IN3190/IN4190 — Digital Signal Processing
Curriculum topics Autumn 2022

Updated in August, 2022. *Note: this list can be further updated during the semester!*

Contents

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1 Discrete-time signals & Systems

1.1 Discrete-time signals

- **Discrete time signal,** $x(n) = \{\ldots, x(-1), \underset{\uparrow}{x(0)}, x(1), \ldots\}$**.**

  Alternative notation in some literature: $x[n]$ instead of $x(n)$

- **Unit sample sequence – Dirac's delta function – unit step function,** $\delta(n)$**.**

  $$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise.} \end{cases}$$

- **Any abritary sequence** $x(n)$ **can be synthesized as** $x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$**.**

- **Unit step sequence,** $u(n)$

  $$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

- **Unit ramp sequence,** $u_r(n)$

  $$u_r(n) = \begin{cases} n, & n \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

- **Exponential sequences**

  $x(n) = a^n \ \forall n$
  A one-sided exponential sequence $\alpha^n, \ n \geq 0; \ \alpha \in \Re$ is called a geometric series.
  $$\sum_{n=0}^{\infty} \alpha^n \longrightarrow \frac{1}{1-\alpha}, \ |\alpha| < 1.$$
  $$\sum_{n=0}^{N} \alpha^n \longrightarrow \frac{1-\alpha^N}{1-\alpha}, \forall \alpha.$$

- **Sinusoidal sequences**

  $x(n) = \cos(w_0 n + \Phi), \ \forall n, \ \ w_0, \ \Phi \in \Re.$

- **Periodic sequences**

  $x(n)$ periodic iff $x(n) = x(n+N) \ \forall n$.
  Fundamental period: Smallest positive integer N that satisfies the relation.

- **Symmetric sequences**

  A signal is conjugate symmetric (*even* if real) if, for all n, $x(n) = x^*(-n)$.
  A signal is conjugate antisymmetric (*odd* if real) if, for all n, $x(n) = -x^*(-n)$.
  Any signal can be decomposed into a sum of a conjugate symmetric signal and a conjugate anti-symmetric signal.

- **Signal energy**

  $$E = \sum_{-\infty}^{\infty} x(n)x^*(n) = \sum_{-\infty}^{\infty} |x(n)|^2.$$
  Energy signal: Signal with finite enery, i.e. $E < \infty$.

- **Signal power**

Average power defined as $P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} |x(n)|^2$.
Power signal: signal with nonzero and finite average power.
Periodic sequence $\tilde{x}(n)$ with fundamental period N: $P_{\tilde{x}} = \frac{1}{N} \sum_{0}^{N} |\tilde{x}(n)|^2$.

1.2 LTI systems and their properties

(See also aspects related to LTI systms in the z-transform and ROC sections following below.)

- **Discrete-time system characterized through an input-output transformation**

  $y(n) = \mathcal{H}\{x(n)\}$ or $x(n) \longrightarrow \boxed{\mathcal{H}\{\cdot\}} \longrightarrow y(n)$ or $x(n) \xrightarrow{\mathcal{H}} y(n)$.

- **Linear versus nonlinear systems**

  A system T is linear iff $\mathcal{H}\{a_1x_1(n) + a_2x_2(n)\} = a_1\mathcal{H}\{x_1(n)\} + a_2\mathcal{H}\{x_2(n)\}$.

- **Time-invariant versus time-variant systems**

  A linear system T is *time-invariant* or *shift-invariant* iff the following is true:
  $x(n) \longrightarrow \boxed{\mathcal{H}\{\cdot\}} \longrightarrow y(n) \longrightarrow \boxed{\text{Shift by } k} \longrightarrow y(n-k)$.
  $x(n) \longrightarrow \boxed{\text{Shift by } k} \longrightarrow x(n-k) \longrightarrow \boxed{\mathcal{H}\{\cdot\}} \longrightarrow y(n-k)$.
  A linear, time-invariant system is denoted a *LTI-system*.

- **LTI-systems and convolution sum**

  Let $x(n)$ and $y(n)$ be the input-output pair of an LTI-system. Then the output is given by:
  $y(n) = \mathcal{H}\{x(n)\} = \sum_{k=-\infty}^{\infty} x(k)\mathcal{H}\{\delta(n-k)\} \equiv \sum_{k=-\infty}^{\infty} x(k)h(n-k)$.
  $h(n)$ is the response of an LTI system to $\delta(n)$ and called the *impulse response*.
  A LTI-system is completey descried in time-domain by the impulse response, $h(n)$.
  The mathematical operator $y(n) \equiv x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$ is called *linear convolution sum.*

- **Static versus dynamic systems**

  A system is *static* or *memoryless* if the outpt at any time $n = n_0$ depends only on the input at time $n = n_0$.

- **Causal versus noncausal systems**

  A system is *causal* if, for any n_0, the system response at time n_0 only depends on the input up to time $n = n_0$.
  A LTI-system is causal iff $h(n) = 0,\ n < 0$.

- **Stable versus unstable systems**

  A system is *bounded-input bounded-output (BIBO) stable* if, for any input that is bounded, $|x(n)| \leq A < \infty$, the output will be bounded, $|y(n)| \leq B < \infty$.
  A LTI-system is BIBO stable iff $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$.

- **A LTI system as a linear constant-coefficient difference equation**

  The general form:
  $y(n) = \sum_{k=0}^{q} b(k)x(n-k) - \sum_{k=1}^{p} a(k)y(n-k)$,
  where $a(k)$ and $b(k)$ are constants that defines the system.
  If one or more terms $a(k)$ are nonzero; *recursive system.*
  If all coeffisients $a(k)$ equal to zero; *nonrecursive system.*

- **Finite-duration impulse response (FIR) filters**

FIR/nonrecursive/moving-average (MA) filter:

$$h(n) = \begin{cases} \text{Some nonzero value}, & n_1 \le n \le n_2 \\ 0, & \text{otherwise.} \end{cases}$$

Convolution formula, causal FIR filter:

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k) = \sum_{k=0}^{M-1} x(k)h(n-k).$$

- **Infinite-duration impulse response (IIR) filters**

  Causal IIR: $y(n) = \sum_{k=0}^{\infty} h(k)x(n-k).$

1.3 Convolution and correlation

- **Convolution**

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k).$$

- **Convolution properties**

  Commutative property: $x(n) * h(n) = h(n) * x(n)$.
  Assosiative property: $\{x(n) * h_1(n)\} * h_2(n) = x(n) * \{h_1(n) * h_2(n)\}$.
  Distributive property: $x(n) * \{h_1(n) + h_2(n)\} = x(n) * h_1(n) + x(n) * h_2(n)$.

- **Performing convolution**

  Direct evaluation.
  Graphical approach.
  Slide rule method.

2 The z-transform

2.1 The forward z-transform

- **Definition of the two-sided z-transform.**

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

- **Region of convergence (ROC)**

  The region of convergence is the subset of the complex plane where the z-transform converges. For an FIR-filter (or a finite-length signal) it is the entire complex plane with the possible exceptions of 0 or ∞. For an IIR-filter (or infinite-length signal) it is one of three:

  1. The outside of a disc $|z| > a$ for a causal system.
  2. A disc $|z| < a$ for an anti-causal system.
  3. The intersection between the two abovementioned regions $a < |z| < b$ for a two-sided system.

- **Poles and Zeroes**

  A *pole* is a point $z \in \mathbb{C}$ where $H(z) = \infty$. A *zero* is a point $z \in \mathbb{C}$ where $H(z) = 0$.

- **Relationship between the z-transform and the discrete-time Fourier transform (DTFT).**

  $X(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$

- **Some common z-transforms.**

  Being able to read and use tables like these:

TABLE 3.3 SOME COMMON *Z*-TRANSFORM PAIRS

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
7	$(\cos \omega_0 n) u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
8	$(\sin \omega_0 n) u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
9	$(a^n \cos \omega_0 n) u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$
10	$(a^n \sin \omega_0 n) u(n)$	$\frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$

- **Properties of the z-transform.**

  Being able to prove some of these:

TABLE 3.2 PROPERTIES OF THE *Z*-TRANSFORM

Property	Time Domain	z-Domain	ROC
Notation	$x(n)$ $x_1(n)$ $x_2(n)$	$X(z)$ $X_1(z)$ $X_2(z)$	ROC: $r_2 < \lvert z\rvert < r_1$ ROC_1 ROC_2
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC_1 and ROC_2
Time shifting	$x(n-k)$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
Scaling in the z-domain	$a^n x(n)$	$X(a^{-1}z)$	$\lvert a\rvert r_2 < \lvert z\rvert < \lvert a\rvert r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \lvert z\rvert < \frac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real part	$\text{Re}\{x(n)\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Includes ROC
Imaginary part	$\text{Im}\{x(n)\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Includes ROC
Differentiation in the z-domain	$nx(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \lvert z\rvert < r_1$
Convolution	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least, the intersection of ROC_1 and ROC_2
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Initial value theorem	If $x(n)$ causal	$x(0) = \lim_{z\to\infty} X(z)$	
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$	At least $r_{1l}r_{2l} < \lvert z\rvert < r_{1u}r_{2u}$
Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(1/v^*)v^{-1}dv$		

- **Calculating a transfer function/system function from a difference equation.**

  Calculate the z-transform of the difference equation for $y(n)$ and divide by $X(z)$:

$$H(z) = \frac{Y(z)}{X(z)}.$$

- **Determining and interpreting pole-zero-plots with respect to:**

  1. **Stability.**
  2. **Causality.**
  3. **Symmetry.**
  4. **Real or complex (time domain) signals.**
  5. **The connection to the transfer function.**
  6. **Approximating the frequency response, determining the filter type.**

Stability:

The ROC contains the unit circle.

Causality:

The ROC is $|z| > \alpha$ - causal system ($h(n) = 0$ for $n < N$).
The ROC is $|z| < \alpha$ - anti-causal system ($h(n) = 0$ for $n > N$).
The ROC is $\alpha < |z| < |\beta|$ - two-sided system (there is no interval towards pos. or neg. infinity in which $h(n)$ is zero).

Symmetry:

The system/signal is symmetric in the time domain iff poles and zeroes come in reciprocal pairs. *Meaning:* z is a pole of $H(z) \Leftrightarrow z^{-1}$ is a pole of $H(z)$. (The same applies for zeros.)

Real or complex time domain signal:

The signal is real in the time domain iff all poles and zeros come in complex conjugate pairs. *Meaning:* z is a pole of $H(z) \Leftrightarrow z^*$ is a pole of $H(z)$. (The same applies for zeros.)

The connection to the transfer function:
The pole-zero plot tells us where the transfer function is *zero* and *infinite*. In the regions surrounding these points, the transfer function is low or high, respectively.

Approximating the frequency response, determining the filter type:
Use the angles of poles/zeroes to determine the angular frequency of peaks/dips in the spectrum, and use their distance from the unit circle to determine (approximately) the height of the peak or dip.

2.2 The inverse z-transform

- **Inverse z-transform using table lookup**

  Can often identify patterns and relations to make the table of forward z-transforms useful when finding an inverse z-transform

- **Inverse z-transform formally: a Cauchy contour integral for the inversion**

  Often difficult

- **Inverse z-transform by development into a polynomial series**

  Develop in terms of $X(z) = \sum_{n=-\infty}^{\infty} X(n)z^{-n} \; = \ldots + x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + \ldots$

  Often difficult

- **Inverse z-transform of rational transfer functions by partial fraction expansion.**

  Any transfer function in the form of a rational polynomial

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

  can be reduced to the form

$$H(z) = \sum_{l=0}^{L} \sum_{k=0}^{K-1} b_l z^{-l} \frac{c_k}{1 - p_k z^{-1}}.$$

  and inverse transformed using the properties of linearity and time shifts.

3 Frequency analysis of signals

3.1 Continous-time signals

- **The (continous-time) Fourier series**

  Synthesis equation: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath \frac{2\pi}{T_0} kt} = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi F_0 kt}.$

  Analysis equation: $c_k = \frac{1}{T_p} \int_{T_p} x(t) e^{-\jmath \frac{2\pi}{T_0} kt} dt = \frac{1}{T_p} \int_{T_p} x(t) e^{-\jmath 2\pi F_0 kt} dt.$

  - **All periodic signal of practical interest satisfy these conditions.**
  - Other periodic signals may also have a Fourier series representation.

- **The (Continous-Time) Fourier Transform (FT/CTFT)**

  Synthesis equation: $x(t) = \int_{-\infty}^{\infty} X(F) e^{\jmath 2\pi F t} dF.$

  Analysis equation: $X(F) = \int_{-\infty}^{\infty} x(t) e^{-\jmath 2\pi F t} dt.$

  - The above is a unitary convention for CTFT. In some other literature, the forward/backward (analysis/synthesis) the Fourier integral kernel is without 2π in the exponent and the CTFT couples instead can become:
    $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(F) e^{\jmath F t} dF$ and the corresponding $X(F) = \int_{-\infty}^{\infty} x(t) e^{-\jmath F t} dt.$

3.2 Discrete-time signals

- **The discrete-time Fourier series (almost DFT !!!)**

  Synthesis equation: $x(n) = \sum_{k=0}^{N-1} c_k e^{\jmath 2\pi kn/N}.$

  Analysis equation: $c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-\jmath 2\pi kn/N}.$

  - $\{c_k\}$ represents the amplitude and phase associated with the frequency component $s_k(n) = e^{\jmath 2\pi kn/N} = e^{\jmath w_k n},\ w_k = 2\pi k/N.$
  - $\{c_k\}$ periodic with period N.

- **The discrete-time Fourier transform (DTFT)**

  Synthesis equation: $x(n) = \frac{1}{2\pi} \int_{2\pi} X(e^{\jmath\omega}) e^{\jmath\omega n} d\omega$

  Analysis equation: $X(e^{\jmath\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-\jmath\omega n}$

  - $X(e^{\jmath\omega})$ unique over the frequency interval $(-\pi, \pi)$, or equivalently, $(0, 2\pi)$.
  - $X(e^{\jmath\omega})$ periodic with period 2π.
  - Convergence: $X_N(e^{\jmath\omega}) = \sum_{n=-N}^{N} x(n) e^{-\jmath\omega n}$ converges uniformly to $X(e^{\jmath\omega})$, i.e. $\lim_{N\to\infty} \{\sup_w |X(e^{\jmath\omega}) - X_N(e^{\jmath\omega})|\} = 0.$

- Guaranteed if $x(n)$ is absolutely summable.

- Possible with square summable sequences if *mean-square* convergence condition.

■ Energy density spectrum

Parseval's relations:

	Continuous time	Discrete-time
Periodic	Infinite energy and $P = \frac{1}{T_p}\int_{T_p} \lvert x(t)\rvert^2\, dt = \sum_{k=-\infty}^{\infty} \lvert c_k\rvert^2$	$P = \frac{1}{N}\sum_{n=0}^{N-1} \lvert x(n)\rvert^2 = \sum_{k=0}^{N-1} \lvert c_k\rvert^2$ $E = \sum_{n=0}^{N-1} \lvert x(n)\rvert^2 = N\sum_{k=0}^{N-1} \lvert c_k\rvert^2$
Aperiodic	$x(t)$ any finite enery signal with FT $X(F)$ $E = \int_{t=-\infty}^{\infty} \lvert x(t)\rvert^2\, dt$ $= \int_{-\infty}^{\infty} \lvert X(F)\rvert^2\, dF$	$E = \sum_{n=-\infty}^{\infty} \lvert x(n)\rvert^2$ $= \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2\, d\omega$

■ The relationship between the Fourier transform and the z-transform

$X(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$

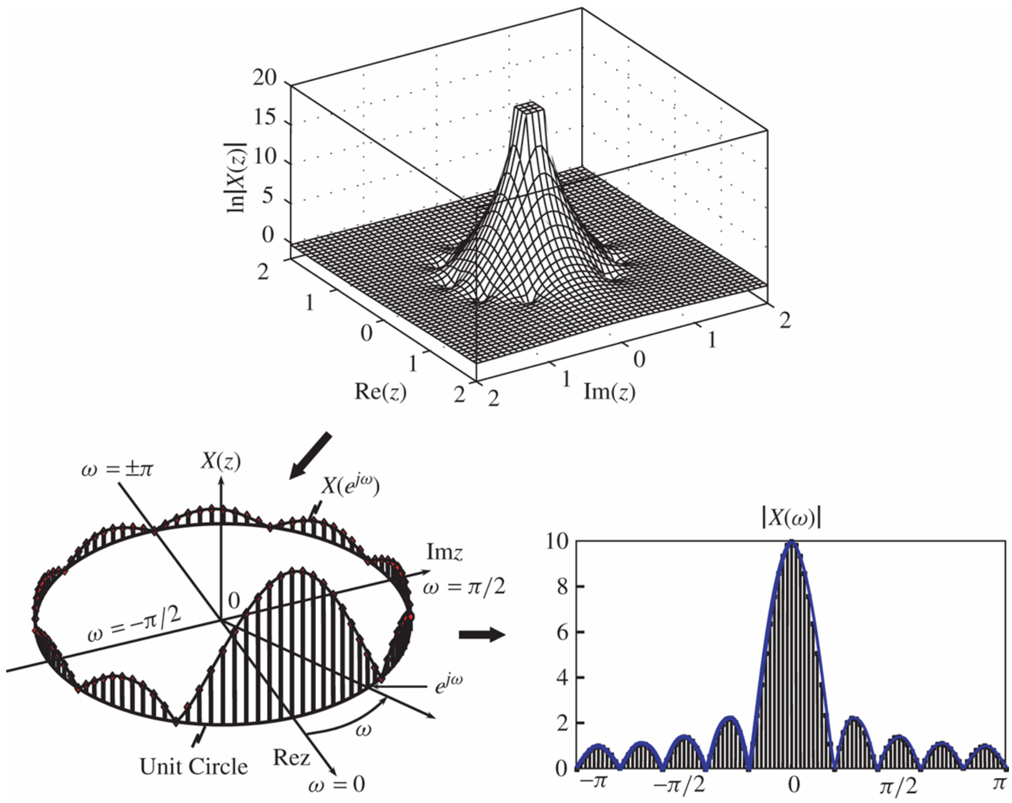


Figure 4.2.9 relationship between $X(z)$ and $X(\omega)$ for the sequence in Example 4.2.4, with $A = 1$ and $L = 10$

■ The four Fourier series/transforms

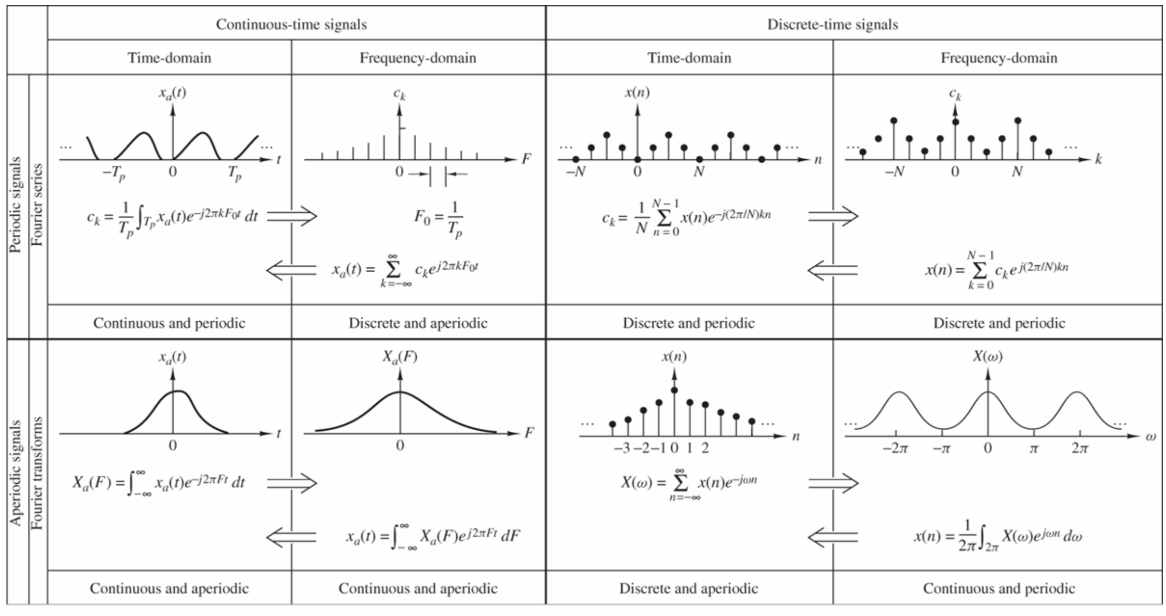


Figure 4.3.1 Summary of analysis and synthesis formulas.

- **Properties of the Fourier transform**

TABLE 4.5 Properties of the Fourier Transform for Discrete-Time Signals

Property	Time Domain	Frequency Domain
Notation	$x(n)$	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_2(\omega)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$ $= X_1(\omega)X_2^*(\omega)$ [if $x_2(n)$ is real]
Wiener–Khintchine theorem	$r_{xx}(l)$	$S_{xx}(\omega)$
Frequency shifting	$e^{j\omega_0 n}x(n)$	$X(\omega - \omega_0)$
Modulation	$x(n)\cos\omega_0 n$	$\frac{1}{2}X(\omega+\omega_0) + \frac{1}{2}X(\omega-\omega_0)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda$
Differentiation in the frequency domain	$nx(n)$	$j\frac{dX(\omega)}{d\omega}$
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$	

3.3 The frequency response function

- **The frequency response function**

- $H(e^{\jmath\omega}) = H(z)|_{z=e^{\jmath w}} \sum_{k=-\infty}^{\infty} h(k)e^{-\jmath k\omega}$.
- $H(e^{\jmath\omega})$ is a function of the frequency variable ω.
- $H(e^{\jmath\omega})$ is, in general, **complex-valued**, and may be written as:
  - Real and imaginary parts: $H(e^{\jmath\omega}) = H_R(e^{\jmath\omega}) + \jmath H_I(e^{\jmath\omega})$ or
  - Magnitude and phase: $H(e^{\jmath\omega}) = |H(w)|e^{\jmath\Theta(\omega)}$,
  - where $|H(e^{\jmath\omega})|^2 = H(e^{\jmath\omega})H^*(e^{\jmath\omega}) = H_R^2(e^{\jmath\omega}) + H_I^2(e^{\jmath\omega})$
  - and $\Theta(e^{\jmath\omega}) = \tan^{-1}\dfrac{H_I(e^{\jmath\omega})}{H_R(e^{\jmath\omega})}$.
- Group delay or envelope delay of H: $\tau_g(e^{\jmath\omega}) = -\dfrac{d\Theta(e^{\jmath\omega})}{d\omega}$.
- Periodicity: Since $x(n) = e^{\jmath n\omega_0} = e^{\jmath n(\omega_0+2\pi)}$, we must have that $H(\omega_0) = H(\omega_0 + 2\pi)$.
- $H(e^{\jmath\omega})$ exists if system is BIBO stable, i.e. $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$.
- LTI-system act as a *filter* to the input signals
  - $|H(e^{\jmath\omega})|$; amplified/attenuated frequencies.
  - $\angle H(e^{\jmath\omega})$; phase shift.

■ **Computation of frequency response function**

- $H(e^{\jmath\omega}) = b_0 \dfrac{\prod_{k=1}^{M}(1 - z_k e^{-\jmath w})}{\prod_{k=1}^{N}(1 - p_k e^{-\jmath w})} = b_0 e^{\jmath w(N-M)} \dfrac{\prod_{k=1}^{M}(e^{\jmath w} - z_k)}{\prod_{k=1}^{N}(e^{\jmath w} - p_k)}$.
- If $\begin{aligned} e^{\jmath w} - z_k &= V_k(\omega)e^{\jmath\Theta_k(\omega)} \\ e^{\jmath w} - p_k &= U_k(\omega)e^{\jmath\Phi_k(\omega)} \end{aligned}$
- then
  $|H(e^{\jmath\omega})| = |b_0|\dfrac{V_1(\omega)\cdots V_M(\omega)}{U_1(\omega)\cdots U_M(\omega)}$
  $\angle H(e^{\jmath\omega}) = \angle b_0 + w(N - M) + (\Theta_1(\omega) + \cdots \Theta_M(e^{\jmath\omega})) - (\Phi_1(\omega)\cdots\Phi_N(\omega))$.

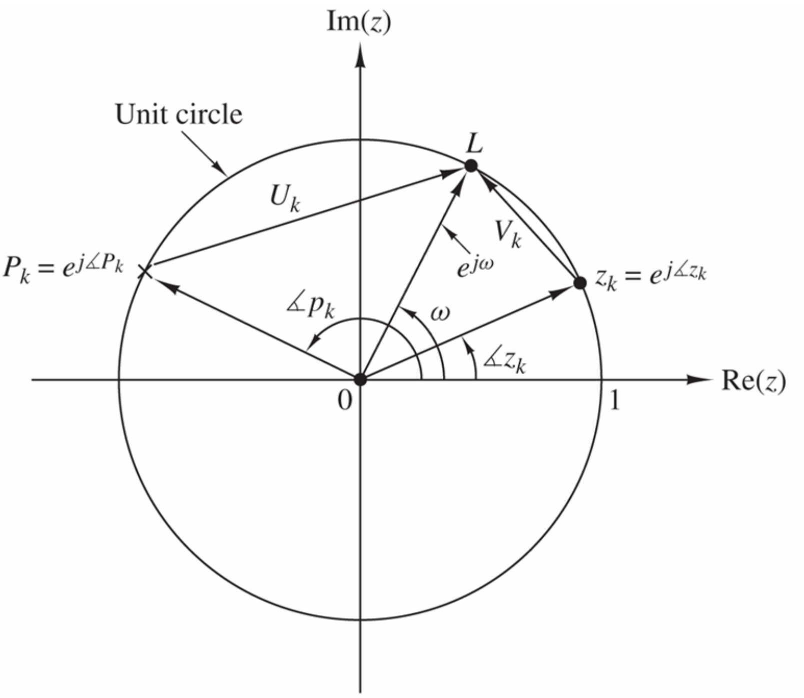


Figure 5.2.2 A zero on the unit circle causes $|H(\omega)| = 0$ and $\omega = \measuredangle z_k$. In contrast, a pole on the unit circle results in $|H(\omega)| = \infty$ at $\omega = \measuredangle p_k$.

3.4 Ideal filters

- **Ideal filter characteristics**
  - Ideal filters have constant magnitude characteristic.
  - Response characteristics of **lowpass**, **highpass**, **bandpass**, **all-pass** and **bandstop** or **band-elimination** filters.
  - Linear phase response
    Ideal filters have linear phase in their passband.
  - In all cases: Ideal filters are not physically realizable.

- **Simple filters**
  - Lowpass
  - Lowpass to highpass transformation
    $H_{\text{hp}}(e^{j\omega}) = H_{\text{lp}}(e^{j(\omega-\pi)})$, i.e.
    $h_{\text{hp}}(n) = (e^{j\pi})^n h_{\text{lp}}(n) = (-1)^n h_{\text{lp}}(n)$.
  - (Digital resonators)
  - (Notch filters)
  - (Comb filters)
  - (All-pass filters)
  - Design of simple digital filters
    1. Based on pole and zero placement
    2. All poles inside unit circle (zeros anywhere).
    3. Complex poles/zeros in complex-conjugate pairs.

3.5 Invertibility of LTI systems

- **Invertibility of LTI systems**
  - A system is **invertible** if there is a one-to-one correspondence between input and output signals
  - LTI systems: $w(n) = h_I(n) * h(n) * x(n) = x(n)$, i.e. $h(n) * h_I(n) = \delta(n)$.
  - Frequency domain: $H(z)H_I(z) = 1$, or $H_I(z) = \frac{1}{H(z)}$.
  - If $H(z)$ rational, that is $H(z) = \frac{B(z)}{A(z)}$, then the inverse is $H_I(z) = \frac{A(z)}{B(z)}$.
    - Zeros of $H(z)$ becomes poles of the inverse system and vice versa.
    - If $H(z)$ is an FIR system, then $H_I(z)$ is an all-pole system.
    - If $H(z)$ is an all-pole system, then $H_I(z)$ is a FIR system.
  - Cannot determine $h_I(n)$ uniquely from $H_I(z)$ without ROC.
- **Minimum-/maximum-/mixed-phase systems**
  - FIR:
    Minimum-phase system: All zeros are inside the unit circle.
    Maximum-phase system: All zeros are outside the unit circle.
    Mixed-phase / nonminimum-phase system otherwise.
  - FIR system: M zeros $\Rightarrow$ 2^M configurations
    One is minimum-phase, one is maximum-phase.
  - IIR:
    Minimum-phase if all poles and zeros are inside the unit circle.
    Maximum-phase if all zeros are outside the unit circle (+all roots inside unit circle, i.e. stable + causal).
    Mixed-phase otherwise (+all roots inside unit circle, i.e. stable + causal).

4 The Discrete Fourier Transform (DFT) and its implementation

4.1 DFT

- **Definition: DFT of length-N signals.**

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, k = 0, 1, \cdots, N-1.$$

- **Definition: Inverse DFT of length-N signals.**

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}, n = 0, 1, \cdots, N-1.$$

- **Relationship to the Fourier-series.**

  The DFT is the Fourier series of the periodic extension, $x_p(n)$, of the length-N signal $x(n)$, multiplied by N,

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) = x(\langle n \rangle_N),$$

  meaning, if the Fourier series coefficients c_k of $x_p(n)$ are

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi kn/N},$$

  then $X(k) = Nc_k$.

- **Relationship to the discrete time Fourier transform (DTFT).**

  The N-point DFT is the spectrum of the DTFT of

$$\bar{X}(e^{j\omega}) = X(e^{j\omega}) * \left(\frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j(L-1)\omega/2} \right)$$

  evaluated at the N points given by $\omega_k = \frac{2\pi}{N} k, \quad k = 0, 1, \cdots, N-1$. In other words:

$$X(k) = \bar{X}\left(\frac{2\pi}{N} k \right).$$

  We can only retrieve $\bar{x}(n)$ from $X(k)$ if $N \geq L$.
  Meaning: we must have at least as many samples in the frequency domain as we do in the time domain.

- **Symmetry properties of the DFT:**

  The DFT inherits all the symmetry properties of the DTFT:

TABLE 5.1 SYMMETRY PROPERTIES OF THE DFT

N-Point Sequence $x(n)$, $0 \le n \le N-1$	N-Point DFT
$x(n)$	$X(k)$
$x^*(n)$	$X^*(N-k)$
$x^*(N-n)$	$X^*(k)$
$x_R(n)$	$X_{ce}(k) = \frac{1}{2}[X(k) + X^*(N-k)]$
$jX_I(n)$	$X_{co}(k) = \frac{1}{2}[X(k) - X^*(N-k)]$
$x_{ce}(n) = \frac{1}{2}[x(n) + x^*(N-n)]$	$X_R(k)$
$x_{co}(n) = \frac{1}{2}[x(n) - x^*(N-n)]$	$jX_I(k)$
Real Signals	
Any real signal $x(n)$	$X(k) = X^*(N-k)$ $X_R(k) = X_R(N-k)$ $X_I(k) = -X_I(N-k)$ $\lvert X(k)\rvert = \lvert X(N-k)\rvert$ $\angle X(k) = -\angle X(N-k)$
$x_{ce}(n) = \frac{1}{2}[x(n) + x(N-n)]$	$X_R(k)$
$x_{co}(n) = \frac{1}{2}[x(n) - x(N-n)]$	$jX_I(k)$

- **Properties of the DFT:**

TABLE 5.2 PROPERTIES OF THE DFT

Property	Time Domain	Frequency Domain
Notation	$x(n), y(n)$	$X(k), Y(k)$
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2X_2(k)$
Time reversal	$x(N-n)$	$X(N-k)$
Circular time shift	$x((n-l))_N$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l))_N$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular convolution	$x_1(n) \textcircled{N} x_2(n)$	$X_1(k)X_2(k)$
Circular correlation	$x(n) \textcircled{N} y^*(-n)$	$X(k)Y^*(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}X_1(k) \textcircled{N} X_2(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

- **Circular convolution and time shifts**

  All operations on finite length (N) sequences are carried out modulo-N:
  Circular convolution:

$$y(n) = \sum_{k=0}^{N-1} x(n)h(\langle k-n\rangle_N).$$

  Circular time shift:

$$x(\langle n-k\rangle_N) = \begin{cases} x(n-k) & \text{for} \quad k \le n \\ x(N+(n-k)) & \text{for} \quad k > n. \end{cases}$$

4.2 The Fast Fourier Transform (FFT)

- **Fast Fourier Transform (FFT) algorithms**

  Reduce the computational complexity from $O(N^2)$ to $O(N \log_2 N)$ operations.

- **Definition: Inverse DFT of length-N signals.**

  Decimation-in-time FFT algorithm: An N-point DFT of a sequence $x(n)$ can be calculated from two $N/2$-point DFTs:

$$
\begin{aligned}
X(k) &= X_{\mathrm{e}}(k) + W_N^k X_{\mathrm{o}}(k), \quad k = 0, 1, \ldots, \frac{N}{2} - 1 \\
X\left(k + \frac{N}{2}\right) &= X_{\mathrm{e}}(k) - W_N^k X_{\mathrm{o}}(k), \quad k = 0, 1, \ldots, \frac{N}{2} - 1.
\end{aligned}
\tag{1}
$$

  where $X_{\mathrm{e}}(k)$ and $X_{\mathrm{o}}(k)$ are the DFTs of the even- and odd-numbered samples of $x(n)$.

5 Design of digital filters

5.1 General concideration and linear phase FIR filters

- **Advantages in using an FIR filter**
  1. Can be designed with exact linear phase.
  2. Filter structure always stable with quantized coefficients.
  3. The design methods are generally linear.
  4. They can be realized efficiently in hardware.
  5. The filter startup transients have finite duration.
- **Commonly used approaches to FIR filter design**
  - FIR filter design is based on a direct approximation of the specified magnitude response, with the often added requirement that the phase be linear.
  - The design of an FIR filter of order $M-1$ may be accomplished by finding either the length-M impulse response samples $\{h(n)\}$ or the M samples of its frequency response $H(e^{j\omega})$.
  - Three commonly used approaches to FIR filter design
    1. Windowed Fourier series approach.
    2. Frequency sampling approach.
    3. Computer-based optimization methods.
- **Advantages in using an IIR filter**
  - Reasons for conversion of analog IIR-filters to a digital filters:
    1. Analog approximation techniques are highly advanced.
    2. They usually yield closed-form solutions.
    3. Extensive tables are available for analog filter design.
    4. Many applications require digital simulation of analog systems.
  - IIR instead of FIR filters:
    Order of an FIR filter, in most cases, is considerably higher than the order of an equivalent IIR filter meeting the same specifications, and FIR filter has thus higher computational complexity.
- **Commonly used approaches to IIR filter design**
  - Most common approach to IIR filter design:
    1. Convert the digital filter specifications into an analog prototype lowpass filter specifications.
    2. Determine the analog lowpass filter transfer function $H_a(s)$.
    3. Transform $H_a(s)$ into the desired digital transfer function $G(z)$.
  - Alternative method: Computer-based optimization method.
- **Systems with linear phase**
  - A LTI system has **linear phase** if

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\alpha\omega}.$$

- A LTI system has **generalized linear phase** if
$$H(e^{j\omega}) = A(e^{j\omega})e^{-j(\alpha\omega-\beta)}.$$
where $A(e^{j\omega})$ is real-valued function of ω and $\beta \in \Re$.

- If system is to be both causal and have linear phase, it must be FIR!

- A sufficient condition for a real-valued FIR filter to have generalized linear phase is that $h(n)$ is
  - Symmetric: $h(n) = h(M-1-n), \quad n = 0, 1, \ldots, M-1$.
    Then $\alpha = (M-1)/2$ and $\beta = 0 \vee \pi$.
  - Antisymmetric: $h(n) = -h(M-1-n), \quad n = 0, 1, \ldots, M-1$.
    Then $\alpha = (M-1)/2$ and $\beta = \pi/2 \vee 3\pi/2$.

■ **Zero-location for linear FIR filters**

- If $h(n)$ symmetric/antisymmetric then
  - $h(n) = \pm h(M-1-n), \quad n = 0, 1, \ldots, M-1$
  - $z^{-(M-1)}H(z^{-1}) = \pm H(z)$.
  - If z_0 root, then $1/z_0$ also root (reciprocal pairs).

- If $h(n)$ real then
  - $H(z) = H^*(z^*)$.
  - If z_1 complex root, then z_1^* also root (complex conjugate roots).

- Linear phase real FIR-filters:
  If z_1 zero, then $1/z_1, z_1^*$ and $1/z_1^*$ are zeros.

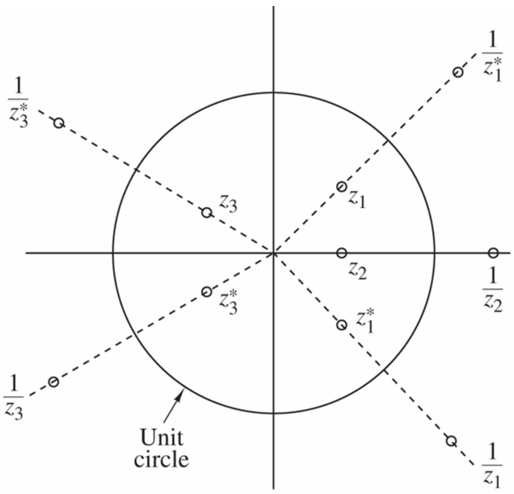


Figure 10.2.1 Symmetry of zero locations for a linear-phase FIR filter.

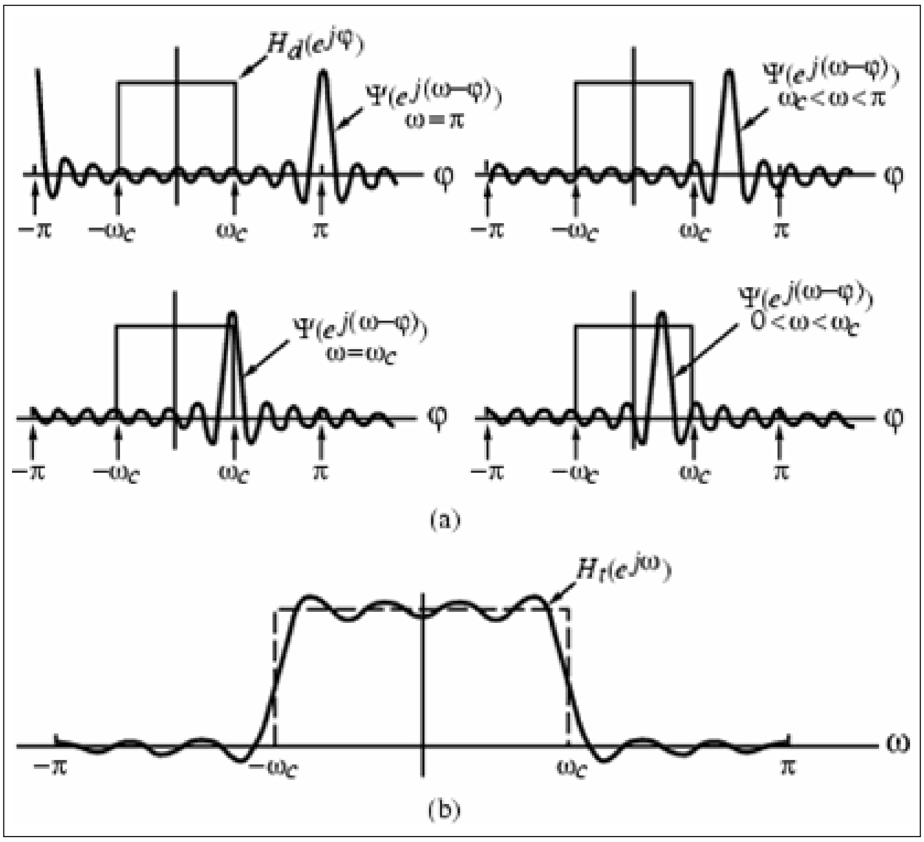
■ **The different linear phase filters (Type I–IV)**

Linear phase filter type	**Filter order**	**Symmetry of Coefficients**	**H(f = 0)**	**H(f = 1)** (Nyquist)
Type I	Even	$h(n) = h(M-1-n), \quad n = 0 \ldots M-1$	No rest.	No rest.
Type II	Odd	$h(n) = h(M-1-n), \quad n = 0 \ldots M-1$	No rest.	$H(1) = 0$
Type III	Even	$h(n) = -h(M-1-n), \quad n = 0 \ldots M-1$	$H(0) = 0$	$H(1) = 0$
Type IV	Odd	$h(n) = -h(M-1-n), \quad n = 0 \ldots M-1$	$H(0) = 0$	No rest.

- The phase delay and group delay are equal and constant over the frequency band.
  - For an order $M-1$ filter (of length M), the delay is $(M-1)/2$.

- In Matlab, the functions `fir1`, `fir2`, `firls`, `firpm`, `fircls`, `fircls1` and `firrcos` design type I and II linear phase FIR filters by default.
  - Both `firls` and `firpm` design type III and IV given a 'hilbert' or 'differentiator' flag.
  - Not possible to design odd-order type II highpass and bandstop filters!

■ **Gibbs effect**



5.2 FIR filter design

■ **The window method**

- Select an ideal filter, $h_d(n)$, and truncate it with a window, $w(n)$.
  - $h(n) = h_d(n)w(n)$.
  - $w(n)$ finite-length window, symmetric about midpoint.
  - $H(e^{j\omega}) = H_d(e^{j\omega}) \circledast W(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\nu})W(e^{j(\omega-\nu)})d\nu.$
- How well $H(e^{j\omega})$ approximates $H_d(e^{j\omega})$ is determined by
  1. The width of the main lobe of $W(e^{j\omega})$.
  2. The peak side-lobe amplitude of $W(e^{j\omega})$.
- Pro: Simple
- Con: Lack of precise control of w_p and w_s.

■ **Frequency sampling method**

- The desired response, $H_d(e^{j\omega})$ is sampled uniformly at $\omega_k = \frac{2\pi}{M}(k+\alpha)$, $M/2$ points (symmetry) between 0 and π.

- From $H_d(e^{j\omega}) = \sum_{n=0}^{M-1} h_d(n)e^{-j\omega n}$ we obtain
  - $h(n) = \frac{1}{M}\sum_{k=0}^{M-1} H_d(e^{j\omega_k})e^{j\omega_k n}$,
  - $y(n) = b_0x(n) + b_1x(n-1) + \cdots + b_{M-1}x(n-M+1)$
    $= \sum_{k=0}^{M-1} b_k x(n-k) \quad n = 0, 1, \ldots, M-1.$
- OK at the frequency samples, but no control in-between.
- Introduction of transition samples (from tables) improves the solution.

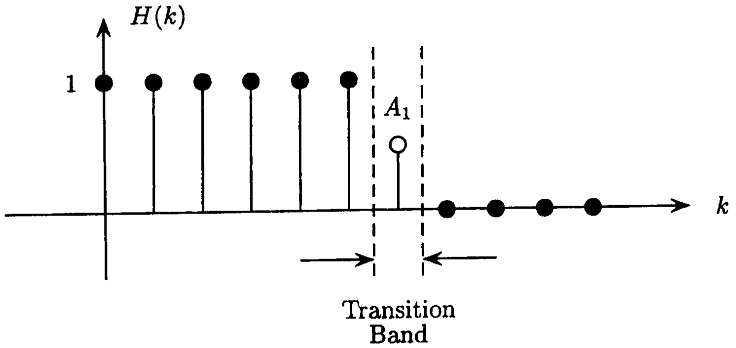


Fig. 9-3. Introducing a transition sample with an amplitude of A_1 in the frequency sampling method.

5.3 IIR filter design

■ **Bilinear transform**

- An analog prototype filter described by $H_a(s)$.
- Mapping from s-plane to z-plane;
  $$H(z) = H_a(s)|_{s=m(z)}$$
  where $s = m(z)$ is the mapping function with the following properties
  - $j\Omega$-axis should be mapped to the unit circle, $|z| = 1$ (one-to-one and onto).
  - Points in the left-half s-plane should be mapped inside the unit circle.
  - $m(z)$ should be rational so that a rational $H_a(s)$ is mapped to a rational $H(z)$.
- Bilinear transform
  - The mapping from s-plane to z-plane defined by $s = \frac{2}{T_s}\frac{1-z^{-1}}{1+z^{-1}}$.
  - I.e. $H(z) = H_a\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$.
  - Rational, one-to-one and onto, but nonlinear relation between the $j\Omega$-axis and the unit circle (warping): $w = 2\arctan\left(\frac{\Omega T_s}{2}\right)$.
  - Result of warping: Bilinear trans. will only preserve the magn. resp. of analog filters that have an ideal response that is piecewise constant.
  - Steps to follow:
    1. Prewarp w_p and w_s. With $T_s = 2$, $\Omega = \tan(w/2)$.
    2. Design an analog lowpass filter
    3. Apply bilinear transf.

6 Sampling and Reconstruction of Signals

- **Frequency and sampling notations**
  - Sampling a continus signal $x_a(t)$: $x(n) = x_a(t)\big|_{t=nT}$
  - Sampling interval: T
  - Sampling rate: $F_T = 1/T$
  - Normalized (digital) frequency: $f = F/F_T$, where F is the physical frequency (Hertz typically). f is unitless (cycles per sample), and its domain is $f \in [0, 1]$ or $f \in [-1/2, 1/2]$
  - Normlized angular frequency: $\omega = \Omega/F_T$, where $\Omega = 2\pi F$ is the physical angular frequency (radians/second typically). ω is unitless (radians per sample), and its domain is $\omega \in [0, 2\pi]$ or $\omega \in [-\pi, \pi]$
  - [Side-note: Sometimes care has to be taken because several numerical processing packages and routines, for example in Matlab, use the Nyquist frequency $F_T/2$ instead of F_T when going from continuous frequency to normalized frequency]

- **Shannon's Sampling Theorem**

  If a continuous-time signal $x(t)$ is to be perfectly reconstructed from a sampled signal $x(n) = x(nT) = x(n/F_T)$, we must have

$$F_T > 2F_{\max}$$

  where $F_{\max}$ is the lowest value for which $|X(|F_{\max}|)| = 0$. (I.e. $F_{\max}$ is the highest frequency component in $x(t)$.)
  $F_T < 2F_{\max}$ is called *undersampling*.
  $F_T = 2F_{\max}$ is called *critical sampling* and $2F_{\max}$ is referred to as the *Nyquist rate* for a signal with maximum frequency component $F_{\max}$.
  $F_T > 2F_{\max}$ is called *oversampling*.

- **Ideal Reconstruction**

  When a sampling $x(n) = x(nT)$ satisfies Shannon's sampling theorem, we can reconstruct $x(t)$ perfectly from $x(n)$ through

$$x(t) = \sum_{k=-\infty}^{\infty} x(k) \frac{\sin\left(\frac{\pi}{T}(t - kT)\right)}{\frac{\pi}{T}(t - nT)}$$

- **Sampling of Bandpass Signals**

  A signal strictly bandlimited to $F \in [F_L, F_H]$ is called a *bandpass signal* with *bandwidth* $B = F_H - F_L$.

  The signal can be perfectly reconstructed if: $2B \leq F_T \leq 4B$

7 Multi-Rate Digital Signal Processing

- **Downsampling by an Integer Factor D**

Rate change: $F_T \to \frac{1}{D}F_T \Rightarrow T \to DT$.
Implementation:

$$y(n) = x(Dn) = x(DTn)$$

Shannon's sampling theorem still applies, $x(t)$ must be band-limited to $\frac{F_T}{2D}$ for zero aliasing. *Equivalent:* $x(n)$ must be bandlimited to $\frac{\pi}{D}$. This is achieved by filtering $x(n)$ with a low-pass filter with cut-off frequency $\frac{\pi}{D}$ before removing all but the D^{th} samples. If $x(n)$ is band limited to $\frac{\pi}{D}$, the spectrum of $y(n)$ is related to that of $x(n)$ through

$$Y(e^{J\omega}) = \frac{1}{D}X\left(e^{J\frac{\omega}{D}}\right)$$

Generally, even if $x(n)$ is not band limited, the spectrum of $y(n)$ is related to that of $x(n)$ through

$$Y(e^{J\omega}) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{J\frac{\omega-2\pi k}{D}}\right)$$

- **Upsampling by an Integer Factor I**

  Rate change: $F_T \to IF_T \Rightarrow T \to \frac{1}{I}T$.
  Implementation:

  $$y(n) = \bar{x}(n) * h(n)$$

  where

  $$\bar{x}(n) = \begin{cases} x(n/I) & \text{for} \quad \langle n \rangle_I = 0 \\ 0 & \text{otherwise} \end{cases}$$

  and $h(n)$ is a low-pass filter with cut-off frequency $\frac{\pi}{I}$. The spectrum of $\bar{x}(n)$ is related to the spectrum of $x(n)$ through

  $$\bar{X}(e^{J\omega}) = X(e^{JI\omega})$$

- **Fractional Rate Change by a Factor $\frac{I}{D}$**

  Combine upsampling and downsampling: $F_T \to \frac{I}{D}F_T \Rightarrow T \to \frac{D}{I}T$. One should always perform upsampling first, because of two reasons.

  1. The two low-pass filters are in cascade, and can therefore be replaced by one filter.
  2. For $\frac{I}{D} > 1$, there is no information loss when upsampling prior to downsampling, while there might be when downsampling prior to upsampling unless the input signal is sufficiently oversampled.

8 Operator notation and diagrams

- **A delay corresponds to multiplication with z^{-1}**
  - Referred as **unit-delay property**: $x(n-1) \overset{\mathcal{Z}}{\longleftrightarrow} z^{-1}X(z)$
  - Graphical representation of addition
  - Graphical representation of multiplication by a factor
  - Block notation of $y(n) = b_0 x(n) + b_1 x(n-1)$: