INF2080

Context-Free Langugaes

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- defined (non)deterministic finite automata (NFAs/DFAs) and the languages they accept: regular languages
- defined regular expressions, useful as a shorthand for describing languages
- a language L is regular $\leftrightarrow$ there exists a regular expression that describes L
- pumping lemma as a useful tool for determining whether a language is *nonregular*

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- grammars describe the *syntax* of a language; they try to describe the relationship of all the parts to one another, such as placement of nouns/verbs in sentences
- useful for programming languages, specifically compilers and parsers: if the grammar of a programming language is available, parsing is very straightforward.

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$\rightarrow$ first example of a context-free language

Context-Free Grammars

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- Every grammar contains a *start variable* (above: variable S). Common convention: the first listed variable is the start variable (if you choose a different start variable, you must specify!).
- Words are generated by starting with the start variable and recursively replacing variables with the righthand side of a rule.

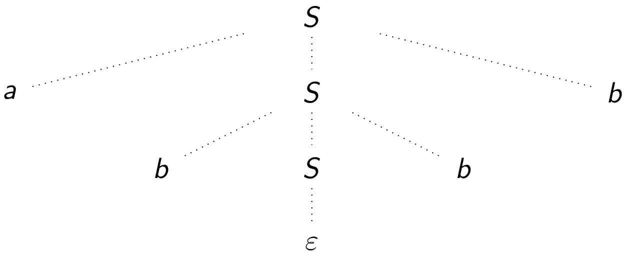
$$S \rightsquigarrow aSb \rightsquigarrow aaSbb \rightsquigarrow aa\varepsilon bb \rightsquigarrow aabb$$

Parse Trees

Derivations of the form

$$S \rightsquigarrow aSb \rightsquigarrow aaSbb \rightsquigarrow aa\varepsilon bb \rightsquigarrow aabb$$

can also be encoded as a parse tree:



Context-Free Grammars

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The symbol $\mid$ takes on the meaning of “or.”
$\to$ palindromes of even length over $\{a, b, c\}$.

Context-Free Grammar

Definition (Context-Free Grammar)

A *context-free grammar* is a 4-tuple (V, Σ, R, S) where

1. V is a finite set of *variables*
2. Σ is a finite set disjoint from V of *terminals*
3. R is a finite set of *rules*, each consisting of a variable and of a string of variables and terminals
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We call $L(G)$ the language generated by a context-free grammar. A language is called a *context-free language* if it is generated by a context-free grammar.

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- Regular languages could be modelled by an automaton with *finite* memory...what about context-free languages?

Answers to these over the course of this and next lecture (and group sessions)

RLs and CFLs

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Theorem

Every regular language is context-free.

Properties of CFLs

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Closure under union/concatanation/Kleene star?
$\rightsquigarrow$ Yes, group sessions!
Closure under complement/intersection?
$\rightsquigarrow$ No, but we need to know more before we can determine if a language is not context-free.

Ambiguity

- Consider the grammar

$$E \to E + E \mid E \times E \mid (E) \mid a$$

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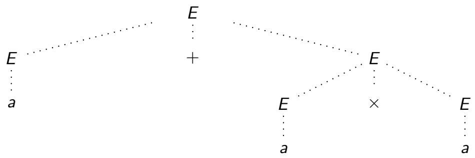
What does the parse tree for the string $a + a \times a$ look like?

Ambiguity



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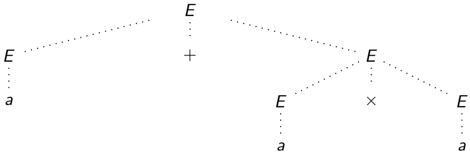
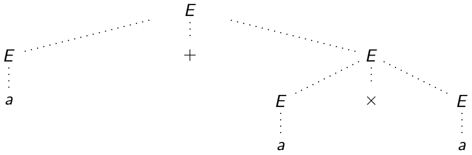
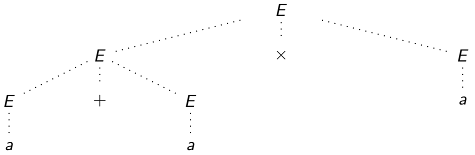


Intuitively corresponds to $a + (a \times a)$





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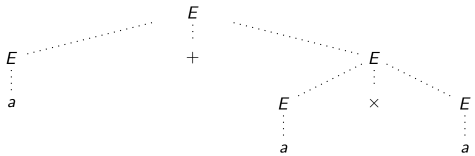

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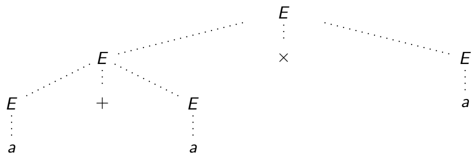

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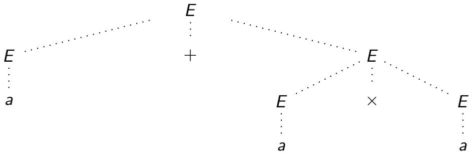

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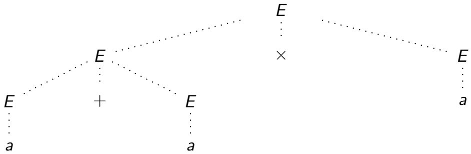

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This is called *ambiguity*

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If $L(G)$ contains a string that is derived ambiguously, we say that G is ambiguous.

Chomsy Normal Form

- Context-free languages have a nice property: Every CFL can be described by a CFG in *Chomsky Normal Form*:

Definition

A grammar is in *Chomsky Normal Form* if every rule is of the form:

$$A \to BC$$
$$A \to a$$

where a is any terminal, A is any variable, B, C are any variables that are not the start variable. In addition the rule $S \to \varepsilon$ is permitted.

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CNF - Example

Grammar;

$$S \to ASA \mid aB$$
$$A \to B \mid S$$
$$B \to b \mid \varepsilon$$

First, add new start variable:

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and you would continue to remove the unit rules $A \to S$, etc....But how to convert, say, $S \to ASA$ into rules with only two symbols on the right? $\rightsquigarrow$ introduce help variables!

$$S \to ASA$$
$$\rightsquigarrow S \to AA_1, A_1 \to SA$$

CNF

- Thus, we see how all CFGs can be converted to CFGs in CNF.
- Useful property to have, both for practical purposes and theoretical work: knowing what the grammar looks like can be very beneficial (we will see an example next week)
- Next time: how can finite automata be enriched so as to accept context-free languages?