INF2080

Context-Free Langugaes

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- a language L is regular $\leftrightarrow$ there exists a regular expression that describes L

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- We've looked at one of the simpler computational models: finite automata
- defined (non)deterministic finite automata (NFAs/DFAs) and the languages they accept: regular languages
- defined regular expressions, useful as a shorthand for describing languages
- a language L is regular $\leftrightarrow$ there exists a regular expression that describes L
- pumping lemma as a useful tool for determining whether a language is *nonregular*

Pumping Lemma revisited

Recall example from last week:

$$L = \{a^n b^n \mid n \geq 0\}$$

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What about the following language, for $\Sigma = \{a, b, c\}$:

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- Union of two languages:
- first language: all words of the form $ab^n c^n$
- second language: all Σ^* words that start with either 0 or 2 or more a's.
  $\rightarrow$ L is a disjoint union

Pumping Lemma revisited

Lemma (Pumping Lemma)

If A is a regular language, then there is a number p, called the pumping length, where if w is a word in A of length $\geq p$ then w can be divided into three parts, $w = xyz$, such that

1. $xy^iz \in A$ *for every* $i \geq 0$,
2. $|y| > 0$,
3. $|xy| \leq p$.

$$L = \{ab^nc^n \mid n \geq 0\} \cup \{a^kw \mid k \neq 1, \text{ and } w \in \Sigma^* \text{ doesn't start with } a\}$$

Does L satisfy the pumping lemma?

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- Let p be the pumping length.
- Each $w \in L$ is either of the form ab^nc^n or a^kw.

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- Assume $s = ab^n c^n$, where n is such that $|s| \geq p$.

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- choose $x = \varepsilon, y = a, z = b^nc^n$. Then $|y| > 0$ and $|xy| \leq p$.

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- The strings xz and $xy^i z$ for $i > 2$ are in Σ^* and don't start with a

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- The strings xz and xy^iz for $i > 2$ are in Σ^* and don't start with a $\Rightarrow xz, xy^iz \in L$.

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- The string $xy^i z$ for $i \geq 1$ starts with 2 or more a's, followed by a word $w \in \Sigma^*$ that does not start with an a.

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- So L is nonregular...is this a counter-example to the pumping lemma? No, pumping lemma is not an if and only if statement!

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- grammars describe the *syntax* of a language; they try to describe the relationship of all the parts to one another, such as placement of nouns/verbs in sentences
- useful for programming languages, specifically compilers and parsers: if the grammar of a programming language is available, parsing is very straightforward.

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First example:

$$S \to aSb$$
$$S \to \varepsilon$$

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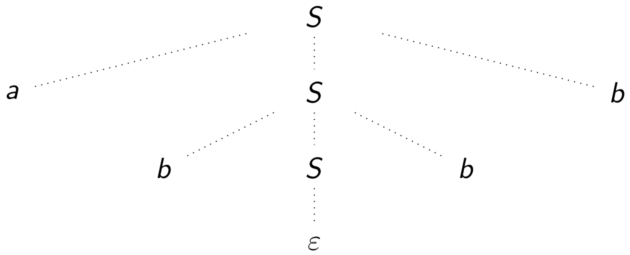
- Every grammar contains a *start variable* (above: variable S). Common convention: the first listed variable is the start variable (if you choose a different start variable, you must specify!).
- Words are generated by starting with the start variable and recursively replacing variables with the righthand side of a rule.

$$S \rightsquigarrow aSb \rightsquigarrow aaSbb \rightsquigarrow aa\varepsilon bb \rightsquigarrow aabb$$

Derivations of the form

$$S \rightsquigarrow aSb \rightsquigarrow aaSbb \rightsquigarrow aa\varepsilon bb \rightsquigarrow aabb$$

can also be encoded as a parse tree:



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Second example:

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The symbol $\mid$ takes on the meaning of "or."
$\to$ palindromes of even length over $\{a, b, c\}$.

Context-Free Grammar

Definition (Context-Free Grammar)

A *context-free grammar* is a 4-tuple (V, Σ, R, S) where

1. V is a finite set of *variables*
2. Σ is a finite set disjoint from V of *terminals*
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We call $L(G)$ the language generated by a context-free grammar. A language is called a *context-free language* if it is generated by a context-free grammar.

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Answers to these over the course of this and next lecture (and group sessions)

RLs and CFLs

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Theorem

Every regular language is context-free.

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- create grammar $G_{L_1 \cup L_2}$ that generates all words $w \in L_1 \cup L_2$.
- Create new start variable S.
- $G_{L_1 \cup L_2} = (V, \Sigma, R, S)$ where
- $V = V_1 \cup V_2 \cup \{S\}$,
- $\Sigma = \Sigma_1 \cup \Sigma_2$, and
- $R = R_1 \cup R_2 \cup \{S \rightarrow S_1 \mid S_2\}$.

CFL Union: Example

$$S_1 \to aS_1b \mid \varepsilon \qquad \cup \qquad S_2 \to aS_2a \mid bS_2b \mid cS_2c \mid \varepsilon$$

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Properties of CFLs: Concatanation

Let $G_1 = (V_1, \Sigma_1, R_1, S_1)$ and $G_2 = (V_2, \Sigma_2, R_2, S_2)$ be two grammars that generate L_1, L_2 respectively.
Concatanation:

- create grammar $G_{L_1 L_2} = (V, \Sigma, R, S)$ that accepts all words $w = w_1 w_2$, where $w_1 \in L_1$ and $w_2 \in L_2$.

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- create grammar $G = (V, \Sigma, R, S)$ that generates all words in L_1^*.

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Example:

$$S_1 \to aS_1b \mid \varepsilon$$

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Properties of CFLs

Closure under complement/intersection?

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$\rightsquigarrow$ No, but we need to know more before we can determine if a language is not context-free.

(next week)

Ambiguity

- Consider the grammar

$$E \rightarrow E + E \mid E \times E \mid (E) \mid a$$

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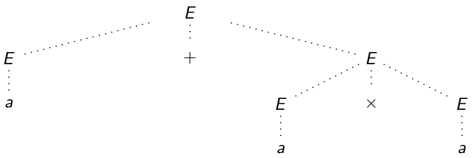
What does the parse tree for the string $a + a \times a$ look like?

Ambiguity



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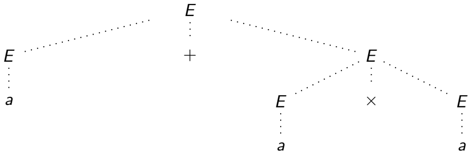

Intuitively corresponds to $a + (a \times a)$

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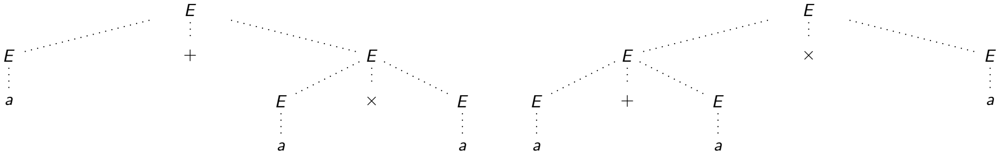

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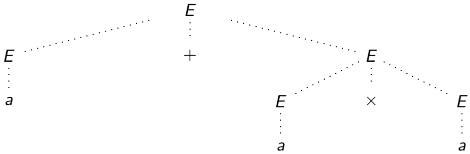

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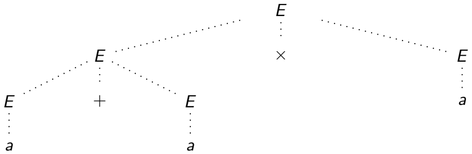

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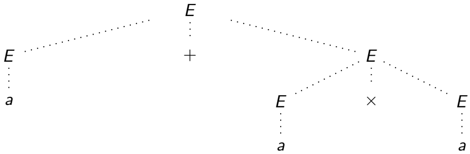

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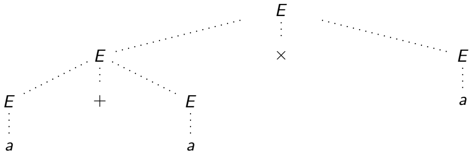

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This is called *ambiguity*

Ambiguity

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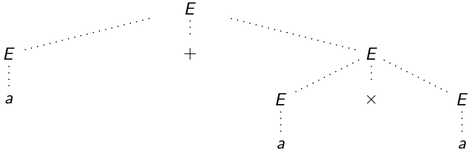
Both have the same parse tree!



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If $L(G)$ contains a string that is derived ambiguously, we say that G is ambiguous.

Chomsy Normal Form

- Context-free languages have a nice property: Every CFL can be described by a CFG in *Chomsky Normal Form*:

Definition

A grammar is in *Chomsky Normal Form* if every rule is of the form:

$$A \to BC$$
$$A \to a$$

where a is any terminal, A is any variable, B, C are any variables that are not the start variable. In addition the rule $S \to \varepsilon$ is permitted.

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CNF - Example

Grammar;

$$S \to ASA \mid aB$$
$$A \to B \mid S$$
$$B \to b \mid \varepsilon$$

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and you would continue to remove the unit rules $A \rightarrow S$, etc....

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and you would continue to remove the unit rules $A \rightarrow S$, etc....But how to convert, say, $S \rightarrow ASA$ into rules with only two symbols on the right? $\rightsquigarrow$ introduce help variables!

$$S \rightarrow ASA$$
$$\rightsquigarrow S \rightarrow AA_1, A_1 \rightarrow SA$$

CNF

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- how can finite automata be enriched so as to accept context-free languages? $\rightarrow$ next week!