UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam:	**INF 4300 / INF 9305 – Digital image analysis**
Date:	**Thursday December 1, 2016**
Exam hours:	**14.30-18.30 (4 hours)**
Number of pages:	**8 pages of sketches to a solution**
Enclosures:	**None**
Allowed aid:	**Calculator**

- Read the entire exercise text before you start solving the exercises. Please check that the exam paper is complete. If you lack information in the exam text or think that some information is missing, you may make your own assumptions, as long as they are not contradictory to the "spirit" of the exercise. In such a case, you should make it clear what assumptions you have made.

- You should spend your time in such a manner that you get to answer all exercises shortly. If you get stuck on one question, move on to the next question.

- Your answers should be **short**, typically a few sentences and / or a sketch should be sufficient.

Good luck!!

Exercise 1: Moments

a) Give the expression for an ordinary moment of order (p+q) of an object in a 2D digital image.

Answer:

$$m_{pq} = \sum_x \sum_y x^p y^q f(x,y)$$

b) Describe, in words or using mathematical expressions, how you get from an ordinary to a central moment.

Answer: Shift the origin to the centre of mass of the object, and compute the moment as before.

$$\mu_{p,q} = \sum_x \sum_y (x - \bar{x})^p (y - \bar{y})^q f(x,y)$$

c) How can you get a central moment that is invariant to scale; and what else are such moments invariant to?

Answer: Normalize by the area(volume) to the power of gamma, as given by the equation below.

$$\eta_{pq} = \frac{\mu_{pq}}{(\mu_{00})^{\gamma}}, \quad \gamma = \frac{p+q}{2} + 1, \quad p+q \geq 2$$

A scale-normalized central moment is also invariant to position.

d) In the table to the right, "+", "-", and "0" indicates a positive, negative, or zero value of seven scale-normalized central moments (η_{11}, η_{20}, η_{02}, η_{21}, η_{12}, η_{30}, η_{03}) for objects symmetric about the y-axis ("M"), the x-axis ("C"), and both ("O") in a continuous analog image.

	η_{11}	η_{20}	η_{02}	η_{21}	η_{12}	η_{30}	η_{03}
M	0	+	+	-	0	0	-
C	0	+	+	0	+	+	0
O	0	+	+	0	0	0	0

Which of Hu's moment combinations in their simplified version (see below) may be useful for the task of obtaining rotation-invariant shape features of elliptical objects in a digital image? Please explain!

Hu's moments; a bit simplified

For second order moments (p+q=2), two invariants are used:

$\varphi_1 = \eta_{20} + \eta_{02}$

$\varphi_2 = (\eta_{20} - \eta_{02})^2 + 4\eta_{11}^2$

For third order moments, (p+q=3), we can use

$a = (\eta_{30} - 3\eta_{12})$, $b = (3\eta_{21} - \eta_{03})$,

$c = (\eta_{30} + \eta_{12})$, and $d = (\eta_{21} + \eta_{03})$

and simplify the five last invariants of the set:

$\varphi_3 = a^2 + b^2$

$\varphi_4 = c^2 + d^2$

$\varphi_5 = ac[c^2 - 3d^2] + bd[3c^2 - d^2]$

$\varphi_6 = (\eta_{20} - \eta_{02})[c^2 - d^2] + 4\eta_{11}cd$

$\varphi_7 = bc[c^2 - 3d^2] - ad[3c^2 - d^2]$

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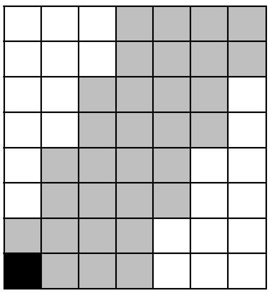
Answer: For an ellipse oriented parallel to axis system; a,b,c, and d are all zero.
This implies that Hu's moments 3 to 7 are zero. So only Hu's first and second moment are useful.
And since the moments are rotation invariant, this will also be true for all other orientations.

e) What may cause zero entries in the table above to show up as non-zero in digital images?

Answer: Noise, caused by the discrete sampling, and by the quantization.

Exercise 2: Chain coding

a) Give the absolute and relative chain code of the binary object below, starting at the origin in the lower left corner having coordinates (0,0), using an 8-directional code, where 2 is up (forward).



Answer: Absolute chain code is 21212120006565656444 (20 elements)
Relative code is **0**1313130220131313022 (also 20 elements)

b) How do we make a relative chain code start point invariant?

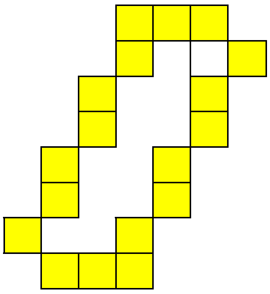
Answer: We obtain start point invariant code by "Minimum circular shift" of the relative code.

c) Discuss why the start point invariance is not unique in this case.

Answer: Since we here have an object that is invariant under point reflection through its center of mass (point symmetry), two symmetric start points on the perimeter give the same relative code.

d) Given the 18 element relative code 021031313021031313 of a new object, how can you use a general property of chain codes to easily determine how it differs from the object in the figure above?

Answer: This is a minimum circular shifted relative code which is shorter than the minimum circular relative code of the original object. We note that both are point symmetric objects.
We do not bother comparing the objects on the code level. But since the relative code is **a reversible transform,** we can reconstruct the object from the given relative code, getting the perimeter of the original object minus its two symmetric start points, and easily see the difference.



e) How do we normalize the absolute chain code in a) to get rotation invariance? Please perform the calculations!

Answer: By finding the first difference of the absolute chain code,
6 7 1 7 1 7 1 6 0 0 6 7 1 7 1 7 1 6 0 0
and then finding the minimum circular shift of the first difference:
0 0 6 7 1 7 1 7 1 6 0 0 6 7 1 7 1 7 1 6

Next three questions for PhD-students only:

f) An alternative way of obtaining a rotation invariant chain code would be to find the orientation of the object, rotate it, and then perform the chain coding. Please explain the steps of finding the orientation of the object in a) !

g) Compute the numerical values that are needed to find the object orientation. *hint:* $(N+0.5)^2 = N(N+1)+0.25$

h) What does it take for this alternative rotation invariance to be valid?

Answer: We would need to find the three second order central moments, to be used in the equation

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\mu_{11}}{\mu_{20}-\mu_{02}}\right)$$

In the two moments of inertia,

$$\mu_{20} = \sum_x \sum_y (x-\bar{x})^2 f(x,y)$$

$$\mu_{02} = \sum_x \sum_y (y-\bar{y})^2 f(x,y)$$

we also need the first order moments and the area of the object,

$$\bar{x} = \frac{m_{10}}{m_{00}}, \qquad \bar{y} = \frac{m_{01}}{m_{00}}$$

Numerical values:

Area=m_{00}=32, m_{10} =2*0+4*1+6*2+8*3+6*4+4*5+2*6=96 => cmx=96/32 =3
m_{01} =4*(1+2+3+4+5+6+7)=112 => cmy =3.5 **(x,y) of center of mass: (3,3.5)**

moments of inertia:

$mu_{20}=(-3)^2*2 + (-2)^2*4 + (-1)^2*6 + (0)^2*8 + (1)^2*6 + (2)^2*4 + (3)^2*2$
=18+16+6+0+6+16+18 = 80
$mu_{02}= 4[(-3.5)^2 + (-2.5)^2 + (-1.5)^2 + (-0.5)^2 + (0.5)^2 + (1.5)^2 + (2.5)^2 + (3.5)^2]$
= 4[12.25 + 6.25 + 2.25 + 0.25 + 0.25 + 2.25 + 6.25 + 12.25] = 4*42 = 168

mu_{11} = -3[-3.5 -2.5] -2[-3.5-2.5-1.5-0.5] -1[-3.5-2.5-1.5-0.5+0.5+1.5] +0[….]
+1[-1.5-0.5+0.5+1.5+2.5+3.5] + 2[0.5+1.5+2.5+3.5] + 3[2.5+3.5] = 80

rotation angle: ≈ 60 degrees

This alternative invariance is only valid if the boundary itself is invariant to the indicated rotation. Otherwise, it will only be an approximation.

Exercise 3: Classification

Consider a two-dimensional feature vector and a set of points from 2 classes in 2D feature space:
Class 1 has points: (3,0), (5,0), (7,0), (5,2), (5, -2)
Class 2 has points: (0,5), (0,3), (0,7), (-2, 5), (2,5)
The discriminant function for Gaussian classifier is in the general form:

$$g(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_i)^T \Sigma_i^{-1}(\mathbf{x}-\boldsymbol{\mu}_i) - \frac{d}{2}\ln 2\pi - \frac{1}{2}\ln|\Sigma_i| + \ln P(\omega_i)$$

a) Compute the mean for each class

Solution:

$$\mu_1 = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \quad \mu_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

b) Show that the covariance between feature 1 and 2 is 0 for both classes.

For class 1: $\sigma_{1,12}$= *1/5[(3-5)(0-0)+(5-5)(0-0)+(7-5)(0-0)+(5-5)(2-0)+(5-5)(-2-0)] = 0*
For class 2: $\sigma_{2,12}$=*1/5[(0-0)(5-5)0(0-0)(3-5)+(0-0)(7-5)+(-2-0)(5-5)+(2-0)(5-5)] = 0*

c) Show how the discriminant function can be simplified in this case with a classifier with equal diagonal covariance matrices.

Can be simplified to

$$g_i(x) = \mu_{i^T} x - \frac{1}{2}\mu_{i^T}\mu_i + \ln P(\omega_i)$$

d) Find an expression for the decision boundary using this simplified discriminant function.

Can be simplified to

$$g_i(x) = g_j(x) \Leftrightarrow \mu_{i^T} x - \frac{1}{2}\mu_{i^T}\mu_i + \ln P(\omega_i) = \mu_{j^T} x - \frac{1}{2}\mu_{j^T}\mu_j + \ln P(\omega($$

e) Compute the decision boundary if we assume equal prior probabilities.
The equation is:

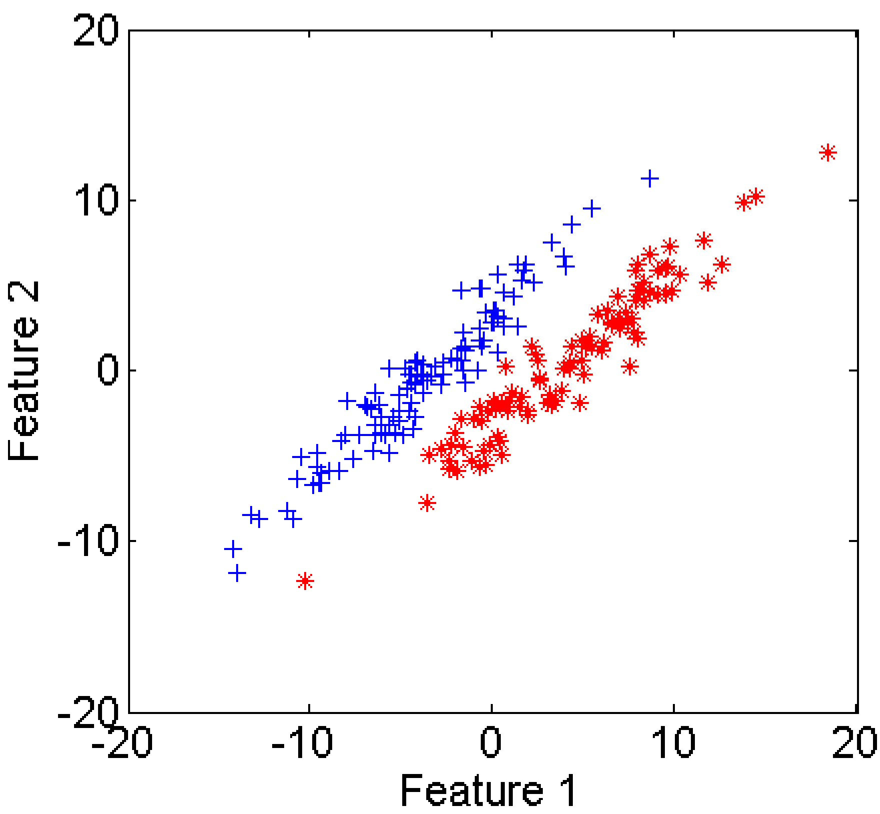
$$5x_1 - 25 = 5x_2 - 25 \Leftrightarrow x_2 = x_1$$

f) Sketch the class means and the decision boundary in a plot if we assume that the two classes have equal prior probability.

g) If $P(\omega_1) = 0.75$, in which direction will the decision boundary change? Indicate this on the plot.

Exercise 4: Linear feature transforms

You are given a 2D dataset with 2 classes as plotted in the figure below.



The data has the following properties:

Covariance matrix C: $\begin{bmatrix} 36 & 21 \\ 21 & 19 \end{bmatrix}$

Eigenvectors of C: $v1 = \begin{bmatrix} -0.55 \\ 0.83 \end{bmatrix}$ $\quad v2 = \begin{bmatrix} 0.83 \\ 0.55 \end{bmatrix}$

Eigenvalues of C: $\lambda 1 = 5, \quad \lambda 2 = 51$

a) Explain the criterion function that principal component analysis (PCA) optimizes

PCA find the direction with maximum variance, which is equivalent to minimizing the signal representation error. (Either answer is correct)

b) Explain if PCA requires any normalization of the input data

PCA normally use the correlation matrix, but we can use the covariance matrix if we subtract the mean.

c) Which direction vector gives the first principal component?

The direction given by the vector [0.83, 0.55]^T

d) PCA is a linear transform $y=A^Tx$ of the input data x. What is A for the data example?

Solution

$$A = \begin{bmatrix} 0.83 & -0..55 \\ 0.55 & 0.83 \end{bmatrix}$$

e) How much of the variance in the data is explained by the first principal component?

Given by the eigenvalues 51/(51+5)=91%

f) Which geometrical relation is there between the first and the second principal component?

They are perpendicular (and the correlation between them is zero)

g) **PhD Students only:**
From the data listed above, we can construct the covariance matrix of the transform data y. What is the covariance matrix of y? No computation is needed.

The variance is equal to the eigenvalues, and the covariance is zero:

$$\text{cov}(y) = \begin{bmatrix} 51 & 0 \\ 0 & 5 \end{bmatrix}$$

h) In this exercise, the dominant direction is found by principal component analysis. Based on other topics in this course, could you suggest another method that could be used to find the dominant direction (unsupervised, no class labels used)?

This is also equal to the direction that has the smallest moment of inertia, and can be found from the equation to find the orientation of an object (formulae not required)

$$\frac{1}{2}\tan^{-1}\left(\frac{2\mu_{11}}{\mu_{20}-\mu_{02}}\right)$$

i) An alternative to PCA is Fisher's linear discriminant. Which criterion function does Fisher use?

Fisher maximize the between-class scatter and minimize the within-class scatter, with function $J= w^T S_W w / w^T S_B w$

j) Do either PCA or Fisher have any limitations regarding how many features the transform can produce? Justify you answer.

No limitations with PCA, but with Fisher max dimension K-1, if K is the number of classes. This is because the rank of SB is K-1