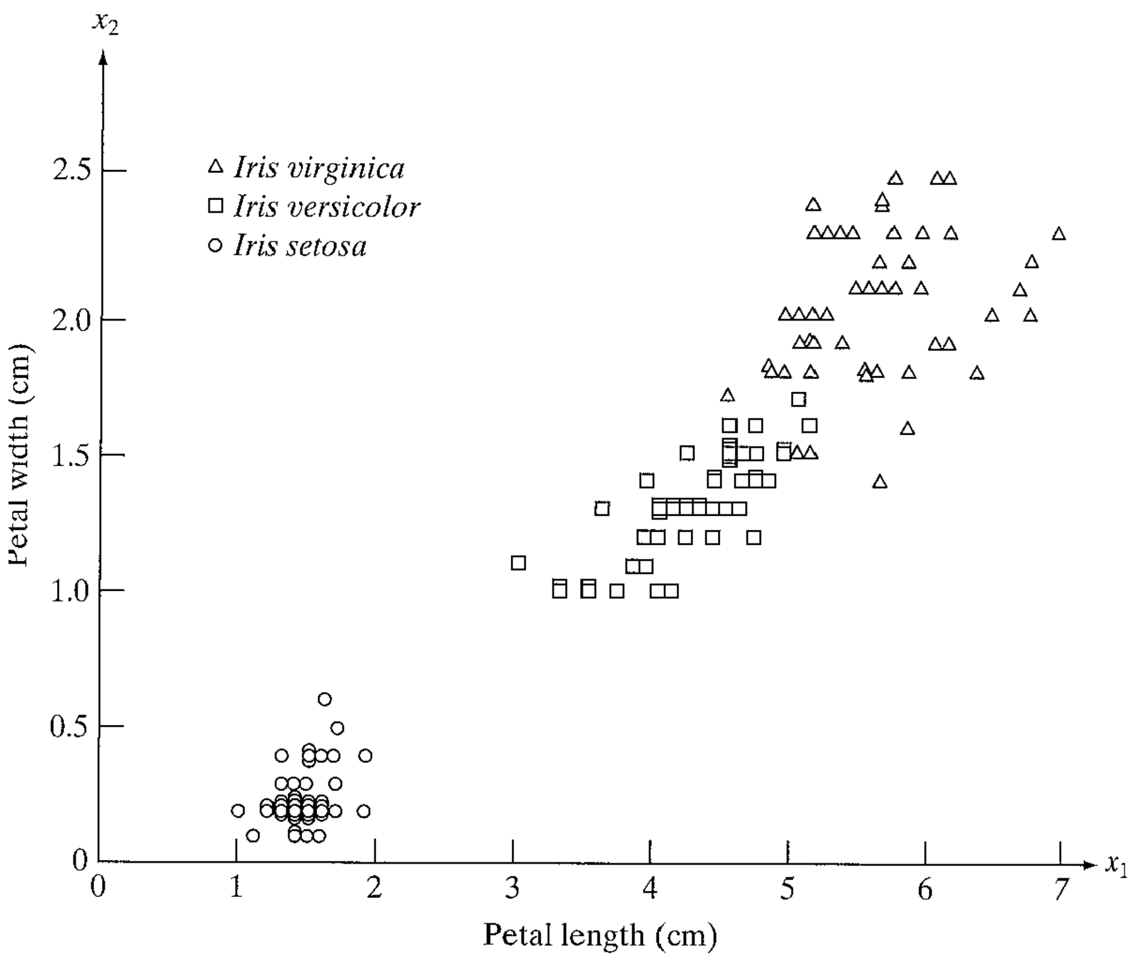
Solution of selected exercises

Exercises INF 4300 related to the lecture 12.10.17

2. Finding the decision functions for a minimum distance classifier.

A classifier that uses diagonal covariance matrices is often called a minimum distance classifier, because a pattern is classified to class that is closest when distance is computed using Euclidean distance.



a. In the above figure, find the class means just by looking at the plot.

b. If this data is classified using a minimum distance classifier, sketch the decision boundaries on the plot.

Solution:

Problem 12.1

(a) By inspection, the mean vectors of the three classes are, approximately, $\mathbf{m}_1 = (1.5, 0.3)^T$, $\mathbf{m}_2 = (4.3, 1.3)^T$, and $\mathbf{m}_3 = (5.5, 2.1)^T$ for the classes Iris setosa, versicolor, and virginica, respectively. The decision functions are of the form given in Eq. (12.2-5). Substituting the above values of mean vectors gives:

$$\begin{aligned} d_1(\mathbf{x}) &= \mathbf{x}^T\mathbf{m}_1 - \frac{1}{2}\mathbf{m}_1^T\mathbf{m}_1 = 1.5x_1 + 0.3x_2 - 1.2 \\ d_2(\mathbf{x}) &= \mathbf{x}^T\mathbf{m}_2 - \frac{1}{2}\mathbf{m}_2^T\mathbf{m}_2 = 4.3x_1 + 1.3x_2 - 10.1 \\ d_3(\mathbf{x}) &= \mathbf{x}^T\mathbf{m}_3 - \frac{1}{2}\mathbf{m}_3^T\mathbf{m}_3 = 5.5x_1 + 2.1x_2 - 17.3 \end{aligned}$$

(b) The decision boundaries are given by the equations

$$\begin{aligned} d_{12}(\mathbf{x}) &= d_1(\mathbf{x}) - d_2(\mathbf{x}) = -2.8x_1 - 1.0x_2 + 8.9 = 0 \\ d_{13}(\mathbf{x}) &= d_1(\mathbf{x}) - d_3(\mathbf{x}) = -4.0x_1 - 1.8x_2 + 16.1 = 0 \\ d_{23}(\mathbf{x}) &= d_2(\mathbf{x}) - d_3(\mathbf{x}) = -1.2x_1 - 0.8x_2 + 7.2 = 0 \end{aligned}$$

A plot of these boundaries is shown in Fig. P12.1.

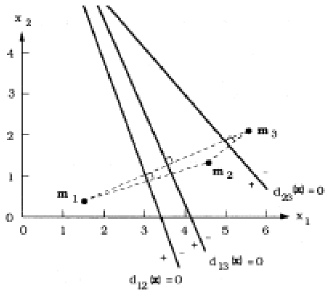


Figure P12.1

3. Discriminant functions

A classifier that uses Euclidean distance computes distance from pattern ***x*** to class j as:

$$D_j(x) = \|x - \mu_j\|$$

Show that classification with this rule is equivalent to using the discriminant function

$$d_j(x) = x^T\mu_j - \frac{1}{2}\mu_j^T\mu_j$$

Solution:

Problem 12.2

From the definition of the Euclidean distance,

$$D_j(\mathbf{x}) = \|\mathbf{x} - \mathbf{m}_j\| = \left[(\mathbf{x} - \mathbf{m}_j)^T(\mathbf{x} - \mathbf{m}_j)\right]^{1/2}$$

Since $D_j(\mathbf{x})$ is non-negative, choosing the smallest $D_j(\mathbf{x})$ is the same as choosing the smallest $D_j^2(\mathbf{x})$, where

$$\begin{aligned} D_j^2(\mathbf{x}) &= \|\mathbf{x} - \mathbf{m}_j\|^2 = (\mathbf{x} - \mathbf{m}_j)^T(\mathbf{x} - \mathbf{m}_j) \\ &= \mathbf{x}^T\mathbf{x} - 2\mathbf{x}^T\mathbf{m}_j + \mathbf{m}_j^T\mathbf{m}_j \\ &= \mathbf{x}^T\mathbf{x} - \mathbf{2}\left(\mathbf{x}^T\mathbf{m}_j - \frac{1}{2}\mathbf{m}_j^T\mathbf{m}_j\right) \end{aligned}$$

We note that the term $\mathbf{x}^T\mathbf{x}$ is independent of j (that is, it is a constant with respect to j in $D_j^2(\mathbf{x})$, $j = 1, 2, \ldots$). Thus, choosing the minimum of $D_j^2(\mathbf{x})$ is equivalent to choosing the maximum of $\left(\mathbf{x}^T\mathbf{m}_j - \frac{1}{2}\mathbf{m}_j^T\mathbf{m}_j\right)$.

❖ Example:

Given ω_1, ω_2 : $P(\omega_1) = P(\omega_2)$ and $p(\underline{x}|\omega_1) = N(\underline{\mu}_1, \Sigma)$,

$p(\underline{x}|\omega_2) = N(\underline{\mu}_2, \Sigma)$, $\underline{\mu}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\underline{\mu}_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 1.1 & 0.3 \\ 0.3 & 1.9 \end{bmatrix}$

classify the vector $\underline{x} = \begin{bmatrix} 1.0 \\ 2.2 \end{bmatrix}$ using Bayesian classification :

- $\Sigma^{-1} = \begin{bmatrix} 0.95 & -0.15 \\ -0.15 & 0.55 \end{bmatrix}$
  - Compute Mahalanobis d_m from μ_1, μ_2 : $d^2{}_{m,1} = [1.0, \quad 2.2] \Sigma^{-1} \begin{bmatrix} 1.0 \\ 2.2 \end{bmatrix} = 2.952$, $d^2{}_{m,2} = [-2.0, \quad -0.8] \Sigma^{-1} \begin{bmatrix} -2.0 \\ -0.8 \end{bmatrix} = 3.672$
- Classify $\underline{x} \rightarrow \omega_1$. Observe that $d_{E,2} < d_{E,1}$

Exercise 3: Classification (From Exam 2015)

a) Consider a two-dimensional feature vector and a set of points in 2D feature space: (-3,6) (-2,4) (-1, 2) (0,0) (1, -2) (2, -4) (3, -6)
Show that the covariance matrix between the two features is :

$$\Sigma = \begin{bmatrix} 4 & -8 \\ -8 & 16 \end{bmatrix}$$

Show all your calculations.

Answer:

$$\mu_1 = (-3-2-1+0+1+2+3)/7 = 0$$

$$\mu 2 = (6+4+2+0+2+4+7)/7 = 0$$

$$\sigma_{12} = \frac{1}{N}\sum_i (x_1-\mu_1)(x_2-\mu_2) =$$

$$\frac{1}{7}[(-3-0)(6-0)+(-2-0)(4-0)+(-1-0)(2-0)+(0-0)(0-0)+(-1)(2)+(-2)(4)+-3)(6)] =$$

$$-56/6 = -8$$

$$\sigma_1^{\ 2} = \frac{1}{N}\sum_i (x_1-\mu_1)^2 = \frac{1}{7}[3*3+2*2+1*1+0+1*1+2*2+3*3] = 28/7 = 4$$

$$\sigma_2^{\ 2} = \frac{1}{N}\sum_i (x_2-\mu_2)^2 = \frac{1}{7}[-6*-6+-4*-4+1*1+0+1*1+4*4+6*6] = 16$$

b) Given the two features defined above, would you base you classification on 1 or 2 features? Justify your answer.

Answer: We note that the points line on a straight line, thus the two features are linear dependent, and there is no need to use more than one of them (and the 2D covariance matrix is singular).

c) The discriminant functions for a multivariate Gaussian classifier are given as:

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_i)^T \Sigma_i^{-1}(\mathbf{x}-\boldsymbol{\mu}_i) - \frac{n}{2}\ln 2\pi - \frac{1}{2}\ln|\Sigma_i| + \ln P(\omega_i)$$

Consider two classes with equal prior probability and

$$\boldsymbol{\mu}_1 = \begin{bmatrix}1\\0\end{bmatrix} \quad \boldsymbol{\mu}_2 = \begin{bmatrix}0\\2\end{bmatrix} \quad \Sigma_1 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \quad \Sigma_1^{-1} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\Sigma_2 = \begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix} \quad \Sigma_2^{-1} = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix} \quad |\Sigma_1| = 1 \quad |\Sigma_2| = 1$$

Can the discriminant function be simplified in this case?

Answer: Somewhat,, we can avoid the ln2π and lnP(ωi) and also the determinant of the covariance matrices, which is equal.

d) Classify the point $x = \begin{bmatrix}3\\0\end{bmatrix}$ by computing the value of the discriminant functions and assign it to the class corresponding to the highest probability.

Answer:

$$g_1(x) = -\frac{1}{2}[3-1, 0-0]^T \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}3-1\\0\end{bmatrix} - 0.5\ln 1 = -2$$

...

$$g_2(x) = -\frac{1}{2}[3-0, 0-2]^T \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}3\\-2\end{bmatrix} - 0.5\ln 1 = -5/2$$

Here we have ignore the two class-independent terms. We should assign it to the class with maximum value, so the pattern will be assigned to class 1.

e) Explain how classifier sensitivity and specitivity are computed, and discuss their importance on a medical classification problem.

Answer: Very short: Sensitivity: TP/(TP+FN) The probability that the test is positive given that the patient is sick. Specifitivity: TN/(TN+FP). The probability that the test is negative given that the patient is not sick. For a sick/healthy scenario we want all sick to be diagnosed as sick, so sensitivity is important and we can allow a lower specitivity.

f) Let us assume that we have a 2-class classification problem with a 1-dimensional feature vector f(x) which is exponentially distributed given the class-conditional parameter λ_i:

$$f(x) = \lambda_i \exp^{-\lambda_i x}$$

Find an expression for the decision boundary for this classification problem.

Answer:
The decision boundary is the point T where

$$\lambda_1 e^{-\lambda_1 T} = \lambda_2 e^{-\lambda_2 T} \Leftrightarrow$$

$$\frac{\lambda_1}{\lambda_2} = e^{(\lambda_1 - \lambda_2)T} \Leftrightarrow$$

$$T = \frac{\ln\left(\frac{\lambda_1}{\lambda_2}\right)}{\lambda_1 - \lambda_2}$$

Exercise 4: Classification (From Exam 2016)

Consider a two-dimensional feature vector and a set of points from 2 classes in 2D feature space:
Class 1 has points: (3,0), (5,0), (7,0), (5,2), (5, -2)
Class 2 has points: (0,5), (0,3), (0,7), (-2, 5), (2,5)
The discriminant function for Gaussian classifier is in the general form:

$$g(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_i)^T \Sigma_i^{-1}(\mathbf{x}-\boldsymbol{\mu}_i) - \frac{d}{2}\ln 2\pi - \frac{1}{2}\ln|\Sigma_i| + \ln P(\omega_i)$$

a) Compute the mean for each class

Solution:

$$\mu_1 = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \quad \mu_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

b) Show that the covariance between feature 1 and 2 is 0 for both classes.

For class 1: $\sigma_{1,12}$= 1/5[(3-5)(0-0)+(5-5)(0-0)+(7-5)(0-0)+(5-5)(2-0)+(5-5)(-2-0)] = 0
For class 2: $\sigma_{2,12}$=1/5[(0-0)(5-5)0(0-0)(3-5)+(0-0)(7-5)+(-2-0)(5-5)+(2-0)(5-5)] = 0

c) Show how the discriminant function can be simplified in this case with a classifier with equal diagonal covariance matrices.

Can be simplified to

$$g_i(x) = \mu_i^T x - \frac{1}{2}\mu_i^T \mu_i + \ln P(\omega_i)$$

d) Find an expression for the decision boundary using this simplified discriminant function.

Can be simplified to

$$g_i(x) = g_j(x) \Leftrightarrow \mu_i^T x - \frac{1}{2}\mu_i^T \mu_i + \ln P(\omega_i) = \mu_j^T x - \frac{1}{2}\mu_j^T \mu_j + \ln P(\omega($$

e) Compute the decision boundary if we assume equal prior probabilities.
The equation is:

$$5x_1 - 25 = 5x_2 - 25 \Leftrightarrow x_2 = x_1$$

f) Sketch the class means and the decision boundary in a plot if we assume that the two classes have equal prior probability.

g) If $P(\omega_1) = 0.75$, in which direction will the decision boundary change? Indicate this on the plot.