INF-GEO4310 2011
Solutions, Geometrical Optics, Part II

1 Image formation by refraction

A cylindrical glass rod of diameter 4.00 cm has an index of refraction of 1.50. One end of the rod is shaped like a hemisphere having a radius of 2.00 cm. A small object is placed on the axis of the rod, 8.00 cm left of the vertex (see figure below). We are going to find the position of the image of the object, as well as the magnification.

a. *Which expressions from which section of the distributed text are relevant?*

Section 2.3.4.2 Refraction at a single spherical surface.
The general object-image relation for a single spherical surface of radius R, given the two refractive indices n_a and n_b:

$$\frac{n_a}{s} + \frac{n_b}{s'} = \frac{n_b - n_a}{R}$$

The lateral magnification $m = y'/y = -(n_a s')/(n_b s)$.
The sign of m indicates if the image is upright or inverted relative to the object.

b. *Find the image distance of the object from the vertex if the surrounding medium is air (refractive index n =1.00).*

We get

$$\frac{1.00}{8.00\ cm} + \frac{1.50}{s'} = \frac{1.50 - 1.00}{2.00\ cm} \Rightarrow \underline{\underline{s' = 12.00\ cm}}$$

So the rays will converge when they travel from air into the glass rod, and an image is formed 12 cm inside the glass rod.

c. *Find the lateral magnification.*

The lateral magnification is $m = y'/y = -(n_a s')/(n_b s)$
$= -(1.00 \times 12.00)/(1.50 \times 8.00) = \underline{-1.0}$.
So the image is the same size as the object, but it is inverted.

d. *What is the image distance if the surrounding medium is salt water having a salinity of 14.05 ? (The index of refraction of salt water is $n = 1.3326 + 0.00195\ s$, where s is the salinity).*

First, if the salinity is 14.05, the index of refraction of the water is

N = 1.3326 + 0.00195 x 14.05 ≈ 1.36
Then we get

$$\frac{1.36}{8.00\ cm} + \frac{1.50}{s'} = \frac{1.50 - 1.36}{2.00\ cm} \Rightarrow \underline{\underline{s' = -15.00\ cm}}$$

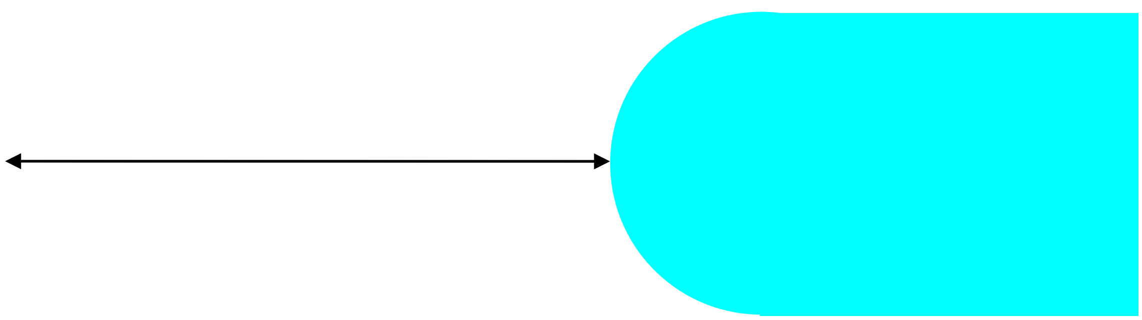
So the rays will *diverge* when they enter the glass rod, and a *virtual* image is formed 15 cm *in front of* the glass rod.

e. *What is the lateral magnification now?*

The lateral magnification is $m = y'/y = -(n_a s')/(n_b s)$
$= -(1.36 \times -15.00)/(1.50 \times 8.00) = \underline{1.7.}$
So the image is 1.7 times the size of the object, and it is erect.

f. *What are the main differences between the two situations (air/water)?*

In air, the spherical end of the rod acts as a converging lens, and an inverted image is formed inside the glass rod. In (salty) water, the rays are diverging inside the rod, resulting in an erect virtual image behind the object in the water.



2 Santa's image problem

Your exam will take place some time before Christmas, and the following problem may therefore be relevant:
Santa checks himself for soot from the chimney, using his reflection in a shiny spherical Christmas tree ornament having a diameter of 7.2 cm, at a distance of 75 cm.

a. *Where is the image, and is it real or virtual?*

The spherical ornament acts as a convex mirror, having a radius of R =-3.6 cm.
The general object-image relation for a spherical mirror is

$$\frac{1}{s} + \frac{1}{s'} = \frac{2}{R} \Rightarrow \frac{1}{75\ cm} + \frac{1}{s'} = \frac{2}{-3.6\ cm} \Rightarrow \underline{\underline{s' = -1.76 cm.}}$$

So the image is formed behind the mirror surface, that is, on the opposite side of the of the outgoing light, and it is *virtual*.

b. *If Santa is "a jolly old elf" 1.6 m high, how tall is the image, and is it erect or inverted?*

The lateral magnification m = y'/y = - s'/s
= - (-1.76 cm) /(75 cm) = 0.0234.
Since m is positive, the image is *erect*. But ist is only 0.0234 as tall as the small elf, or y' = m y = 0.0234 x 1.6 m = 37 mm.

3 Determining the focal length of a lens

Assume that the absolute values of the radii of curvature of the two surfaces of a thin lens in air are 10.0 cm and 5.0 cm and that the index of refraction is 1.5. (Hint: sign rules.)

a. *What is the focal length of the lens if both surfaces are convex ?*

The lensmakers equation for a thin lens:

$$\frac{1}{f} = (n-1)(\frac{1}{R_1} - \frac{1}{R_2})$$

Sign rule for radius of curvature:
"When the centre of curvature C is on the same side as the outgoing light, the radius of curvature is positive; otherwise, it is negative."
Therefore, when both surfaces are convex:

$$\frac{1}{f} = (1.5-1)(\frac{1}{10.0\,cm} - \frac{1}{-5.0\,cm}) \Rightarrow \underline{\underline{f = 6.67\,cm}}$$

b. *What is the focal length if one surface is convex and the other concave?*

If the first surface is convex and the second concave:

$$\frac{1}{f} = (1.5-1)(\frac{1}{10.0\,cm} - \frac{1}{5.0\,cm}) \Rightarrow \underline{\underline{f = -20.0\,cm}}$$

c. *What is the focal length if the lens is double-concave?*

$$\frac{1}{f} = (1.5-1)(\frac{1}{-10.0\,cm} - \frac{1}{5.0\,cm}) \Rightarrow \underline{\underline{f = -6.67\,cm}}$$

d. *Does it matter if we interchange the left and right surface (R1=5, R2=10)?*

Double-convex:

$$\frac{1}{f} = (1.5-1)\left(\frac{1}{5.0\,cm} - \frac{1}{-10.0\,cm}\right) \Rightarrow \underline{\underline{f = 6.67\,cm}}$$

Convex-concave:

$$\frac{1}{f} = (1.5-1)\left(\frac{1}{5.0\,cm} - \frac{1}{10.0\,cm}\right) \Rightarrow \underline{\underline{f = 20.0\,cm}}$$

Double-concave:

$$\frac{1}{f} = (1.5-1)\left(\frac{1}{-5.0\,cm} - \frac{1}{10.0\,cm}\right) \Rightarrow \underline{\underline{f = -6.67\,cm}}$$

The double-convex and the double-concave lenses are mirror-images of a) and c), and the focal lengths are the same whether light enters left-to-right or right-to-left. The convex-concave lens is NOT a mirror-image of b), and has a different focal length.

4 Image of an image

An object 8.0 cm high is placed 12.0 cm to the left of a converging thin lens having a focal length of 8.0 cm. A second converging lens having a focal length of 6.0 cm is placed 36.0 cm to the right of the first lens, on the same optical axis.

a. *Find the position, size and orientation of the image produced by the combination of the two lenses.*

The object-image relation applied to the first lens:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \Rightarrow \frac{1}{12\ cm} + \frac{1}{s'} = \frac{1}{8\ cm} \Rightarrow s' = 24\,cm$$

So the first image is 24 cm to the left of the first lens.
The magnification is m = - 24/12 = - 2, so the height of the first image is –16 cm.
The first image is 36 cm – 24 cm = 12 cm to the left of the second lens, so the object-image relation applied to the second lens gives:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \Rightarrow \frac{1}{12\ cm} + \frac{1}{s'} = \frac{1}{6\ cm} \Rightarrow \underline{\underline{s' = 12\,cm.}}$$

The magnification is m = -12/12 = - 1, so the height of the second image is 16 cm.
The total magnification is M = (-2)(-1) = 2. <u>So the final image is upright, 2 times the size of the object, and located 12 cm to the right of the second lens.</u>

5 Single-slit diffraction

You pass 633 nm laser light through a narrow slit and observe the diffraction pattern on a screen 6.0 m away. You find that the distance between the centers of the first minima is 32 mm.

a. How wide is the slit?

The first minimum of the Fraunhofer diffraction pattern

$$I(\theta \mid a, \lambda) = \left(\frac{\sin[\pi a(\sin\theta)/\lambda]}{\pi a(\sin\theta)/\lambda} \right)^2 I_0$$

occurs at at an angle $\theta = \pm \lambda/a$, where a is the width of the slit. For small angles, the angle θ between the first minima as seen from the slit is given by $\theta = x/2d$, where x is the distance between the minima and d is the distance from the slit to the screen.
This gives
$x / 2d = \lambda / a$
=> $a = 2d\lambda / x = (2 \times 6.0 \text{ m}) (633 \times 10^{-9} \text{ m}) / (32 \times 10^{-3} \text{ m}) = 2.4 \times 10^{-4} \text{ m} =$ 0.24 mm.

b. What is the distance between the second minima?

The second minima occur at $\theta = \pm 2 \lambda/a$, so the distance between the second minima is 2 x 32 mm = 64 mm.

6 Width of a grating spectrum

The wavelengths of the visible spectrum are approximately 400 to 700 nm. Assume that you are using a plane grating with 600 slits per mm and that white light falls perpendicular on the grating.

a. Find the angular width of the first order visible spectrum.

The grating spacing is $d = (1/600) \times 10^{-3}$ m.
The positions of the intensity maxima for a given wavelength λ, distance between slits d, and spectral order m, is given by

$$d \cdot \sin(\theta) = m\lambda, \quad m = 0, \pm 1, \pm 2, \ldots$$

So the angular deviation θ_v of violet light at 400 nm in the first order spectrum is

$$\sin(\theta_v) = \frac{400\ 10^{-9}}{\left(1/600\right) 10^{-3}} = 0.24 \Rightarrow \theta_v = 13.9^\circ$$

Similarly, the angular deviation θ_r of red light at 700 nm in the first order spectrum is

$$\sin(\theta_r) = \frac{700\ 10^{-9}}{\left(1/600\right)10^{-3}} = 0.42 \Rightarrow \theta_v = 24.8^\circ$$

So the angular width of the first order visible spectrum is 24.8 -13.9 = 10.9 degrees.

b. Do the first and second order spectra overlap?

The angular deviation θ_{vm} of violet light at 400 nm in the m-th order spectrum is

$$\sin(\theta_{vm}) = \frac{m\ x\ 400\ 10^{-9}}{d}$$

The angular deviation θ_{rm} of red light at 700 nm in the m-th order spectrum is

$$\sin(\theta_{rm}) = \frac{m\ x\ 700\ 10^{-9}}{d}$$

The first order spectrum (m = 1) spans the angular range given by

$$\sin(\theta_{m=1}) = [4 \text{ x } 10^{-7} / d,\ 7 \text{ x } 10^{-7} / d].$$

The second order spectrum (m = 2) spans the angular range given by

$$\sin(\theta_{m=2}) = [8 \text{ x } 10^{-7} / d,\ 1.4 \text{ x } 10^{-6} / d].$$

We observe that the largest angle (at the red end) of the first order spectrum is smaller than the smallest angle (violet end) of the second order spectrum, regardless of the value of the grating spacing, d.
So for ANY value of the grating spacing d, there is no overlap between the first and second order visible spectra.

c. What about the second and third order spectra?

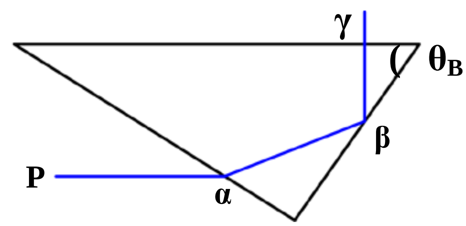
The third order spectrum (m = 3) spans the angular range given by

$$\sin(\theta_{m=3}) = [1.2 \text{ x } 10^{-6} / d,\ 2.1 \text{ x } 10^{-6} / d].$$

Now we see that the smallest angle (violet end) of the third order spectrum is always smaller than the largest angle (red end) of the second order spectrum, regardless of the value of the grating spacing, d.
We conclude that for ALL possible values of the grating spacing d, there is an overlap between the second and third order visible spectra.

The reason why we have overlap between m=2 and m=3, but not between m=1 and m=2, is the spectral sensitivity of the human eye (400-700 nm). If we restrict the spectral range by a filter, there will be no overlap between m=2 and m=3, and if we use a detector having a broader spectral sensitivity than 400-700 nm, we may get overlap between m=1 and m=2 spectra.

7 The Brewster prism from the previous exercise (from the written exam 2009)



d) We put a thin plano-convex lens as a *collimator* between P and the prism. If the radius of curvature of the spherical lens surface is R_3 and the refractive index of the lens material is $n_3 = 1.5$, at what distance from P should we place the lens?

Does it make any difference whether the planar or curved surface faces P?

A thin lens having the light source in its primary focal point will *collimate* the beam, i.e., make the light rays that exit the lens be parallel.
So we need to determine the focal length of the given plano-convex thin lens.
The inverse of the focal length of the thin lens is given by:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

where R1 = R3, R2 = ∞, and n = 1.5 . The simple sign rule for the radius of curvature of the lens surface is that when the centre of curvature is on the same side as the outgoing light, the radius of curvature is positive; otherwise, it is negative.

If we let light pass through the curved surface first,
we have $f = [(n_3-1)((1/R_3) - 0)]^{-1} = R_3/(n_3-1) = 2R_3$.

If we let light pass through the plane surface first,
we have $f = [(n_3-1)(0 - (1/-R_3))]^{-1} = R_3/(n_3-1) = 2R_3$.

The lens must be placed at a distance from P equal to twice the radius of curvature of the curved lens surface, and it does not matter whether the planar or curved surface faces P.

Thank you for your attention!