Solving a 1D Poisson equation with the finite element method (P2 elements)

Obligatory project No. 2 in IN5270/IN9270

General information

- This project has the aim of strengthening IN5270 students' hands-on skills of applying the finite element method.
- Collaboration/discussion among students is encouraged, but each student should submit a small set of files, including a simple project report and Python program(s), which are written/programmed by her/himself.
- Please organize the your submission as a directory named `FEM1D_project` to hold all your files of this project. Include a `README` file with a short overview of the different files. (Info about when, where and how to make the submission will be given at the semester webpage.)
- Write a short report answering the non-programming tasks (see below). LaTeX is probably the preferred format, but there are several other options[1] too. Regardless of format, the report must be in an easy-to-read format like PDF or HTML (or Jupyter notebook).

1 The 1D PDE

We want to use the finite element to numerically solve the following 1D Poisson equation:

$$-u''(x) = 2x - 1, \quad x \in \Omega = [0, 1], \quad u'(0) = C,\ u(1) = D,$$

where C and D are two prescribed scalar constants.

2 P2 elements

You are required to use a *uniform* mesh consisting of N_e elements. The finite basis functions to be used are piecewise quadratic polynomials (also called *P2* elements).

[1] `http://hplgit.github.io/teamods/writing_reports/index.html`

3 Tasks

Task 1

Derive the weak variational formulation of the 1D PDE.

Task 2

Show how to compute the element matrix and vector for

- the leftmost element;
- the rightmost element
- an arbitrary interior element with index e (where $1 \leq e \leq N_e - 2$).

Task 3

Write a Python program, which accepts C, D and N_e as input parameters, for finding the numerical solution using the finite element method with P2 elements. (Cellwise computations and a subsequent assembly process should be used to set up the resulting linear system. It is free to choose any Python module for solving the linear system.)

Task 4

Derive the exact solution $u_e(x)$ of the 1D PDE, which is also dependent on the values of C and D. Choose a few combinations of specific C and D values, study how the L_2 norm of the error $e(x) = u_e(x) - u(x)$ (where $\|e(x)\|_{L_2(\Omega)} = \sqrt{\int_\Omega e^2(x)\,\mathrm{d}x}$) changes with the element size $h = 1/N_e$. What is the convergence rate observed?