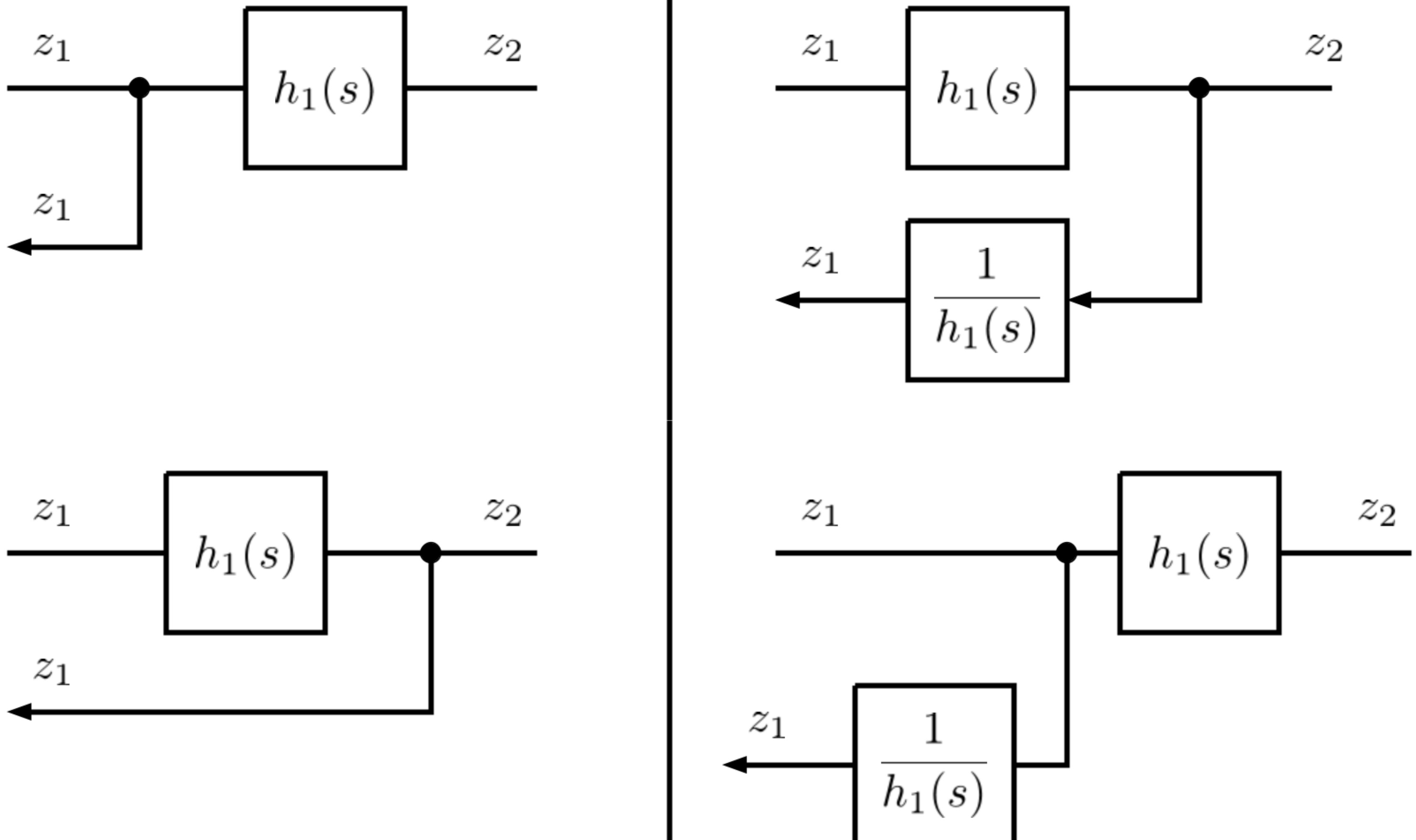
Manipulation of block diagrams



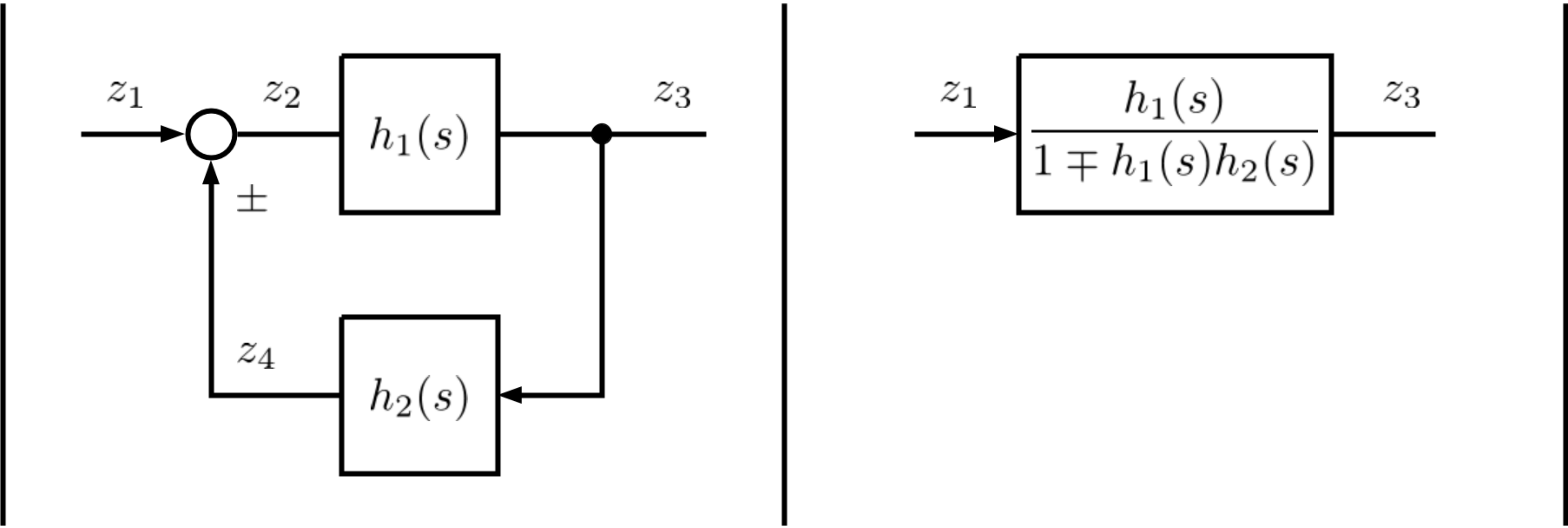
Manipulation of block diagrams



Manipulation of block diagrams



Manipulation of block diagrams



Prove on blackboard

Zeros and poles of the transfer functions

- For rational transferfunctions we denote the roots of the nominator zeros and roots of the denominator poles
- The poles gives important characteristics about the transfer function

$$h(s) = \frac{\rho_p s^p + \cdots + \rho_1 s^1 + \rho_0}{s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0}$$

$$h(s) = \frac{\rho_p (s - v_1) \cdots (s - v_p)}{(s - \lambda_1) \cdots (s - \lambda_n)}$$

- We call the polynomial in the denominator the characteristic polynomial

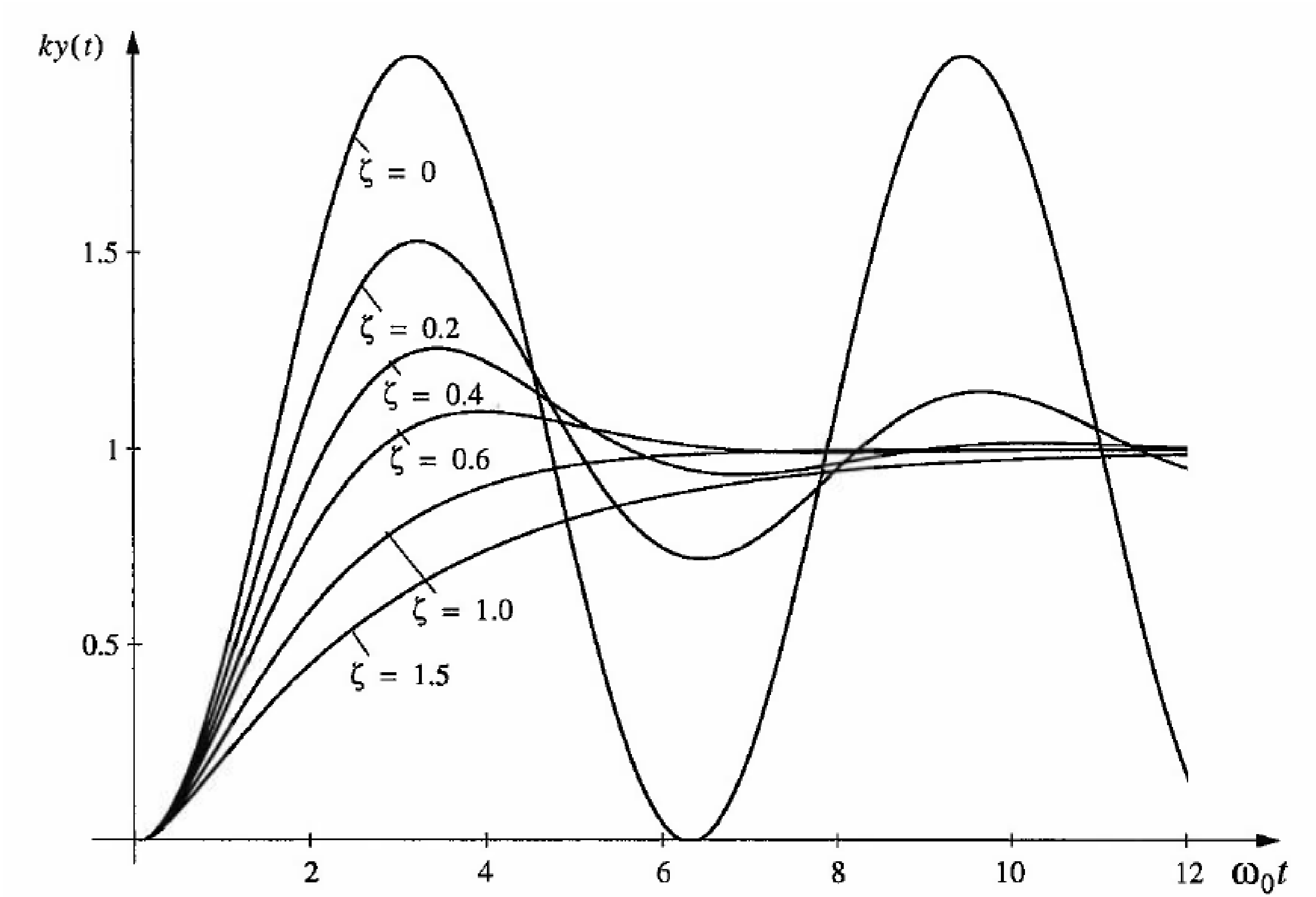
Examples – Finding roots and zeros

- Example 1: Given transfer function
- Example 2: Mass-spring-damper system

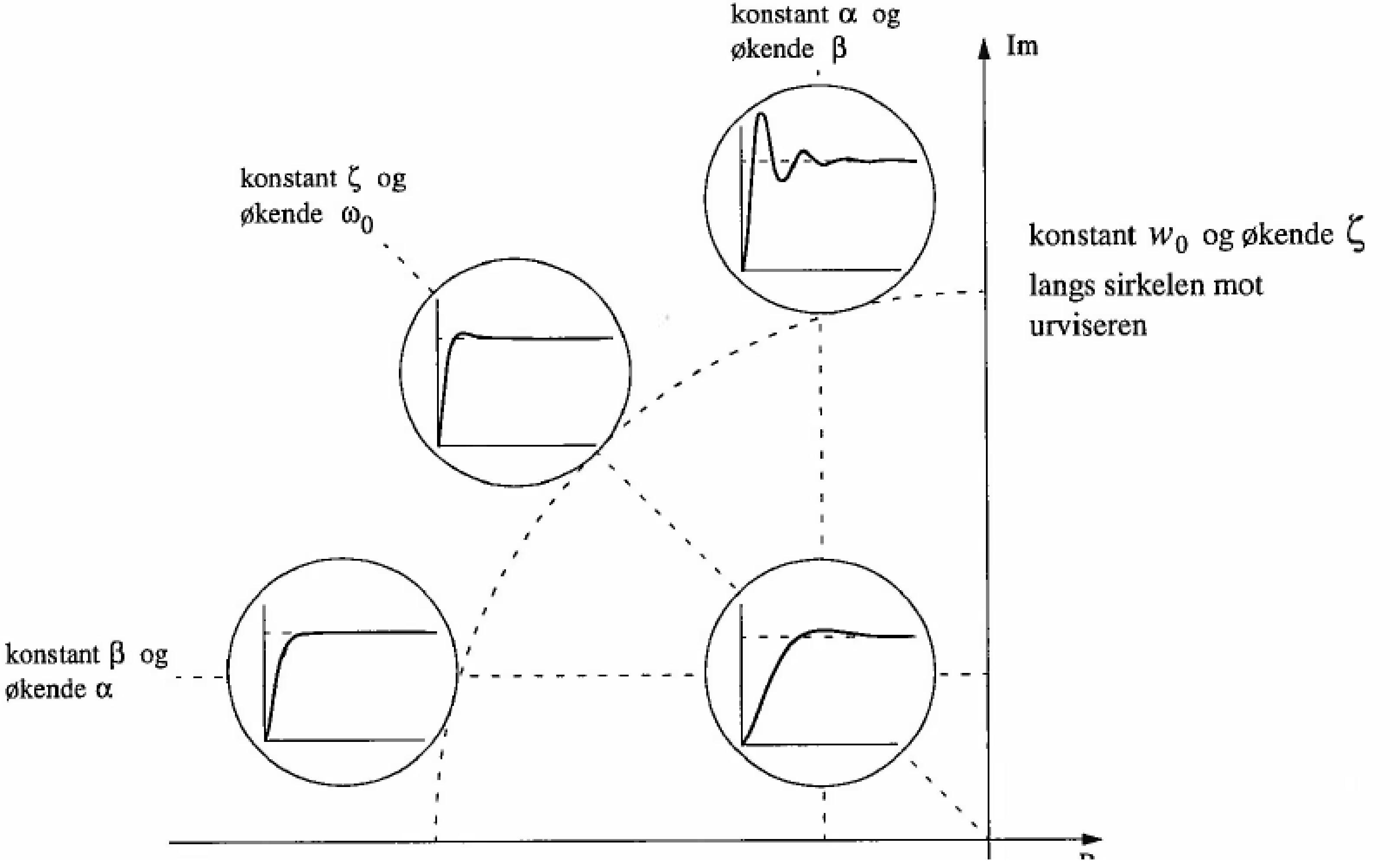
Example cases

- Three cases depending on the poles
- Case I: Poles are real and distinct
  - Over-damped system
- Case II: Poles are real and equal
  - Critically damped system
- Case III: Poles are complex conjugates
  - Under-damped system

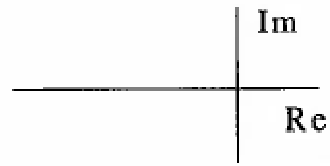
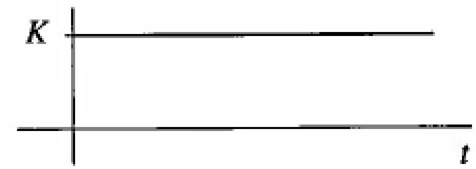
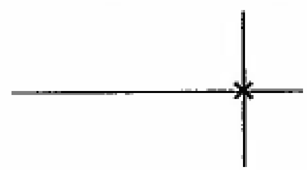
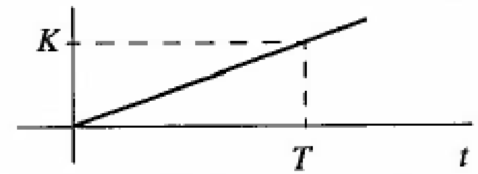
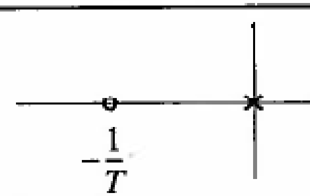
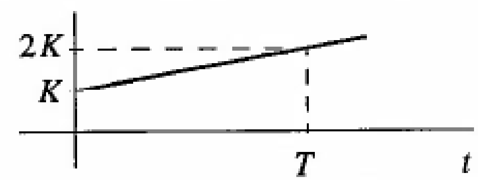
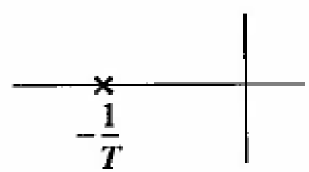
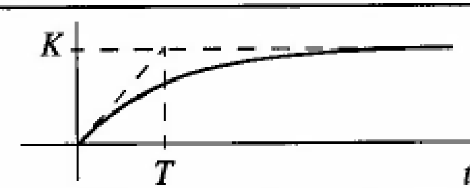
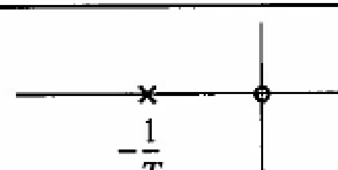
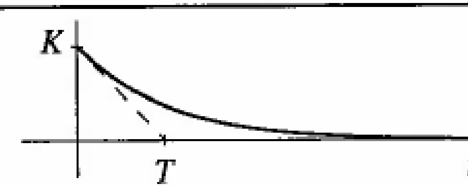
Time responses

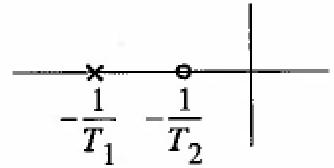
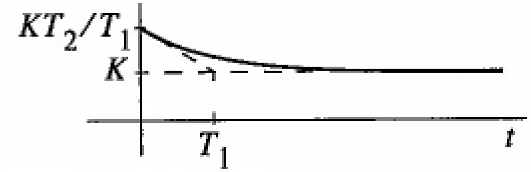
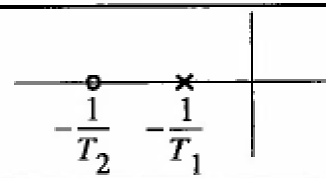
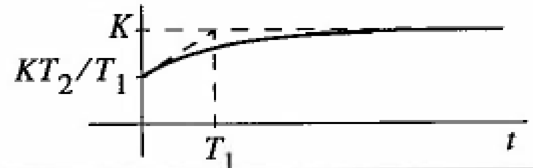
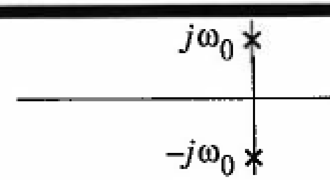
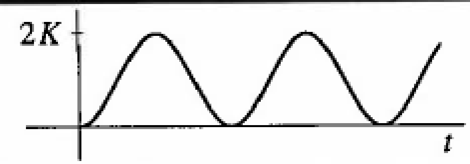
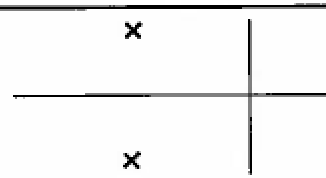
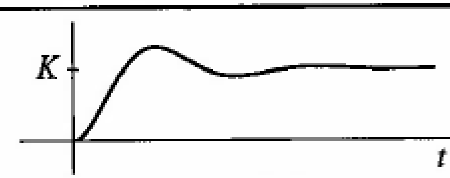
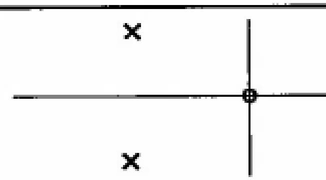
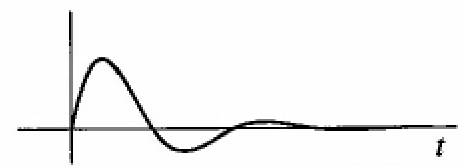
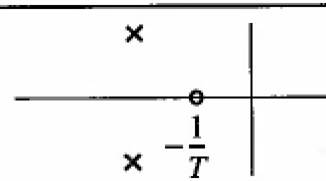


Effect of changes in poles



Common transfer functions and their poles and step respones

Transfer-funksjon $h(s)$	Nullpunkter og poler	Sprangrepons
K		
$K\frac{1}{Ts}$		
$K\frac{1+Ts}{Ts}$		
$K\frac{1}{1+Ts}$		
$K\frac{Ts}{1+Ts}$		

Transfer-funksjon $h(s)$	Nullpunkter og poler	Sprangrepons
$K\frac{1+T_2 s}{1+T_1 s}$ $T_2 > T_1$		
$K\frac{1+T_2 s}{1+T_1 s}$ $T_2 < T_1$		
$\frac{K}{1+\left(\frac{s}{\omega_0}\right)^2}$		
$\frac{K}{1+2\zeta\frac{s}{\omega_0}+\left(\frac{s}{\omega_0}\right)^2}$		
$\frac{Ks}{1+2\zeta\frac{s}{\omega_0}+\left(\frac{s}{\omega_0}\right)^2}$		
$\frac{K(1+Ts)}{1+2\zeta\frac{s}{\omega_0}+\left(\frac{s}{\omega_0}\right)^2}$		

Root locus plots

- Instead of using constant values in the transfer function we now assume that we can *vary* one (or more) parameters
- Changing this parameter will move the poles and zeros *in the complex plane*
- The paths of zeros and poles in the complex plane as a function of changed controller parameters are called **root locus plots**
- We can investigate the stability of our system by looking at how these poles moves based on the control parameters

Example - Root Locus Plot

Drawing on blackboard

Stability – Frequency domain

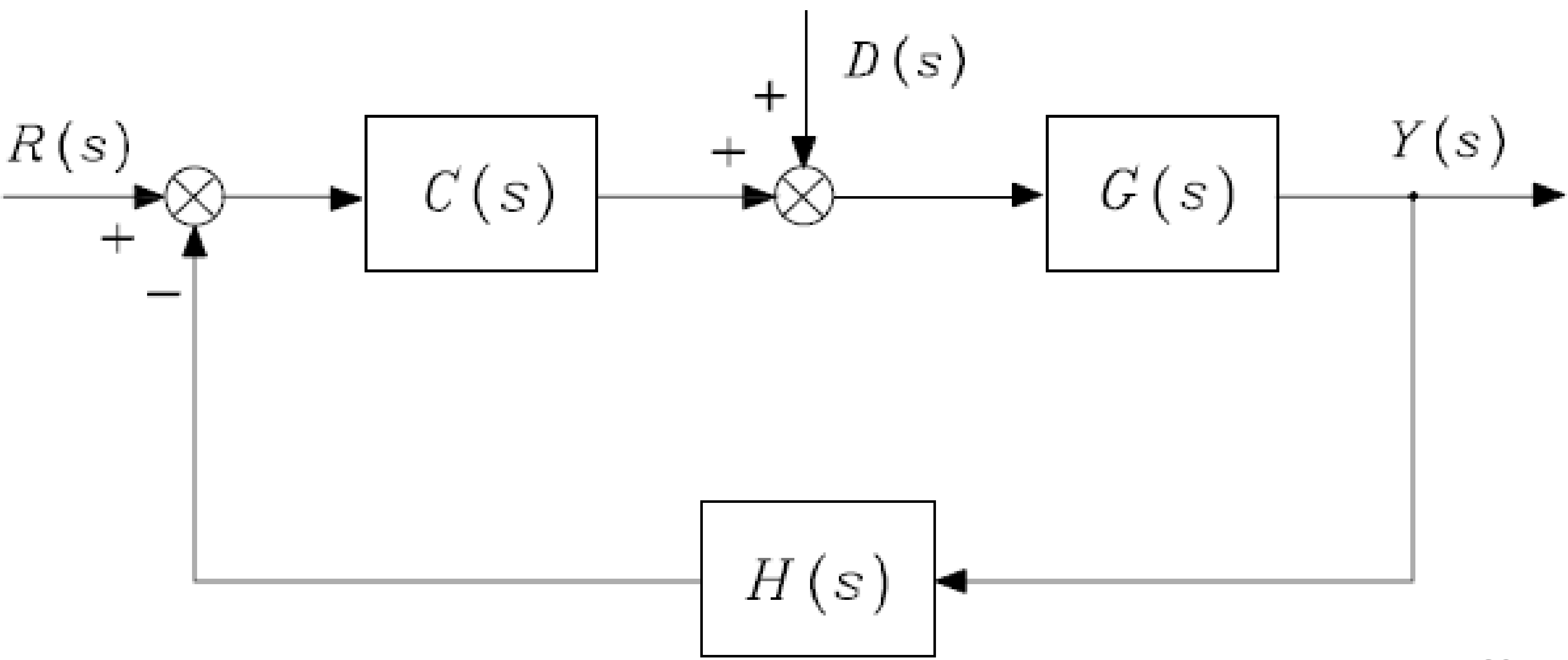
- Find the poles (λ_i) of the transfer function
- If $Re(\lambda_i) < 0$ for all λ_i in H(s) the system is *asymptotically stable*
- If one or more poles has $Re(\lambda_i) = 0$, but they are not in the same point the system is *marginally stable*
- If on or more poles has $Re(\lambda_i) > 0$ the system is *unstable*

Example - Stability

Drawing on blackboard

Feedback systems

- G(s) – System Model
- C(s) – Controller
- D(s) – Disturbance
- H(s) – Transducer (sensor model)
- R(s) – Reference Input
- Y(s) – Output Variable



Feedback systems

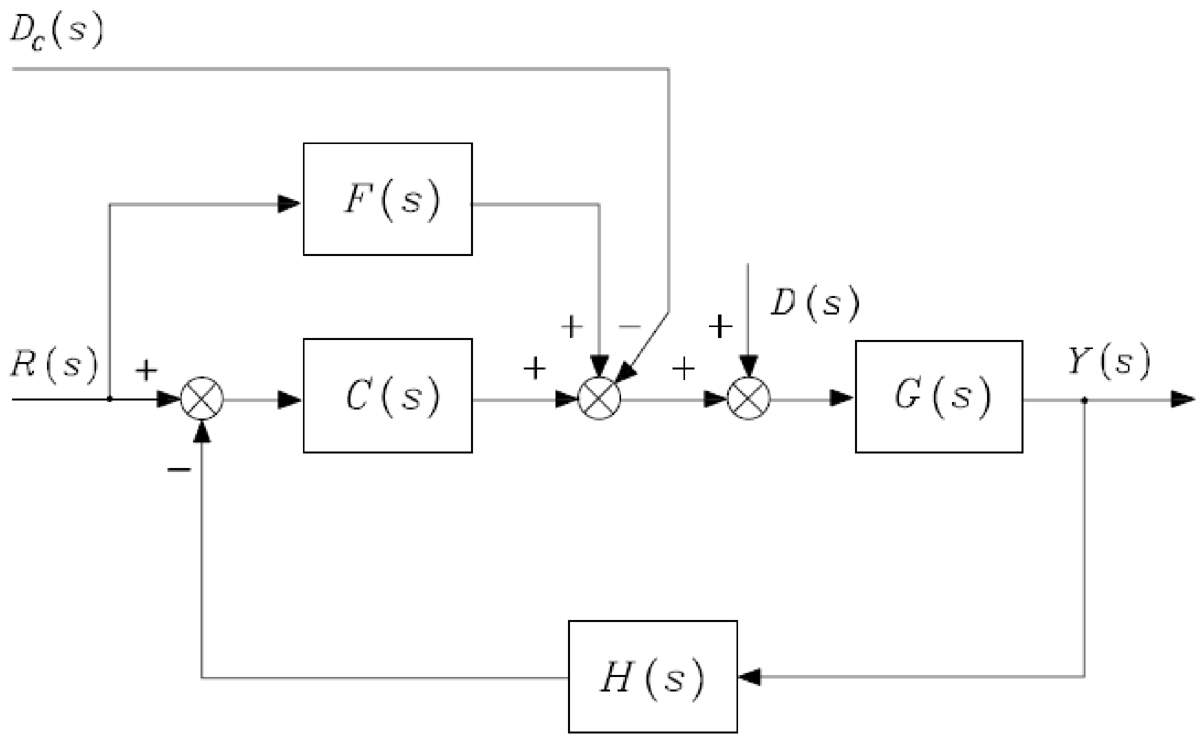
$$Y(s) = W(s)R(s) + W_D(s)D(s),$$

$$W(s) = \frac{C(s)G(s)}{1 + C(s)G(s)H(s)}$$

$$W_D(s) = \frac{G(s)}{1 + C(s)G(s)H(s)}$$

Feedback + Feedforward

- We want to regulate the system modeling error, therefore we add the feed forward parts F(s) and Dc(s), where Dc(s) is a model of the systems disturbances.
- Assuming we can model the system accurately a convenient choice is F(s) = 1/G(s)
- This decreases the systems time response (feedback systems can be slower)



Feedback + Feedforward

- The systems transfer function is then given as

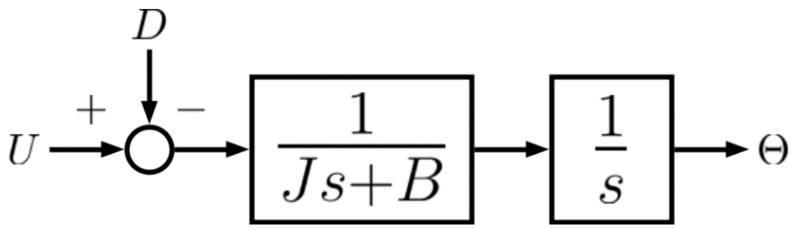
$$Y(s) = \left(\frac{C(s)G(s)}{1 + C(s)G(s)H(s)} + \frac{F(s)G(s)}{1 + C(s)G(s)H(s)} \right) R(s) \qquad \text{(C.8)}$$

$$+ \frac{G(s)}{1 + C(s)G(s)H(s)} \left(D(s) - D_c(s) \right).$$

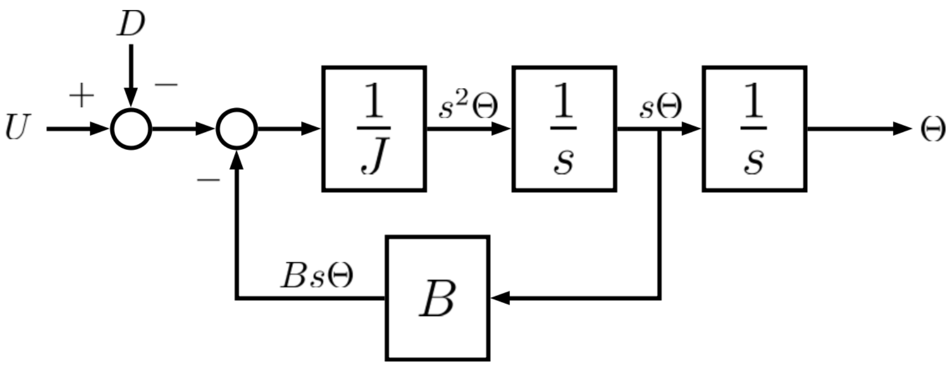
Setpoint Controllers

- We will discuss three common controllers: P, PD and PID
  - All controllers attempt to drive the error (between a desired trajectory and the actual trajectory) to zero
- The system (G(s)) can have any dynamics, but we will use the following system as an example

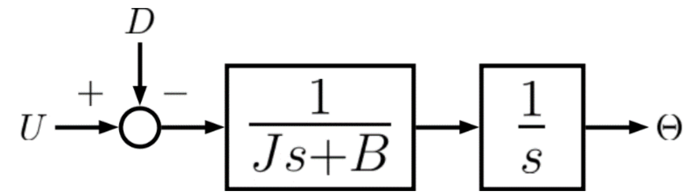
$$\theta = \frac{U - D}{(Js^2 + Bs)}$$



Compact block diagram



Block diagram with basic building blocks

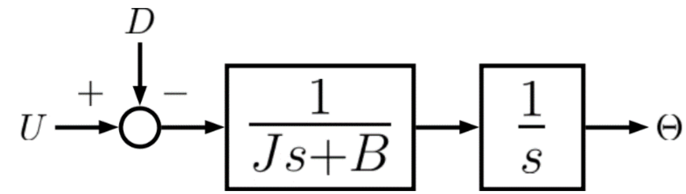


Setpoint Controllers – System Model

- A generic robot model is given as

$$J(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + g(q) = \tau$$

- $J(q)\ddot{q}$ - Inertial forces
- $C(q,\dot{q})\dot{q}$ - Coriolis and centrifugal forces
- $B\dot{q}$ - Viscous friction (damping)
- $g(q)$ - Gravitational forces
- τ - Torque/Force from actuators



Setpoint Controllers – System Model

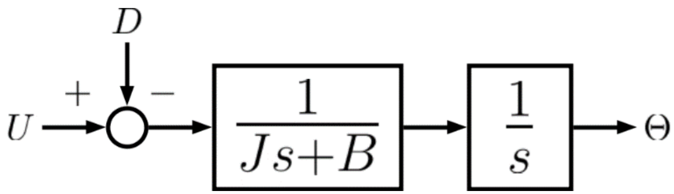
- In our example we assume that the following are treated as a disturbance D
  - Coriolis and centrifigal forces
  - Gravitational forces
  - Coupling between joints ($J(q)\ddot{q} \rightarrow J\ddot{q}$, Ineria is no longer dependent on the joint variables)

$$J(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + g(q) = \tau$$

$$J\ddot{q} + B\dot{q} + D = \tau$$

- Transforming into the frequency domain (with Laplace) gives (remember that θ is our joint variable q)

$$Js^2\theta + Bs\theta + D = \tau$$



Setpoint Controllers – Motivation

- We will now look at different controllers for this system.
- We want the error between the reference (desired) value and the actual output value to go to zero

- The error is defined as $\mathrm{e}(t) = \theta^d(t) - \theta(t)$

- The controller use the error $\mathrm{e}(t)$ to calculate its output, also called *control effort*

- We denote the *control effort* as *u ("U" in the block diagram above)*

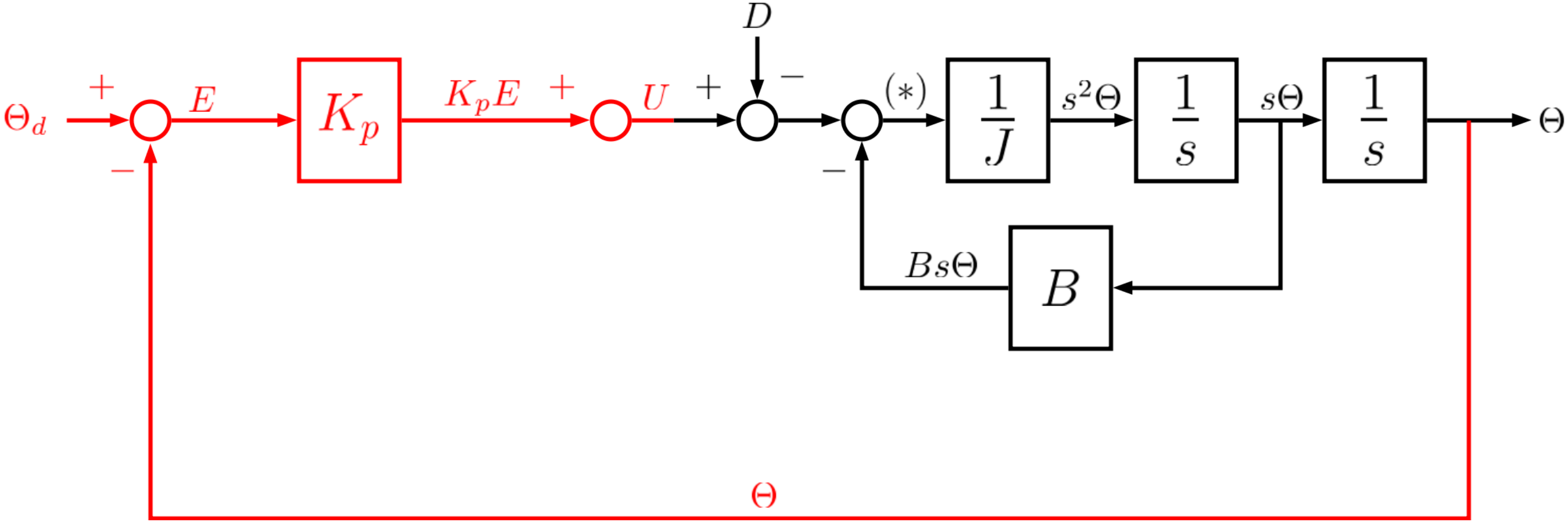
- We will look at the following controllers:
  1. Proportional (P) controller
  2. Proportional Derivative (PD) controller
  3. Proportional Integral Derivative (PID) controller

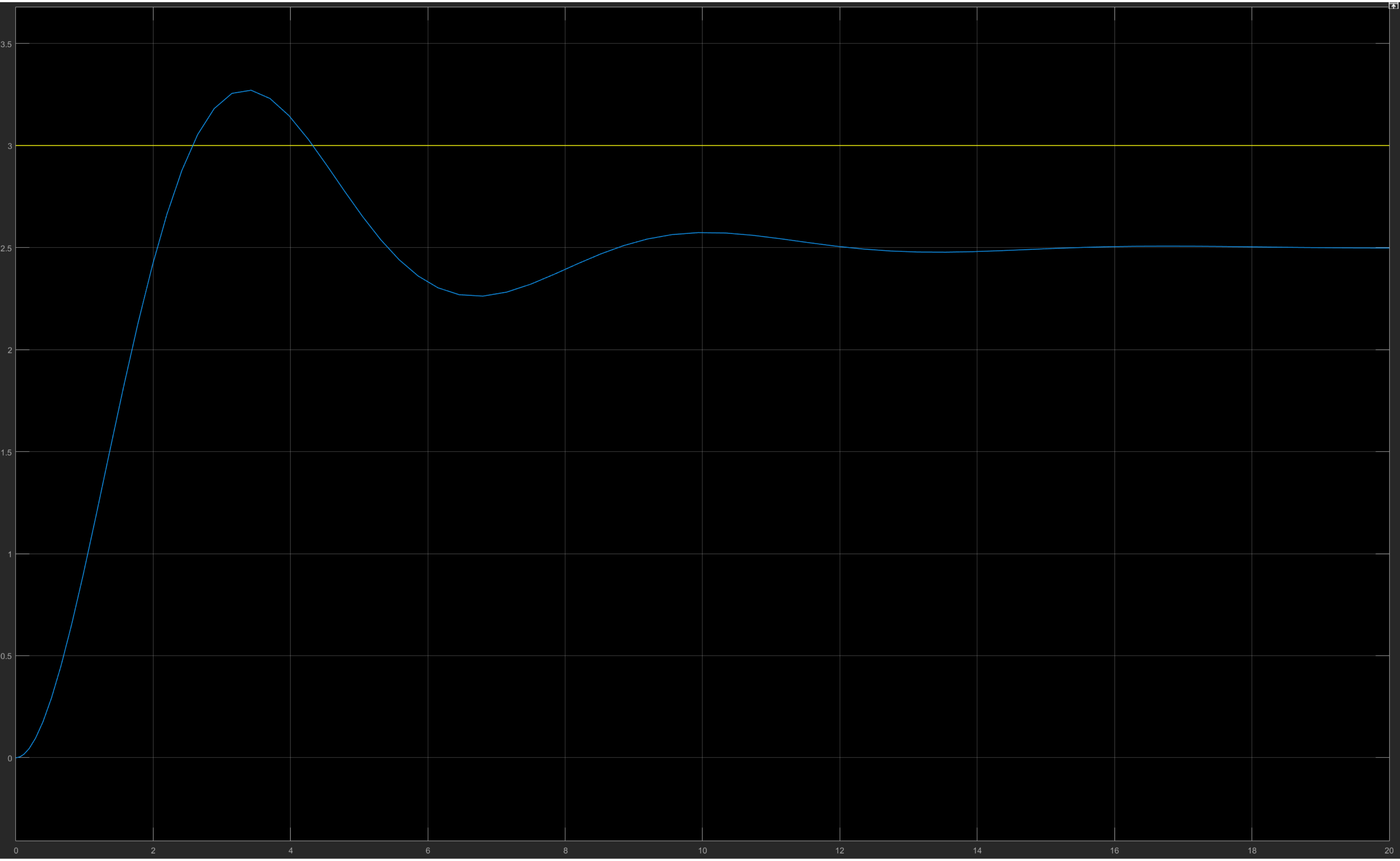
Proportional (P) Controller

- Control law: $U(t) = K_P e(t)$
  - Where $\mathrm{e}(t) = \theta^d(t) - \theta(t)$
- Taking the Laplace transformation gives:

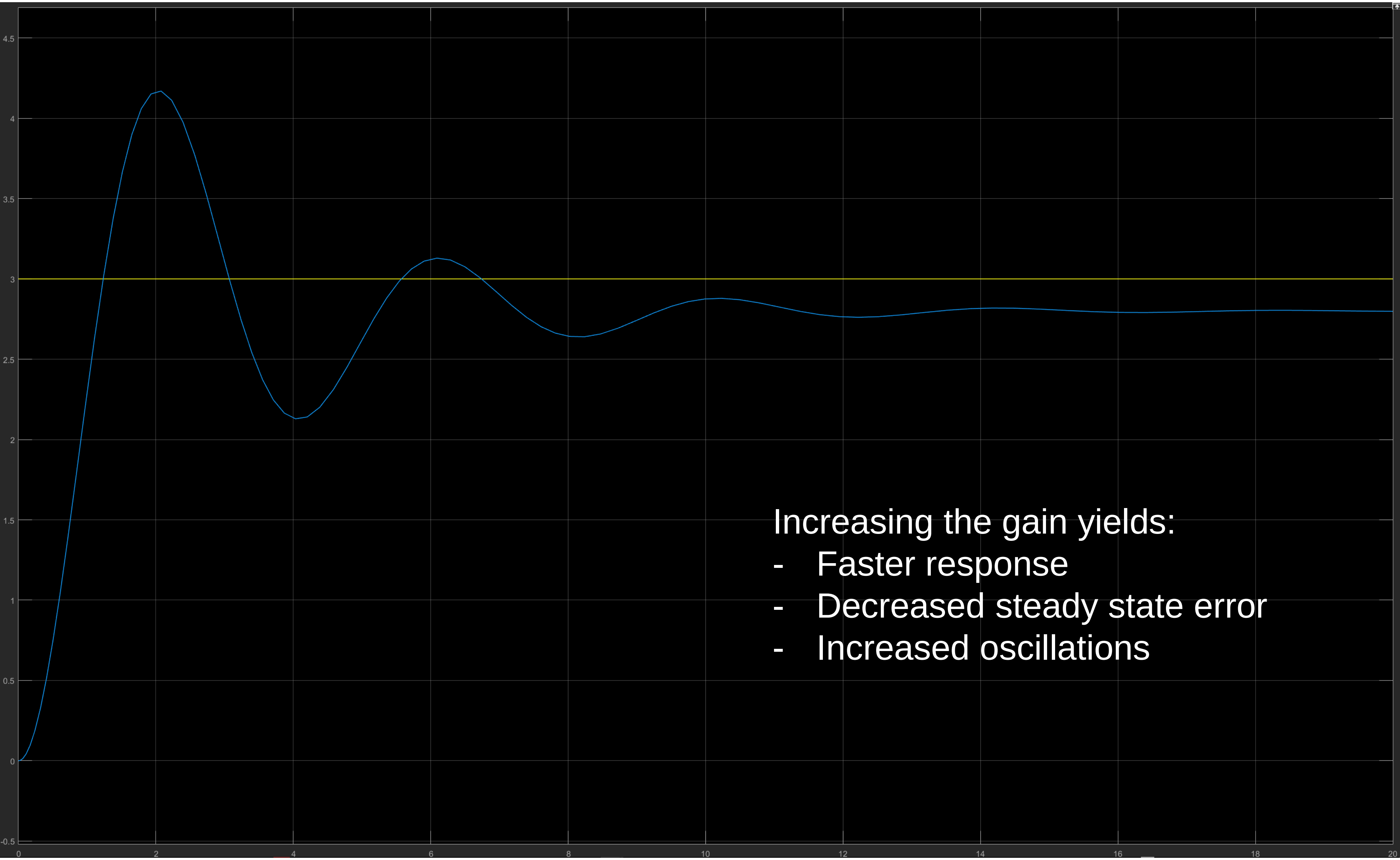
$$U(s) = K_P E(s)$$

- Adding this controller to our system gives the following closed-loop system





J = 1 B = 0.7 D = 0.5 Kp = 1



Increasing the gain yields:

- Faster response
- Decreased steady state error
- Increased oscillations

J = 1 B = 0.7 D = 0.5 Kp = 2.5

Proportional Derivative (PD) Controller

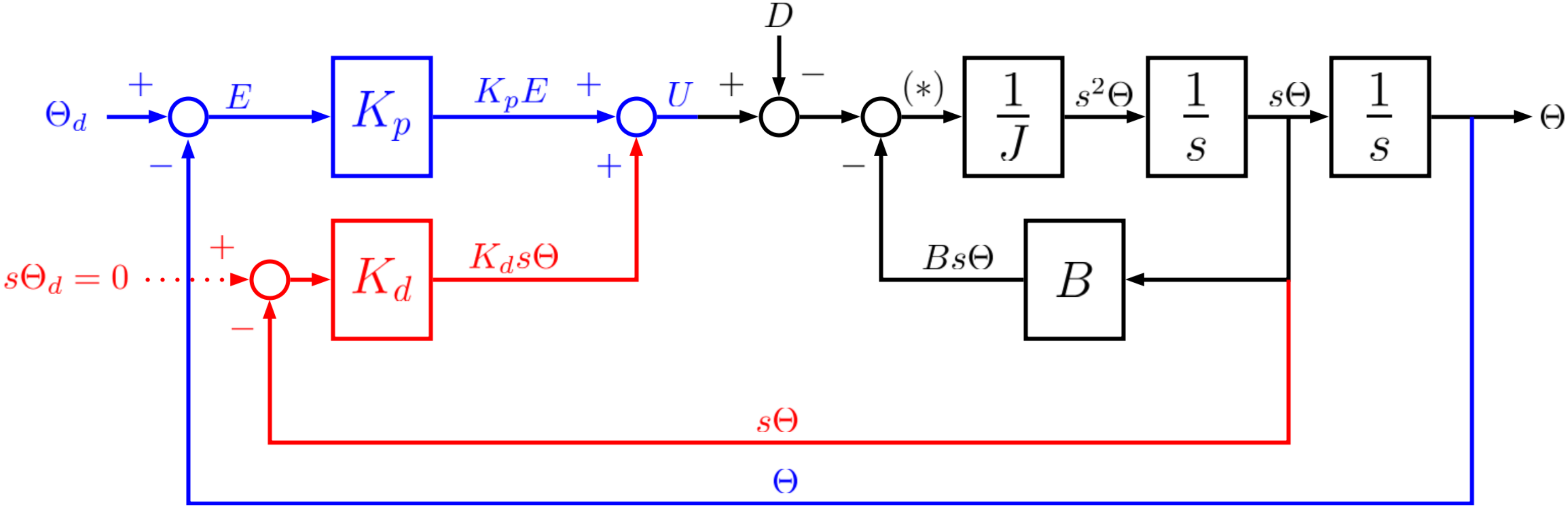
- Control law: $U(t) = K_P e(t) + K_d \dot{e}(t)$

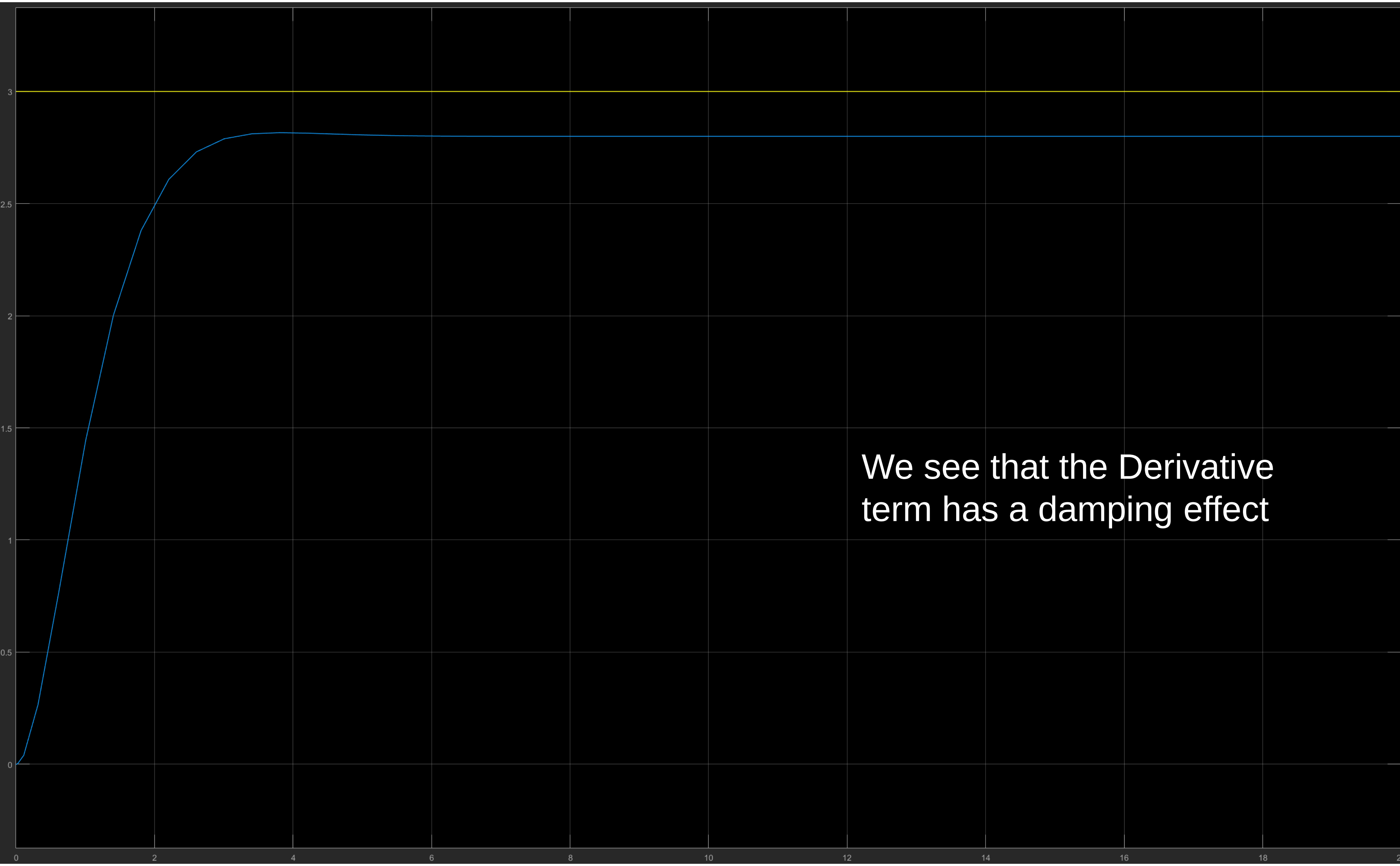
  - Where $e(t) = \theta^d(t) - \theta(t)$

- Taking the Laplace transformation gives:

$$U(s) = (K_P + K_d s)E(s)$$

- Adding this controller to our system gives the following closed-loop system:





J = 1 B = 0.7 D = 0.5 Kp = 2.5 Kd=2

Proportional Derivative (PD) Controller

- Recall that this system can be described by: $$\Theta(s) = \frac{U(s) - D(s)}{Js^2 + Bs}$$

- Where, again, $U(s)$ is: $$U(s) = (K_p + sK_d)(\Theta^d(s) - \Theta(s))$$

- Combining these gives us: $$\Theta(s) = \frac{(K_p + sK_d)(\Theta^d(s) - \Theta(s)) - D(s)}{Js^2 + Bs}$$

- Solving for Θ gives:

$$(Js^2 + Bs)\Theta(s) + (K_p + sK_d)\Theta(s) = (K_p + sK_d)\Theta^d(s) - D(s)$$
$$\Rightarrow (Js^2 + (B + K_d)s + K_p)\Theta(s) = (K_p + sK_d)\Theta^d(s) - D(s)$$
$$\Rightarrow \Theta(s) = \frac{(K_p + sK_d)\Theta^d(s) - D(s)}{Js^2 + (B + K_d)s + K_p}$$

Proportional Derivative (PD) Controller

- The denominator is the *characteristic polynomial*
- The roots of the characteristic polynomial determine the performance of the system

$$s^2 + \frac{(B + K_d)}{J} s + \frac{K_p}{J} = 0$$

- If we think of the closed-loop system as a damped second order system, this allows us to choose values of K_p and K_d

$$s^2 + 2\zeta\omega s + \omega^2 = 0$$

- Thus K_p and K_d are:

$$K_p = \omega^2 J$$

$$K_d = 2\varsigma\omega J - B$$

- A natural choice is $\zeta = 1$ (critically damped)
  - $\zeta < 1$ – underdamped system
  - $\zeta > 1$ – overdamped system

Proportional Derivative (PD) Controller

- Limitations of the PD controller:
  - for illustration, let our desired trajectory be a step input and our disturbance be a constant as well:

$$\Theta^d(s) = \frac{C}{s}, D(s) = \frac{D}{s}$$

  - Plugging this into our system description gives:

$$\Theta(s) = \frac{(K_p + sK_d)C - D}{s(Js^2 + (B + K_d)s + K_p)}$$

  - For these conditions, what is the steady-state value of the displacement?

$$\theta_{ss} = \lim_{s\to 0} \frac{s(K_p + sK_d)C - sD}{s(Js^2 + (B + K_d)s + K_p)} = \lim_{s\to 0} \frac{(K_p + sK_d)C - D}{Js^2 + (B + K_d)s + K_p} = \frac{K_pC - D}{K_p} = C - \frac{D}{K_p}$$

  - **Thus the steady state error is –*D*/*K*$_p$**
  - Therefore to drive the error to zero in the presence of large disturbances, we need large gains… so we turn to another controller

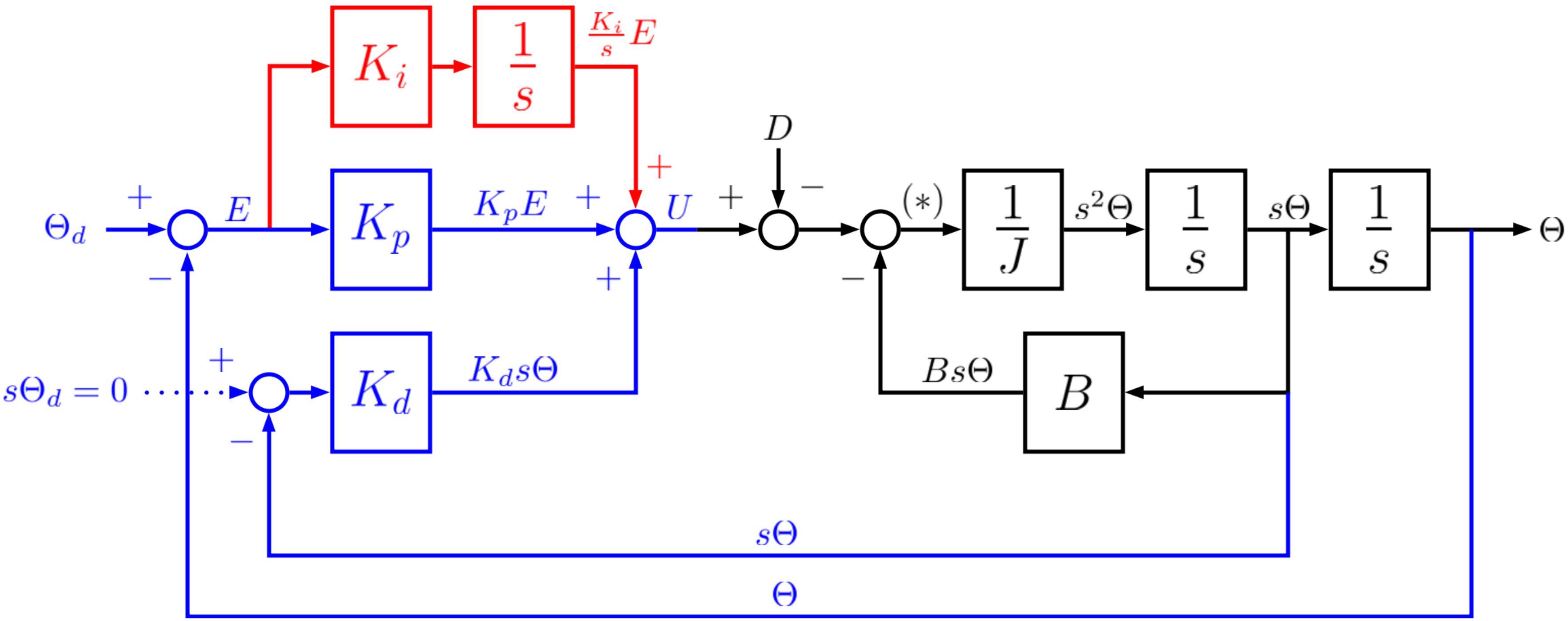
$$\Theta(s)=\frac{\left(K_d s^2+K_p s+K_i\right)\Theta^d(s)-sD(s)}{Js^3+\left(B+K_d\right)s^2+K_p s+K_i}$$

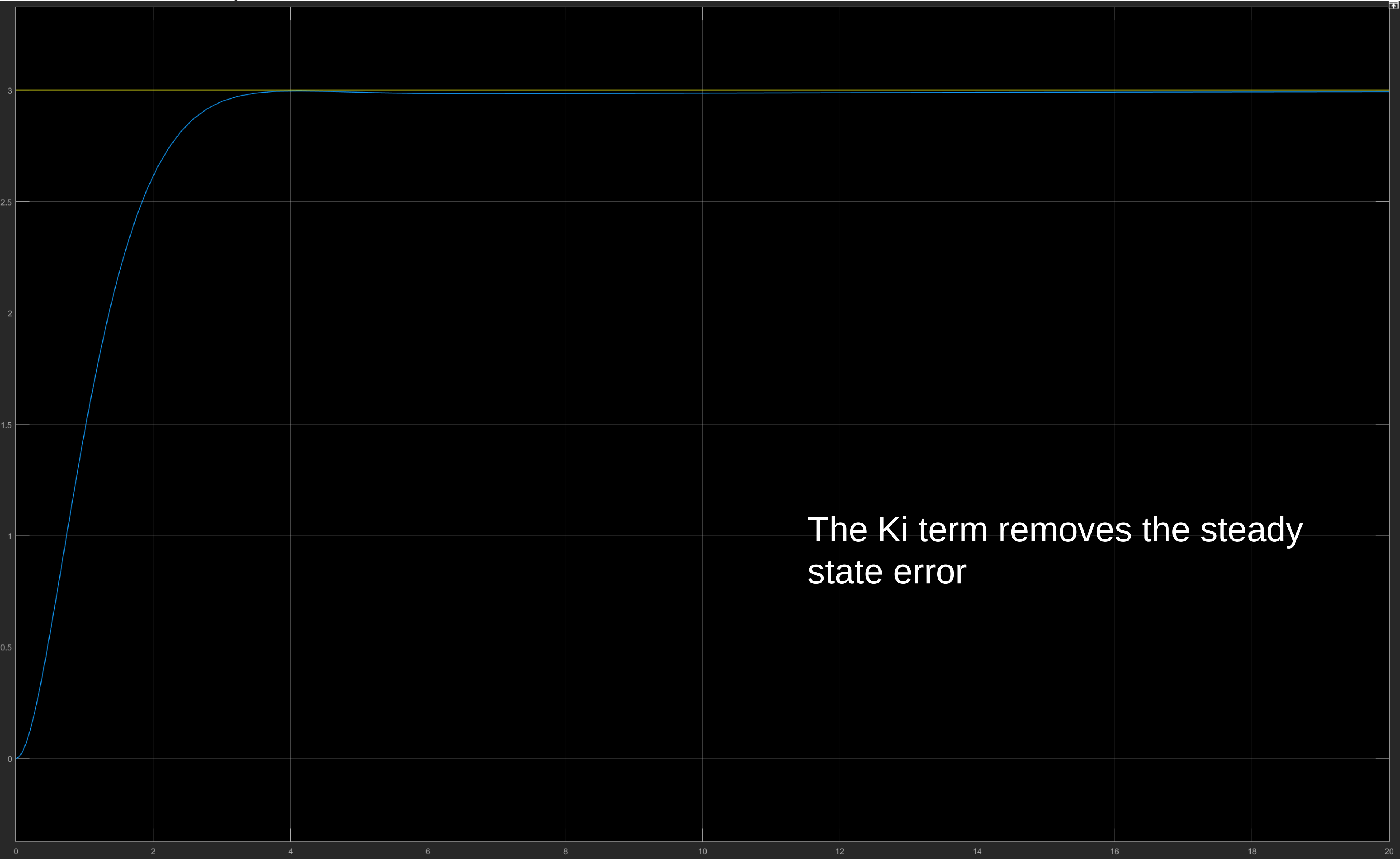
Proportional Integral Derivative (PID) controller

- Control law: $$u(t)=K_p e(t)+K_d\dot{e}(t)+K_i\int e(t)dt$$
- Taking the Laplace transformation gives:

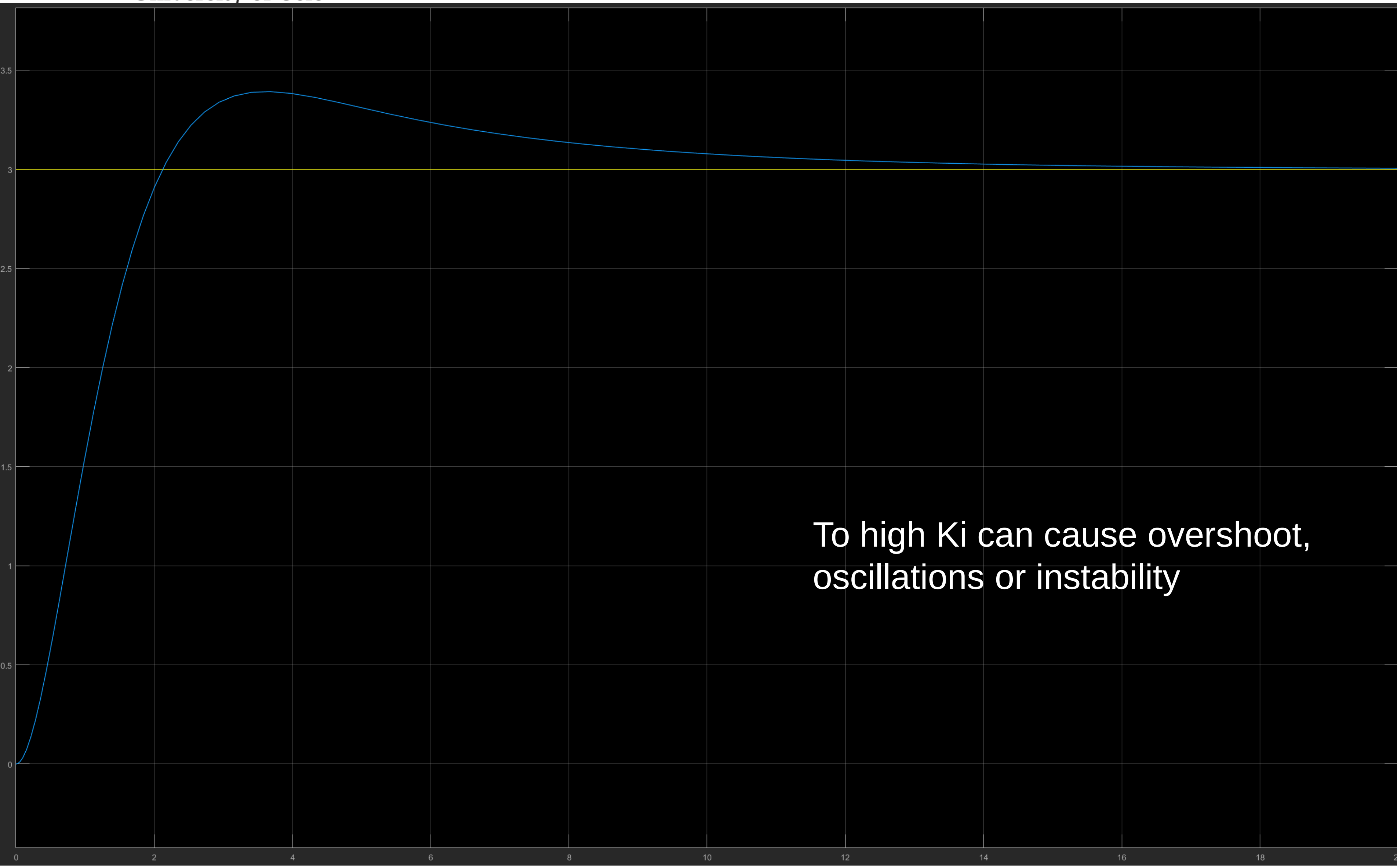
$$U(s)=\left(K_p+K_d s+\frac{K_i}{s}\right)E(s)$$

- Adding this controller to our system gives the following closed-loop system:





J = 1 B = 0.7 D = 0.5 Kp = 2.5 Kd=2 Ki = 0.133



J = 1 B = 0.7 D = 0.5 Kp = 2.5 Kd=2 Ki = 0.5

Proportional Integral Derivative (PID) controller

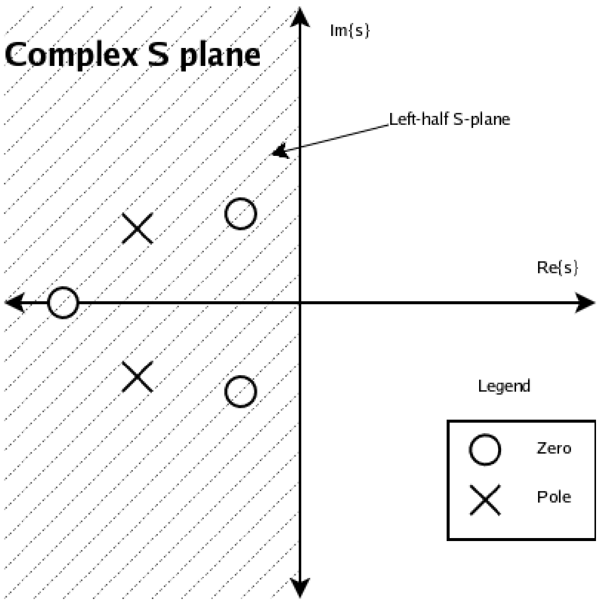
- How to determine PID gains
  1. Set $K_i = 0$ and solve for K_p and K_d
  2. Determine K_i to eliminate steady state error
     - However, we need to be careful of the stability conditions

$$K_i < \frac{(B + K_d)K_p}{J}$$

  – In general real world testing we always start with determining K_p
  – There are general methods for finding controller gains that could be used (ziegler nichols methods etc.)

Proportional Integral Derivative (PID) controller

- Stability
  - The closed-loop stability of these systems is determined by the roots of the characteristic polynomial



  - If all roots (potentially complex) are in the 'left-half' plane, our system is stable
    - for any bounded input and disturbance
  - A description of how the roots of the characteristic equation change (as a function of controller gains) is very valuable
    - Called the *root locus (see example slide 36)*

Setpoint Controllers – Summary

- **P**roportional
  - A pure proportional controller will have a steady-state error
  - High gain (Kp) will produce a fast system
  - High gain may cause oscillations and may make the system unstable
  - High gain reduces the steady-state error

- **I**ntegral
  - Removes steady-state error
  - Increasing Ki accelerates the controller
  - High Ki may give oscillations
  - Increasing Ki will increase the settling time

- **D**erivative
  - Larger Kd decreases oscillations
  - Improves stability for low values of Kd
  - May be highly sensitive to noise if one takes the derivative of a noisy error
  - High noise leads to instability

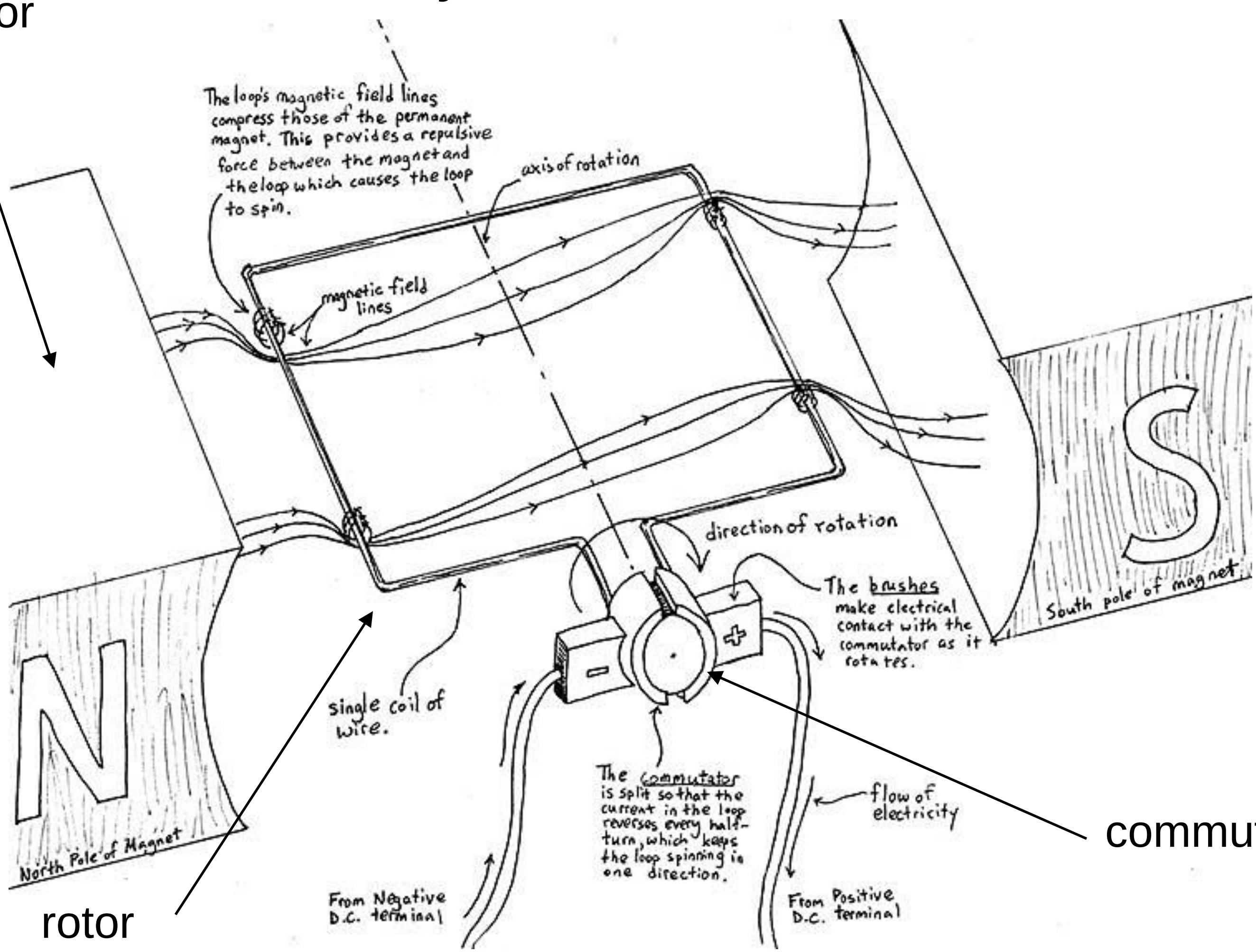
Example - Motor dynamics

- DC motors are ubiquitous in robotics applications
- Here, we develop a transfer function that describes the relationship between the input voltage and the output angular displacement
- First, a physical description of the most common motor: permanent magnet…

torque on the rotor:

$$\tau_m = K_1 \phi i_a$$

Physical instantiation

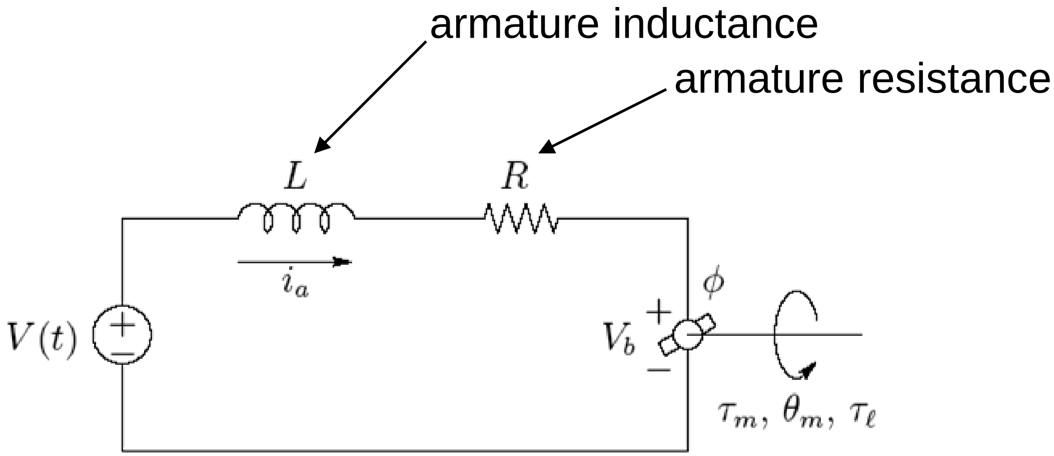


Motor dynamics

- When a conductor moves in a magnetic field, a voltage is generated
  - Called *back EMF*:

$$V_b = K_2 \phi \omega_m$$

  - Where ω_m is the rotor angular velocity



$$L \frac{di_a}{dt} + Ri_a = V - V_b$$

Motor dynamics

- Since this is a permanent magnet motor, the magnetic flux is constant, we can write:

$$\tau_m = K_1 \phi i_a = K_m i_a$$

torque constant

- Similarly:

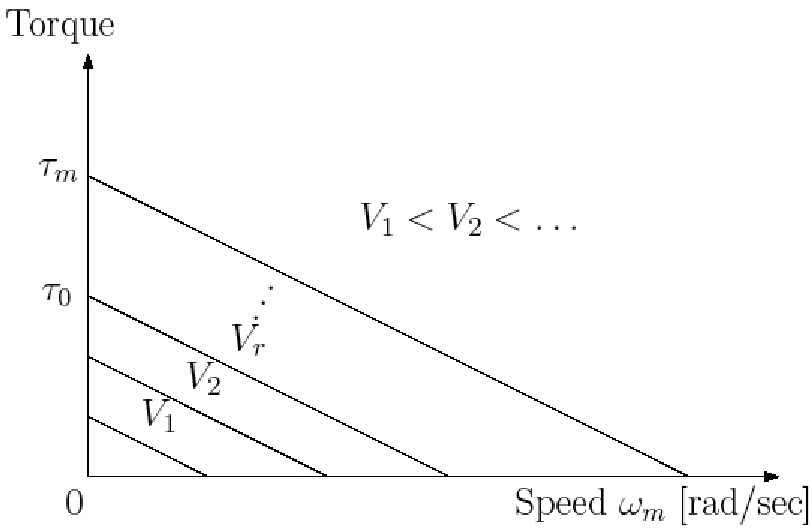
$$V_b = K_2 \phi \omega_m = K_b \frac{d\theta_m}{dt}$$

back EMF constant

- K_m and K_b are numerically equivalent, thus there is one constant needed to characterize a motor

Motor dynamics

- This constant is determined from torque-speed curves
  - Remember, torque and displacement are work conjugates

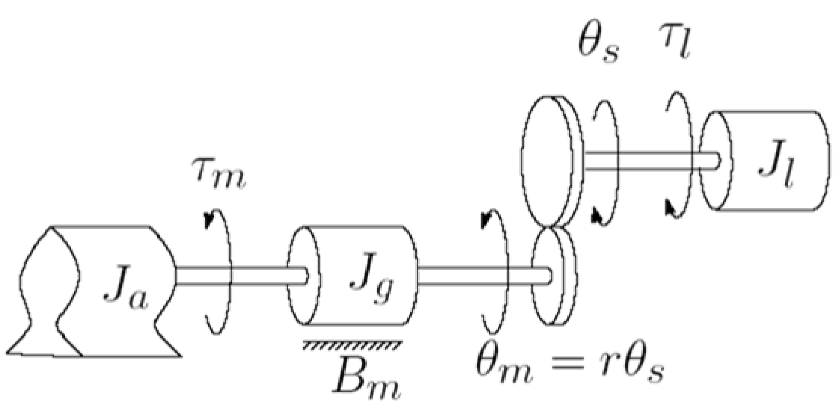


  - τ_0 is the *blocked torque*

Single link/joint dynamics

- Now, lets take our motor and connect it to a link
- Between the motor and link there is a gear such that: $\theta_m = r\theta_L$
- Lump the actuator and gear inertias: $J_m = J_a + J_g$
- Now we can write the dynamics of this mechanical system:

$$J_m \frac{d^2\theta_m}{dt^2} + B_m \frac{d\theta_m}{dt} = \tau_m - \frac{\tau_L}{r} = K_m i_a - \frac{\tau_L}{r}$$



Motor dynamics

- Now we have the ODEs describing this system in both the electrical and mechanical domains:

$$L\frac{di_a}{dt} + Ri_a = V - K_b\frac{d\theta_m}{dt}$$

$$J_m\frac{d^2\theta_m}{dt^2} + B_m\frac{d\theta_m}{dt} = K_m i_a - \frac{\tau_L}{r}$$

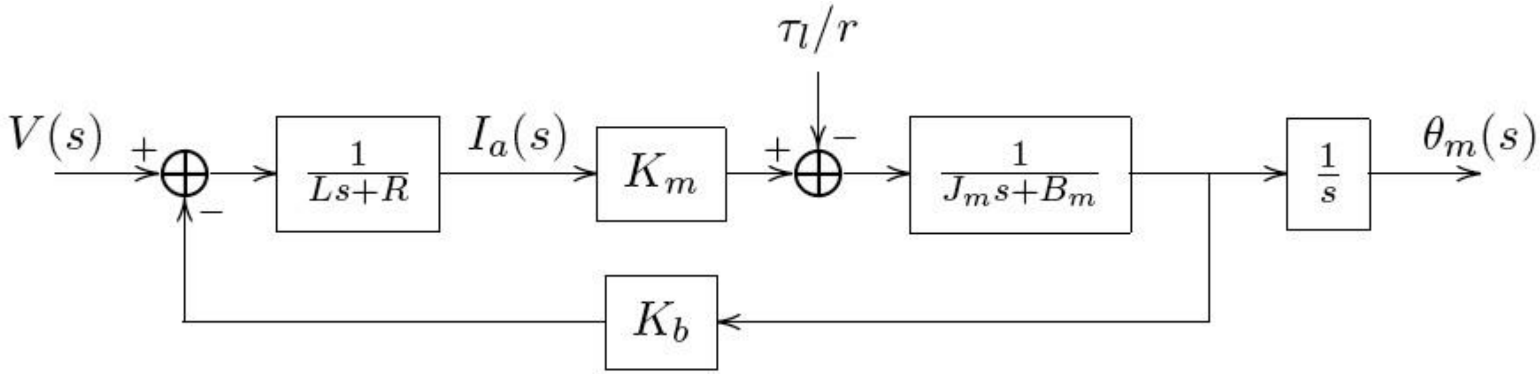
- In the Laplace domain:

$$(Ls + R)I_a(s) = V(s) - K_b s\Theta_m(s)$$

$$(J_m s^2 + B_m s)\Theta_m(s) = K_m I_a(s) - \frac{\tau_L(s)}{r}$$

Motor dynamics

- These two can be combined to define, for example, the input-output relationship for the input voltage, load torque, and output displacement:



Motor dynamics

- Remember, we want to express the system as a transfer function from the input to the output angular displacement
  - But we have two potential inputs: the load torque and the armature voltage
  - First, assume τ_L = 0 and solve for $\Theta_m(s)$:

$$\frac{\left(J_m s^2 + B_m s\right)\Theta_m(s)}{K_m} = I_a(s) \longrightarrow \frac{(Ls+R)\left(J_m s^2 + B_m s\right)}{K_m}\Theta_m(s) = V(s) - K_b s \Theta_m(s)$$

$$\longrightarrow \frac{\Theta_m(s)}{V(s)} = \frac{K_m}{s\left[(Ls+R)(J_m s + B_m) + K_b K_m\right]}$$

Motor dynamics

- Now consider that $V(s) = 0$ and solve for $\Theta_m(s)$:

$$I_a(s) = \frac{-K_b s \Theta_m(s)}{Ls+R} \longrightarrow \left(J_m s^2 + B_m s\right)\Theta_m(s) = \frac{-K_m K_b s \Theta_m(s)}{Ls+R} - \frac{\tau_L(s)}{r}$$

$$\longrightarrow \frac{\Theta_m(s)}{\tau_L(s)} = \frac{-(Ls+R)/r}{s\left[(Ls+R)(J_m s + B_m) + K_b K_m\right]}$$

- Note that this is a function of the gear ratio
  - The larger the gear ratio, the less effect external torques have on the angular displacement

Motor dynamics

- In this system there are two ‘time constants’
  - Electrical: L/R
  - Mechanical: J_m/B_m
- For intuitively obvious reasons, the electrical time constant is assumed to be small compared to the mechanical time constant
  - Thus, ignoring electrical time constant will lead to a simpler version of the previous equations:

$$\frac{\Theta_m(s)}{V(s)} = \frac{K_m / R}{s\left[J_m s + B_m + K_b K_m / R\right]}$$

$$\frac{\Theta_m(s)}{\tau_L(s)} = \frac{-1/r}{s\left[J_m s + B_m + K_b K_m / R\right]}$$

Motor dynamics

- Rewriting these in the time domain gives:

$$\frac{\Theta_m(s)}{V(s)} = \frac{K_m / R}{s\left[J_m s + B_m + K_b K_m / R\right]} \longrightarrow J_m \ddot{\theta}_m(t) + (B_m + K_b K_m / R)\dot{\theta}_m(t) = (K_m / R)V(t)$$

$$\frac{\Theta_m(s)}{\tau_L(s)} = \frac{-1/r}{s\left[J_m s + B_m + K_b K_m / R\right]} \longrightarrow J_m \ddot{\theta}_m(t) + (B_m + K_b K_m / R)\dot{\theta}_m(t) = -(1/R)\tau_L(t)$$

- By superposition of the solutions of these two linear 2nd order ODEs:

$$\underbrace{J_m}_{J}\ddot{\theta}_m(t) + \underbrace{(B_m + K_b K_m / R)}_{B}\dot{\theta}_m(t) = \underbrace{(K_m / R)V(t)}_{u(t)} - \underbrace{(1/R)\tau_L(t)}_{d(t)}$$

Motor dynamics

- Therefore, we can write the dynamics of a DC motor attached to a load as:

$$J\ddot{\theta}(t) + B\dot{\theta}(t) = u(t) - d(t)$$

  - Note that *u*(*t*) is the input and *d*(*t*) is the disturbance (e.g. the dynamic coupling from motion of other links)
- To represent this as a transfer function, take the Laplace transform:

$$\left(Js^2 + Bs\right)\Theta(s) = U(s) - D(s) \longrightarrow$$