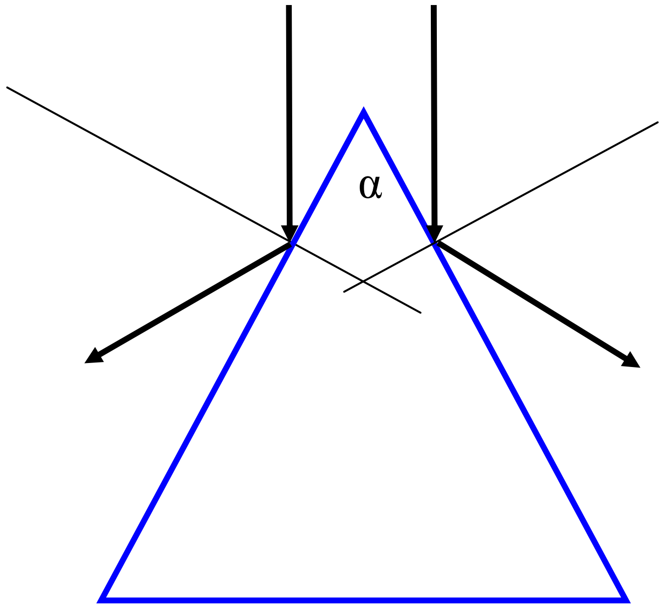
INF-GEO4310 2010
Solutions, Geometrical Optics, Part 1

1 Reflection by a symmetric triangular prism

Let α be the angle between the two faces of a symmetric triangular prism. Let the edge A where the two faces meet be perpendicular to the plane which contains the incident and emergent rays.

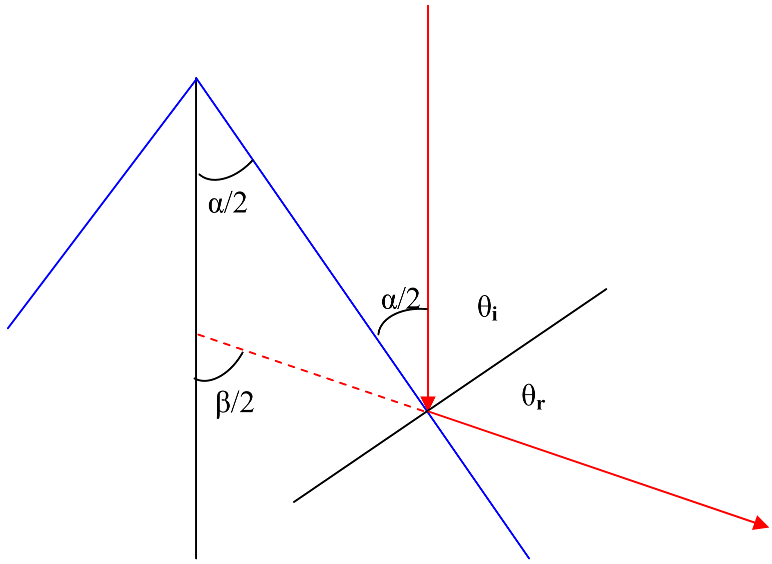


Two parallel beams of light are reflected off the two symmetric faces of the prism.

a. *Show that the angle between the two reflected beams is twice the angle between the two reflecting surfaces.*

First: the simple case where the incident beams are parallel to the symmetry plane of the prism:

Let the plane of symmetry halve the top angle α of the prism.
Let half the angle between the two reflected beams be called $\beta/2$.



The angle of incidence θ_i equals the angle of reflection θ_r

- and the same is true for their 90° complements.

Since the two parallel beams are parallel to the symmetry axis,

$$90° - \theta_r = 90° - \theta_i = \alpha/2.$$

Now we have a triangle with angles $\alpha/2$, $\alpha/2$ and $180° - \beta/2$.

The sum of angles within the triangle is 180°, so

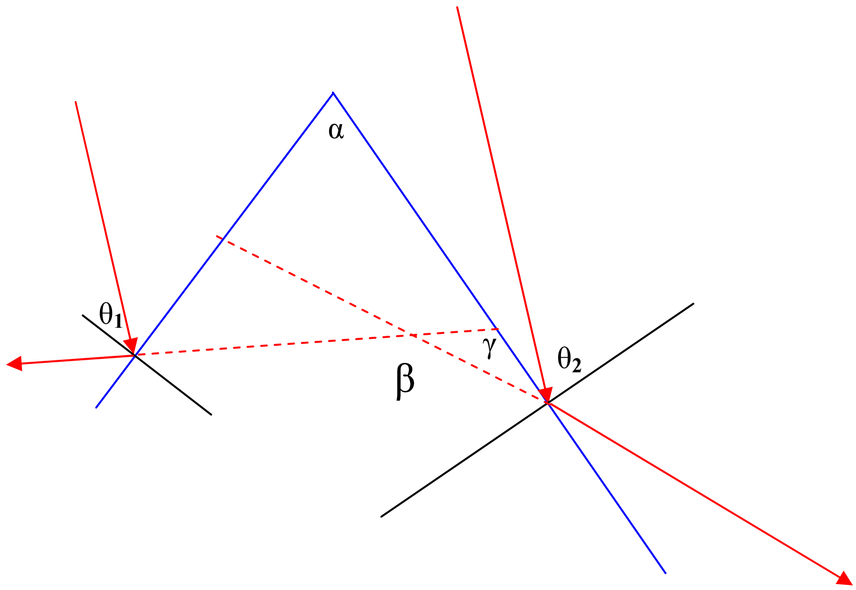
$$180° = \alpha/2 + \alpha/2 + 180° - \beta/2$$

<u>**=> $\beta = 2\,\alpha$.**</u>

Second: The general case of two parallel beams:

One beam strikes the left hand face of the prism under an angle of incidence θ_1, giving the relation $\gamma = \alpha + 90° - \theta_1$.

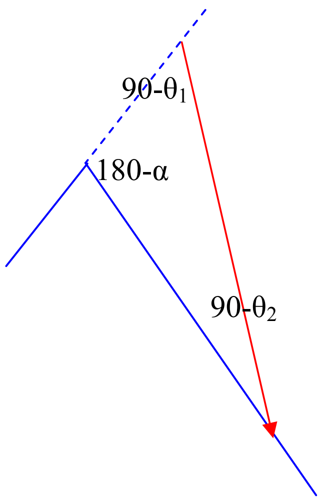
The other beam strikes the right hand face under an angle of incidence θ_2, giving the relation $\gamma = \beta + \theta_2 - 90°$.



The two expressions for γ give:

$\beta + \theta_2 - 90° = \alpha + 90° - \theta_1$

=> $\beta + \theta_1 + \theta_2 = 180° + \alpha$



And since $\theta_1 + \theta_2 = 180° - \alpha$,

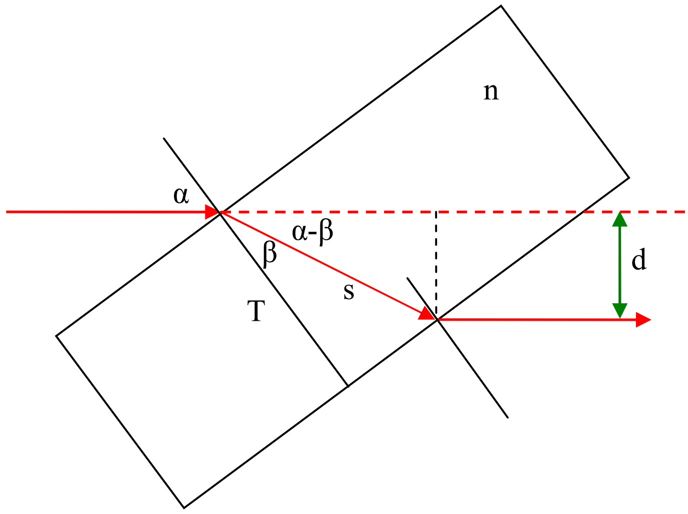
we get

$$\beta + 180° - \alpha = 180° + \alpha$$

=> $\underline{\boldsymbol{\beta = 2\alpha}}$

2 Refraction in plane parallel slab of glass

a. Verify the expressions for the displacements d and l in section 2.3.3.1



Let the angle of incidence be α, while β is the angle of refraction as the beam enters the plane parallel slab. The index of refraction is n, and the thickness of the slab is T.

The displacement, d, given relative to the thickness T of the slab, is

$$\frac{d}{T} = \frac{s\ \sin(\alpha-\beta)}{T}$$

And (T/s) = sin (90 – β). So we get

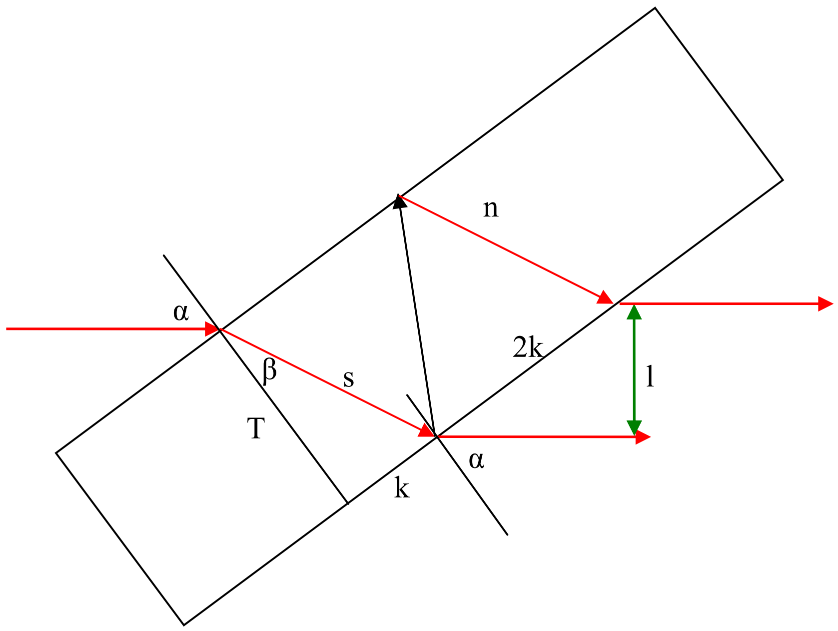
$$\frac{d}{T} = \frac{s\ \sin(\alpha-\beta)}{s\ \sin(90-\beta)} = \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\beta}$$

Snell's law gives:

$$\sin\alpha = n\sin\beta \quad \Rightarrow \quad \sin\beta = \frac{1}{n}\sin\alpha,\ \cos\beta = \frac{1}{n}\sqrt{n^2-\sin^2\alpha}$$

$$\frac{d}{T} = \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\beta} = \sin\alpha\left(1 - \frac{\cos\alpha}{\sqrt{n^2-\sin^2\alpha}}\right)$$

The twice reflected beam:



The twice reflected beam will be displaced relative to the first by an amount l, given in units of the thickness of the slab:

$$\frac{l}{T} = \frac{2k}{T}\sin\left(\frac{\pi}{2}-\alpha\right)=\frac{2\,s\sin\beta}{T}\cos\alpha$$

$$=\frac{2\sin\beta}{\cos\beta}\cos\alpha$$

We have established that

$$\sin\alpha = n\sin\beta \quad \Rightarrow \quad \sin\beta=\frac{1}{n}\sin\alpha,\;\; \cos\beta=\frac{1}{n}\sqrt{n^2-\sin^2\alpha}$$

So that

$$\frac{l}{T} = \frac{2\frac{1}{n}\sin\alpha\cos\alpha}{\frac{1}{n}\sqrt{n^2-\sin^2\alpha}}=\frac{2\sin\alpha\cos\alpha}{\sqrt{n^2-\sin^2\alpha}}$$

3 Dispersion in a plane parallel slab of glass

Assume that a thin beam is incident on a plane parallel slab of glass in air, as in section 2.3.3.1. But now the beam is not monochromatic; it is white light, so the beam is spread out into a spectrum as it passes through the slab.

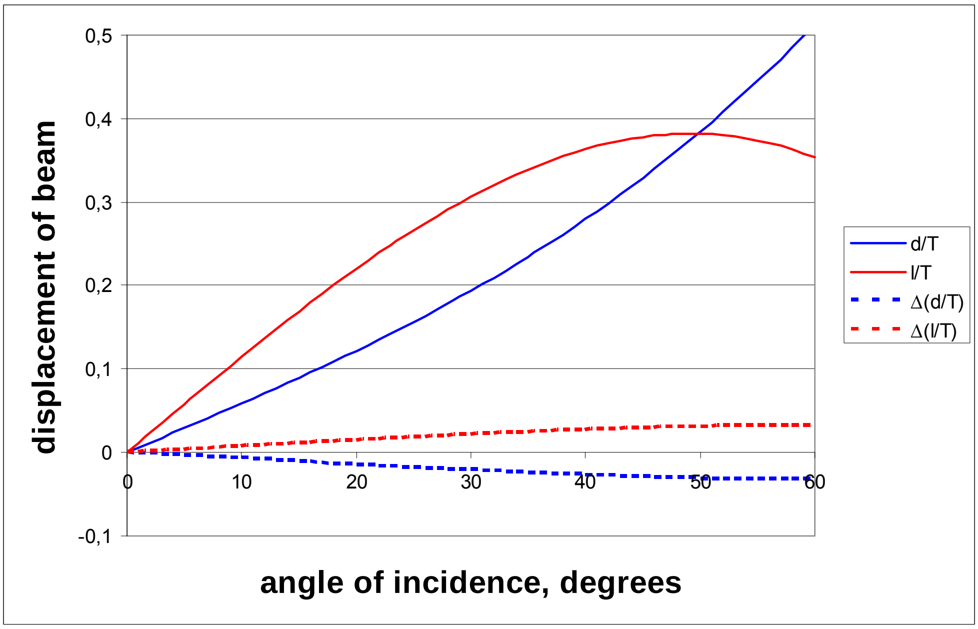
a. *Will the emerging rays of different colors be parallel or not?*

For each color there will be a different value of the index of refraction, n, giving different displacements d for different wavelengths.

But all the displaced beams will be parallel to the input beam,
And therefore also parallel to each other.

b. *What determines the thickness of the beam as it exits the slab?*

- The dispersion of the glass slab used, i.e. the variation of the index of refraction with wavelength,
- The thickness, T, of the slab.
- The angle of incidence.



4 Critical angle and total internal reflection

Assume that we have a a semi-circular bowl of water at 25°C. A light-ray from a 632.8 nm laser enters perpendicular to the surface 4/10 of the radius from the bowl centre.

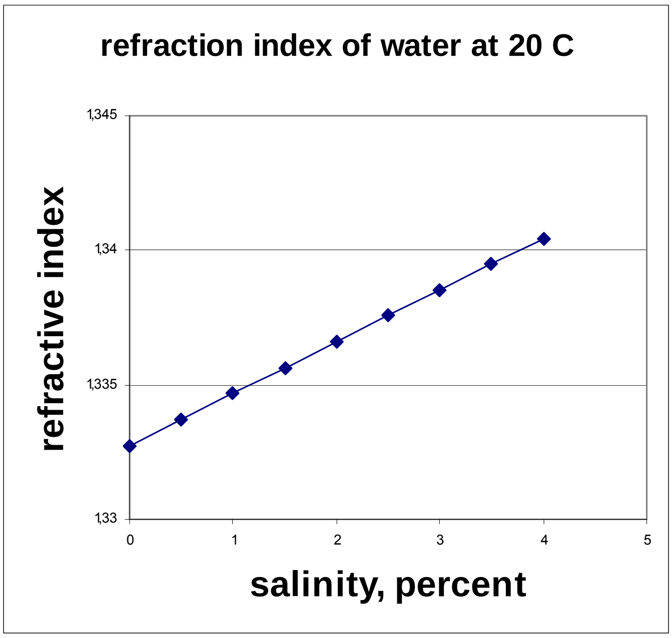
We want to obtain grazing refraction and total internal reflection of the light beam that is reflected towards the water / air interface.

a. *Does the material of the bowl play any role in this?*

No. The material in the bowl only reflects the beam towards the water / air interface.

b. *How much do we have to raise the refractive index of the water by increasing the salinity?*

The refractive index of water can be found at http://www.luxpop.com



It increases approximately linearly from 1.33260 for pure water. The slope of the curve is 0.00195, and we make the assumption that this may be extrapolated linearly.

At the point where the reflected beam hits the water surface, we have

$$\sin(\theta_i/2) = 0.4 \Rightarrow \theta_i = 2 \arcsin(0.4)$$

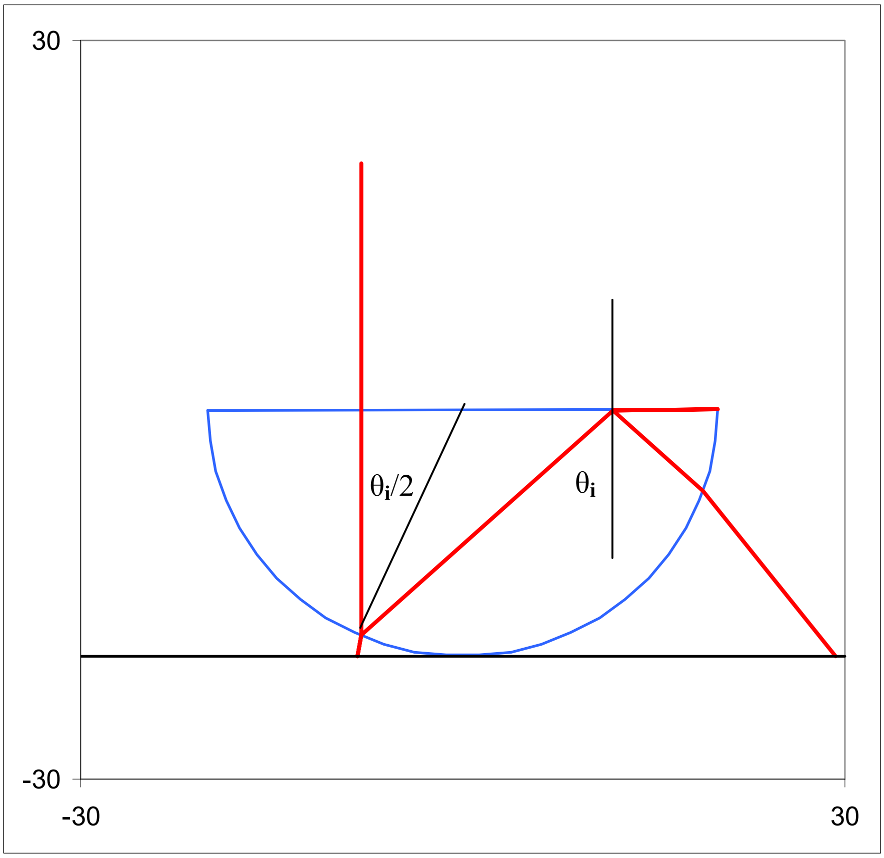
Grazing refraction occurs when

$$n_w(s)\sin(\theta_i) = n_a$$

$$n_w(s) = \frac{n_a}{\sin[2\arcsin(0.4)]} = \frac{1.000267}{0.733212} = 1.364226$$

$$1.3326 + 0.00195 \cdot s = 1.364226$$

$$s = 16.2$$

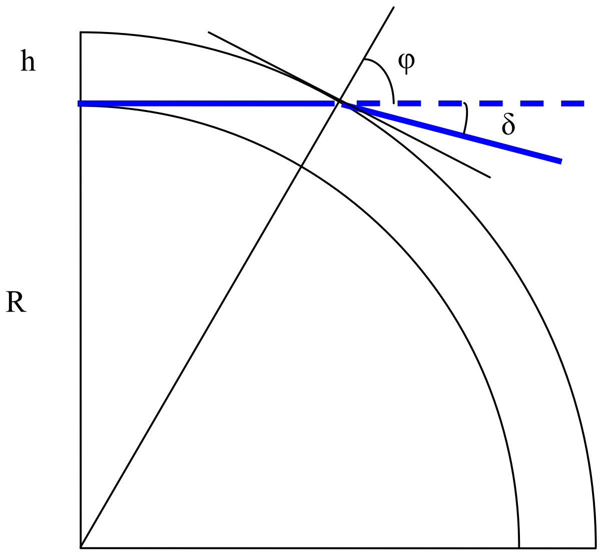


5 Atmospheric refraction

Make the simplifying assumption that the Earth's atmosphere is uniform (thus having a uniform index of refraction), and that it extends to a height h. Beyond that, we assume that there is vacuum.The Earth's radius is R.

a. *Verify that as we observe an object setting on the horizon, under these assumptions it is actually an angle δ below the horizon, given by*

$$\delta = \arcsin\left(\frac{nR}{R+h}\right) - \arcsin\left(\frac{R}{R+h}\right)$$



An object on the horizon is lifted by refraction by an angle δ. Assuming vacuum outside a uniform atmosphere, Snell's law gives:

$$\sin(\delta+\varphi) = n \sin \varphi$$

$$\Rightarrow \delta+\varphi = \arcsin(n \sin \varphi)$$

But sin φ is given by $\sin \varphi = R/(R+h) \Rightarrow \varphi = \arcsin[R/(R+h)]$
so

$$\delta + \arcsin[R/(R+h)] = \arcsin[nR/(R+h)]$$

$$\Rightarrow \underline{\delta = \arcsin[nR/(R+h)] - \arcsin[R/(R+h)]}$$

b. *Calculate δ for R = 6378 km and h = 20 km. Assume that n = 1.0003.*

$$\delta = \arcsin[nR/(R+h)] - \arcsin[R/(R+h)]$$
$$= \arcsin[1.0003 * 6378 / 6398] - \arcsin [6378 / 6398]$$
$$= 85{,}69080° – 85.46848°$$
$$= \underline{0.22°}$$
$$= 0.22°*60 = \underline{13.2'}$$

c. *How does this compare to the statements about atmospheric refraction in section 2.3.3.9 ?*

"On the horizon itself refraction is about 34', …
but only 29' half a degree (one solar diameter) above it."

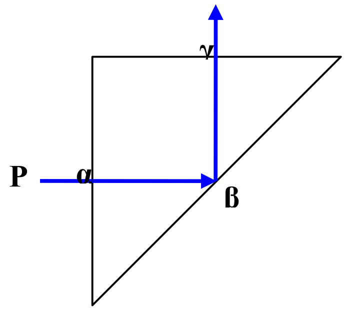
Our simple model where refraction only occurs at the top of a uniform atmosphere is clearly too simple, as it underestimates the horizontal refraction.

Multiple choice – geometrical optics

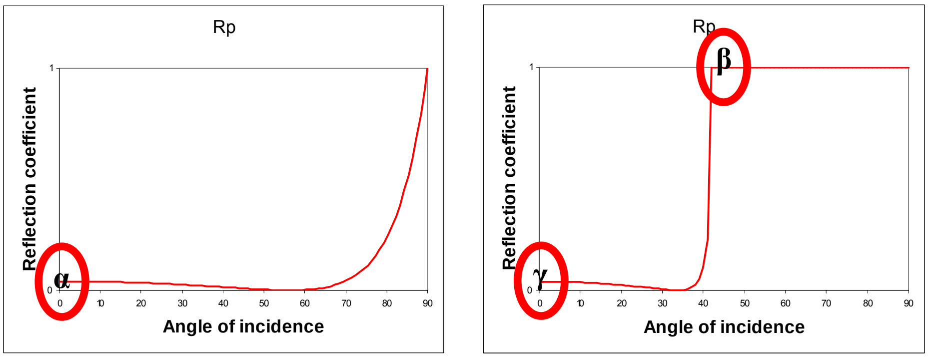
1. **What do we mean by "critical angle" at a boundary between two optical media?**
   - ❑ The angle of incidence where equal parts of refraction and reflection occurs
   - ❑ The largest angle of incidence where all light is reflected
   - ❑ The smallest angle of incidence where no light is reflected
   - ❑ The smallest angle of incidence where all light is refracted
   - ❑ The angle of incidence where refracted light is tangent to the boundary

2. Geometrical optics (20 points).

We can use a number of optical prisms to alter the direction of a light beam. An equilateral right angle prism will change the direction by 90°, as shown in the sketch to the right.



a) Below we give you two figures from the curriculum text showing the reflection coefficient of p-polarized light (polarized in the plane of the sketch) at the transition from air to glass (left) and glass to air (right).



Mark which part of the figures that describe the situation at points α, β, and γ in the first sketch. What do we call the phenomenon that occurs at the point β?

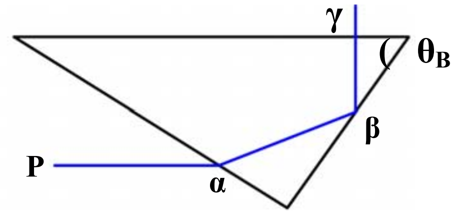
At α, the light beam goes from air to glass at an incidence angle of zero, and a small fraction of the incident light is reflected (R = 0.04), as indicated by the circled point to the left in the left hand figure above.

At β, the light beam is reflected at the glass/air interface at an incidence angle of 45°. When moving from a more dense medium into a less dense one (i.e. $n_1 > n_2$), above an incidence angle known as the *critical angle*, all light is reflected and $R = 1$, as illustrated in the right hand figure. This is known as **”total internal reflection”**. The critical angle is approximately 41° for glass in air. Thus, the reflection coefficient is exactly 1.0 at β, as indicated by the circled point in the right hand figure above.

The reflection angle is equal to the incidence angle at β (45°). Therefore, the beam strikes the glass/air interface orthogonaly at γ, so the reflection coefficient (R = 0.04) here is found in the left hand circle of the right hand figure above.

b) We substitute the prism above by a right angle *Brewster* prism, where one angle is given by $\theta_B = \text{arctg}(n_2/n_1)$, where n_2 is the refractive index of glass, and n_1 the refractive index of air. We place the prism in the light path from P, as shown in the figure to the right, so that the incidence angle is $\theta_i = \theta_B \approx 56°$.

Now the refraction angle θ_r is given by $\theta_i + \theta_r = \pi/2$.
Draw and explain the path of the light beam through the prism.



At an incidence angle equal to the *Brewster* angle, p-polarized light going from air to glass is not reflected, so there is purely refraction at α.

The incidence angle at β is given by $\pi/2 - \theta_r = \pi/2 - (\pi/2 - \theta_i) = \theta_i = \theta_B$ which is larger than the critical angle (41°). So there is total reflection at β.

The reflection angle at β is equal to the incidence angle. So the angle between the incident ray and the glass/air interface at γ is $\pi - \theta_B - \theta_r = \pi - \theta_B - (\pi/2 - \theta_B) = \pi/2$. Which means that the incidence angle at γ is 0°, and a small fraction R is reflected while (1-R) is transmitted, orthogonal to the interface.

c) How much light is reflected back to P in exercise b, compared to the equilateral prism in exercise a, if $n_1 = 1$ and $n_2 = 1.5$?

At normal incidence ($\theta_i = 0$), the reflection coefficient in the two figures is given by $R = [(n_1-n_2)/(n_1+n_2)]^2$. For $n_1 = 1$ and $n_2 = 1.5$ we get R = 0.25/6.25 = 0.04.

In exercise b, no light is lost from α through γ. At γ, 4% (R) is reflected back to β. At β there is only reflection to α, and at α there is no reflection (see right hand figure for incidence angle = 34°), only refraction to P.
So 4% (R) is reflected and refracted back to P.

In exercise a, R is reflected at α. At γ, R(1-R) is reflected via β to α. $R(1-R)^2$ goes to P while $R^2(1-R)$ is reflected back to γ via β. From γ, $R^3(1-R)$ is reflected via β to α.
Now $R^3(1-R)^2$ goes to P and $R^4(1-R)^2$ goes to γ.
So $R + R(1-R)^2 + R^3(1-R)^2 + R^5(1-R)^2 + \ldots$ should be summed at P, giving
$R + R - 2R^2 + R^3 + R^3 - 2R^4 + R^5 + R^5 - 2R^6 + R^7 + \ldots$
$= 2R(1 - R + R^2 - R^3 + R^4 - R^5 + R^6 - \ldots) = 2R/(1+R)$.

So the ratio of the reflected light in b) to the reflected light in a) is
$R / (2R/(1+R)) = (1 + R)/2$, or 0.52.

If we just consider the first two contributions in exercise a, $R + R(1-R)^2$, the ratio becomes $1/(2 - 2R + R^2)$ = 0,5204, which is a little more than 0.5, and very close to the final sum.