

INF2080

Non-Context-Free Languages

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Informatics



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Oslo

Definition (Context-Free Grammar)

A *context-free grammar* is a 4-tuple (V, Σ, R, S) where

- 1 V is a finite set of *variables*
- 2 Σ is a finite set disjoint from V of *terminals*
- 3 R is a finite set of *rules*, each consisting of a variable and of a string of variables and terminals
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A language generated by a context-free grammar is a *context-free language*

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All $\{0, 1\}$ words beginning and ending with the same digit

Chomsky Normal Form

Every context-free grammar can be rewritten into Chomsky normal form:

Definition (Chomsky Normal Form)

A grammar in Chomsky normal form consists only of rules of the following form:

$$S \rightarrow \varepsilon$$

$$A \rightarrow BC$$

$$A \rightarrow d,$$

where S is the start variable, A, B, C are variables with B, C distinct from S , and d is a terminal.

PDA

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Definition (PDA)

A PDA is a tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q, Σ, Γ, F are finite sets and

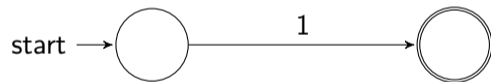
- 1 Q is a set of states,
- 2 Σ is the input alphabet,
- 3 Γ is the stack alphabet,
- 4 $\delta : Q \times \Sigma_\epsilon \times \Gamma_\epsilon \rightarrow \mathcal{P}(Q \times \Gamma_\epsilon)$ is the transition function
- 5 $q_0 \in Q$ is the start state, and
- 6 $F \subseteq Q$ is the set of accepting states.

PDA

Examples:

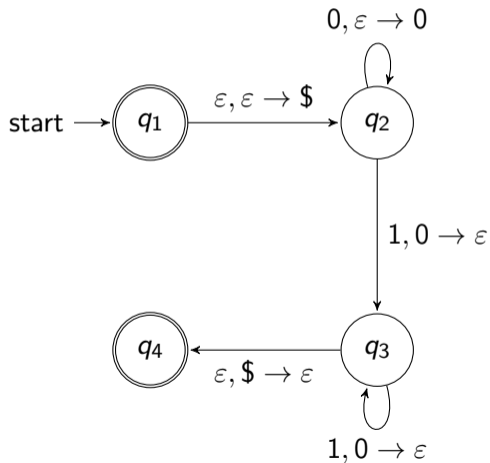
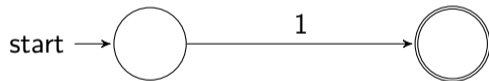
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- So PDAs and CFLs have a similar relationship as NFAs and regular languages
- What about DPDAs (deterministic PDAs)? Are they the same as PDAs? → Nope! Have, in fact, different expressive power. See book (not exam-relevant).

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Lemma (Pumping Lemma for Regular Languages)

If A is a regular language, then there is a number p , called the pumping length, where if s is a word in A of length $\geq p$ then s can be divided into three parts, $s = xyz$, such that

- 1 $xy^iz \in A$ for every $i \geq 0$,
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- y corresponded to a cycle in the DFA
- x corresponded to a path from the initial node to the start of the cycle.
- z corresponded to a path from the end of the cycle to an accepting state.
- Condition (3) in the lemma was a useful tool when proving the nonregularity of a language.

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For every context-free language A there exists a number p (called the pumping length) where, if s is a word in A of length $\geq p$, then s can be divided into five parts, $s = uvxyz$, satisfying the following conditions:

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→ Things are a bit more involved this time!

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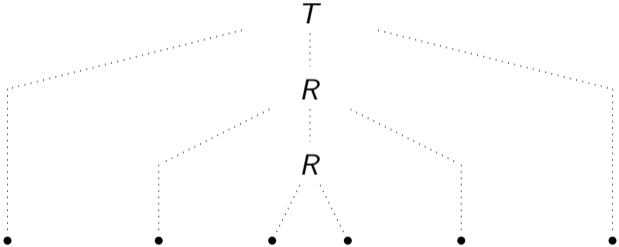
For CFL A there exists a pumping length p where, if $s \in A$ of length $\geq p$, then s can be divided into five parts, $s = uvxyz$, such that:

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- Proof idea: Let G be the CFG generating A and τ a parse tree of s . If s is sufficiently large, we can argue that, along a path, a variable must occur twice.

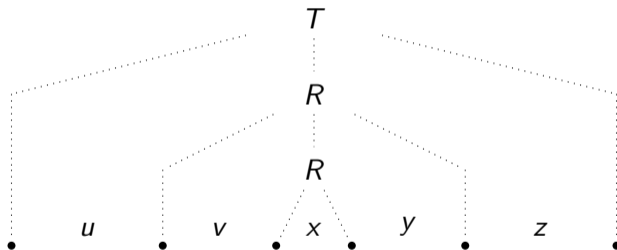
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Parse tree for s :

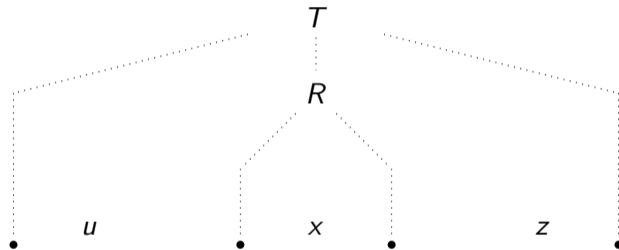


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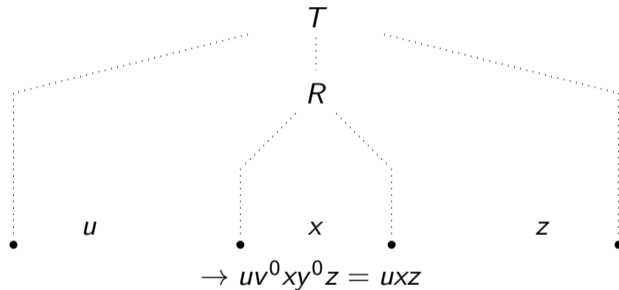
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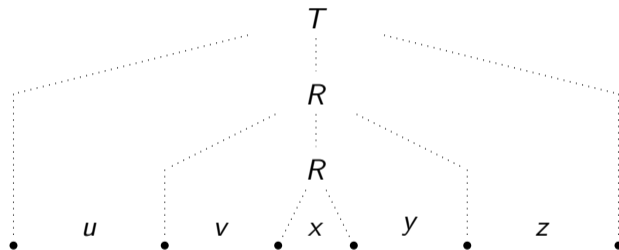
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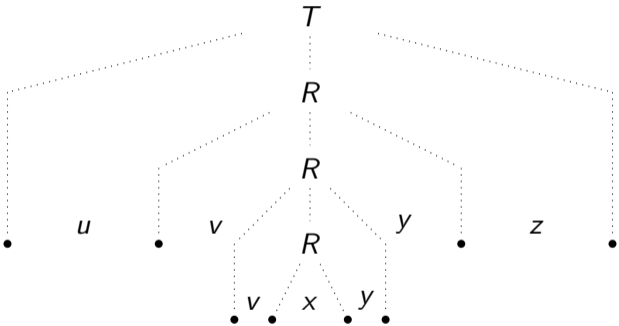
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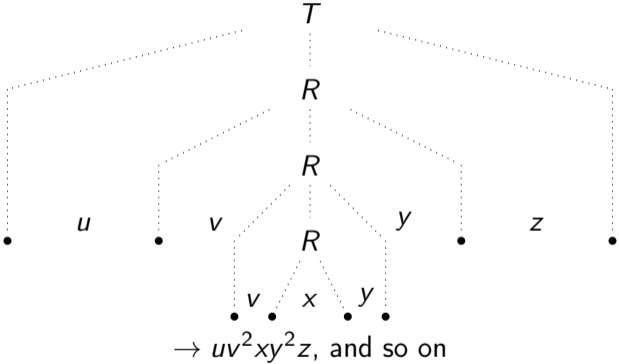
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- \Rightarrow If a parse tree has height h , then it has at most b^h leaves.

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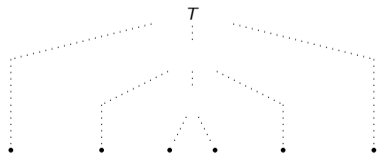
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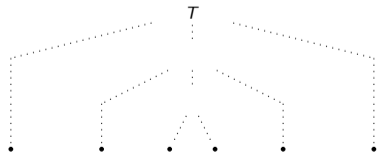
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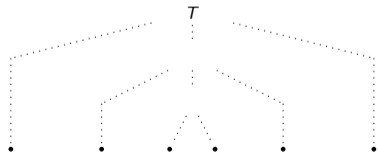
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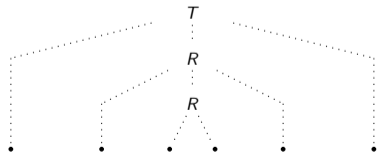
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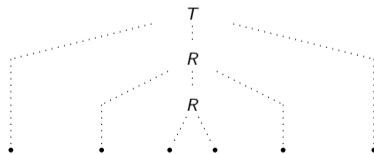
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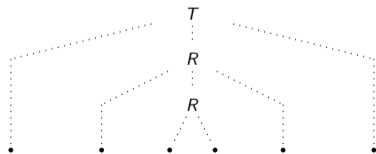
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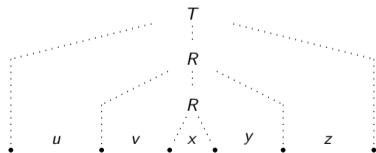
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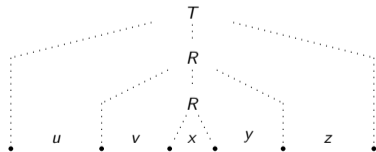
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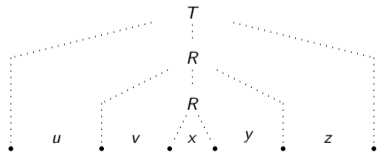
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- The first occurrence of R generates vxy , while the second generates x (remember, only using rules from G),
- Thus we can replace one subtree with the other and get valid parse trees for uxz and $uv^i xy^i z$, $i > 1$.



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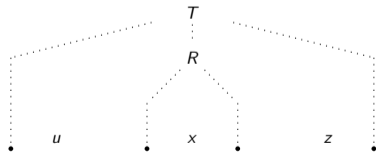
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- 3 $|vxy| \leq p$.

Proof:

- The first occurrence of R generates vxy , while the second generates x (remember, only using rules from G),
- Thus we can replace one subtree with the other and get valid parse trees for uxz and $uv^i xy^i z$, $i > 1$.



Pumping Lemma - Proof

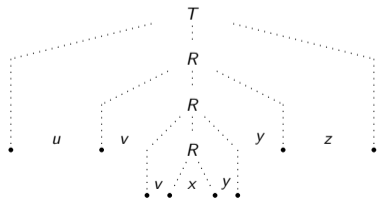
Lemma (Pumping Lemma for CFLs)

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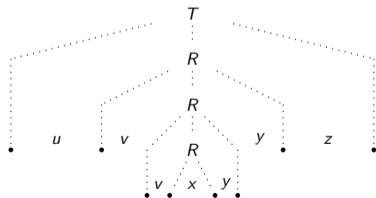
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- This proves condition 1.



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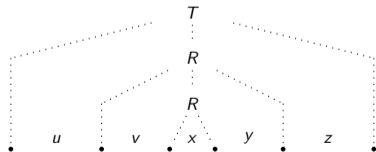
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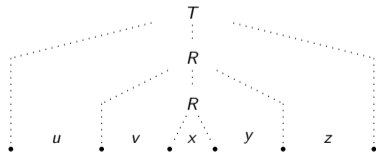
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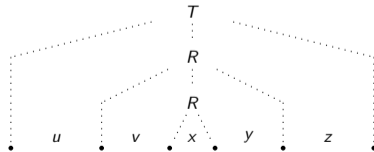
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Contradiction!



Pumping Lemma - Proof

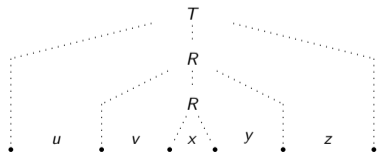
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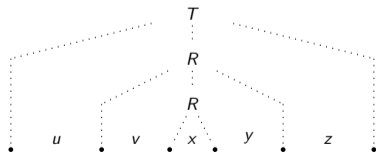
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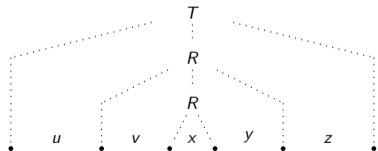
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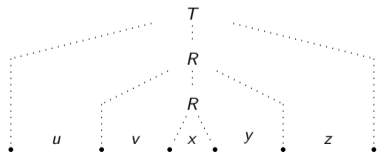
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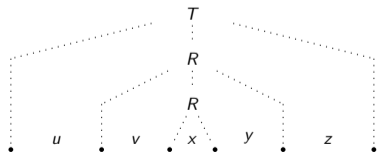
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Pumping Lemma - Examples

Lemma (Pumping Lemma for CFLs)

- 1 $uv^i xy^i z \in A$ for all $i \geq 0$,
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Language: $A_1 = \{a^n b^n c^n \mid n \geq 0\}$.

- Assume A_1 is context-free.

Pumping Lemma - Examples

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- Assume A_1 is context-free. Then let p be its pumping length and $s = a^p b^p c^p$.
- $|s| = 3p \geq p$ and $s \in A_1$

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- Assume A_1 is context-free. Then let p be its pumping length and $s = a^p b^p c^p$.
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- Now we consider how we can subdivide s into $uvxyz$.
- Condition 2: v or y must be nonempty.

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- Case 1: Either v or y contains more than one type of symbol.

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- Case 1: Either v or y contains more than one type of symbol.
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- Case 2: Both v or y contains only one type of symbol.

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- But then $uv^2 xy^2 z$ cannot have an equal number of a 's, b 's, and c 's.

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- s cannot be pumped.
- A_3 is, in fact, *context-sensitive* (more on that tomorrow)

Pumping Lemma - Examples

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- 1 $uv^i xy^i z \in A$ for all $i \geq 0$,
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Language: $A_2 = \{a^i b^j c^k \mid 0 \leq i \leq j \leq k\}$.

- Same as before, assume A_2 is context-free and let $s = a^p b^p c^p$. Again, $s = uvxyz$.

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- Case 2: If both v and y contain only one type of symbol, then we cannot use the same reasoning as in the last example (the number of a 's, b 's, and c 's are different now!)
- However, since each contains only one type of symbol, one symbol a , b , or c does not occur in v or y .

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- Case 2a: a does not appear.

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- Case 2a: a does not appear. Then consider the string uxz . This has the same amount (p) of a 's as s , but fewer b 's and c 's.

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- Case 2b: b does not appear. Then either a 's or c 's must occur in v or y (both cannot be empty). If a 's appear, then $uv^2 xy^2 z$ has more a 's than b 's.

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- Case 2c: c does not appear.

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Language: $A_3 = \{\omega\omega \mid \omega \in \{0, 1\}^*\}$.

- Assume A_3 is CF, and choose $s = 0^p 1 0^p 1$.

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- But this can be pumped! $uv^i xy^i z$ is still a word in A_3 .

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- We need to be more careful with our choice of s .

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- Let $s = 0^p 1^p 0^p 1^p$. We will make use of condition 3, so $|vxy| \leq p$.

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- Let $s = 0^p 1^p 0^p 1^p$. We will make use of condition 3, so $|vxy| \leq p$.
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- If vxy is contained in the second half, then pumping up to $uv^2 xy^2 z$ “pushes” a 0 to the last position of the first half.

Pumping Lemma - Examples

Lemma (Pumping Lemma for CFLs)

- 1 $uv^i xy^i z \in A$ for all $i \geq 0$,
- 2 $|vy| > 0$,
- 3 $|vxy| \leq p$.

Language: $A_3 = \{\omega\omega \mid \omega \in \{0, 1\}^*\}$.

- Let $s = 0^p 1^p 0^p 1^p$. We will make use of condition 3, so $|vxy| \leq p$.
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- Hence s cannot be pumped. $\Rightarrow A_3$ is not context-free