INF2080

Church Turing Thesis and Decidability

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Short Recap

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- We have looked at Turing machines as a computational model
- a finite state machine with an infinite tape, upon which a head can move, read, and write
- have looked at Turing machine variants, seen that they are equivalent:
- the LRS Turing machine (the head can move left, right, or stay put)
- the multitape Turing machine (multiple tapes, multiple heads)
- the nondeterministic Turing machine
- the enumerator

Church Turing Thesis

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- So, all variants of Turing machines we've seen are equivalent in expressivity.
- This is no coincidence: all can perform finite work in a single step, all have unlimited access to infinite memory.
- In fact, Turing machines capture *all* such computational models

Church-Turing Thesis

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- yet a formal description of what an algorithm is, or what is solvable using algorithms, did not appear until the 20th century.
- Many mathematicians assumed that one needed only to *find* the right "method", did not even consider something might be unsolvable.

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- Previous, intuitive notion: a method according to which after a finite number of operations an answer is given (paraphrased, many formulations)
- Formal: an algorithm is a decidable Turing machine (deciders)
- Church Turing thesis: each intuitive definition of algorithms can be described by decidable Turing machines

Decidability

Definition

A language L is *decidable* if a Turing machine M_L exists that *decides* it, that is, if M_L either accepts or rejects any input w.

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- **Acceptance problem:** Given a DFA/NFA/CFG/PDA/TM/... and an input w, does the machine/grammar accept w?
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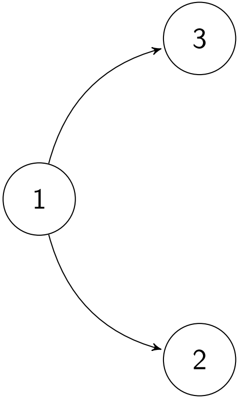
- **Emptiness problem:** Given a DFA/NFA/CFG/PDA/TM/..., is its generated language empty?
- **Equality problem:** Given two DFA/NFA/CFG/PDA/TM/..., are the two generated languages equal?

Notation

For an object O (graph, automaton, Turing machine, etc.), let $\langle O \rangle$ represent its string representation. For example:



can be represented as the string $\{1, 2, 3, (1, 2), (1, 3)\}$

Acceptance problem - DFA

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Theorem

A_{DFA} is a decidable language.

Proof idea: We create a Turing machine that simulates B on w:

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Proof idea: We create a Turing machine that simulates B on w:

M_{DFA} = On input $\langle B, w\rangle$

1. Simulate B on w.
2. If the simulation ends in an accept state, *accept*,
   if it ends in a nonaccepting state, *reject*.

Acceptance problem - DFA

Corollary

The class of regular languages is decidable.

Proof:

- For a given regular language L, we need to construct a decider M_L that accepts all $w \in L$ and rejects all $s \notin L$.

Acceptance problem - DFA

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Proof:

- For a given regular language L, we need to construct a decider M_L that accepts all $w \in L$ and rejects all $s \notin L$.
- we can encode its DFA B into a decider for L:

$M_L =$ On input w

1. Simulate M_{DFA} on $\langle B, w \rangle$.
2. If M_{DFA} accepts, *accept*,
   if it rejects, *reject*.

Acceptance problem - NFA/RE

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Theorem

The language A_{NFA} is decidable.

Proof:

M_{NFA} = On input $\langle B, w\rangle$

1. Convert B to an equivalent DFA C.
2. Simulate M_{DFA} on input $\langle B, w\rangle$

   if it accepts, *accept*; if it rejects, *reject*.

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$A_{RE} = \{\langle R, w\rangle \mid B \text{ is a regular expression that generates } w\}$

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The language A_{RE} is decidable.

Proof: Similar to before, however now we reduce to NFA case:

M_{RE} = On input $\langle R, w\rangle$

1. Convert R to an equivalent NFA B.
2. Simulate M_{NFA} on input $\langle B, w\rangle$

   if it accepts, *accept*; if it rejects, *reject*.

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- Recall the Church-Turing thesis: intuitive notion of algorithm/procedure $\Leftrightarrow$ Turing machine algorithm

Acceptance problem - Regular languages

- So we see that it is does not matter which computational model we use to represent the regular language; this has no effect on decidabillity
- Recall the Church-Turing thesis: intuitive notion of algorithm/procedure $\Leftrightarrow$ Turing machine algorithm
- Our "procedures" of converting NFA$\rightarrow$DFA, RE$\rightarrow$NFA, CFG$\leftrightarrow$PDA can be formally described using a decidable TM!

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- When does a DFA accept a string w? When it reaches an accept state!
- So all the TM has to do is check whether an accept state is reachable from the start state.
- We use the "marking" technique we have previously seen to keep track of the DFA's states that have been reached.

Emptiness problem - Regular languages

Theorem

The language E_{DFA} is decidable.

Proof:

N_{DFA} = On input $\langle A \rangle$

1. Mark the start state of A.
2. Repeat 3. until no new states are marked:
3. Mark any state with an incoming transition from a marked state.
4. If no accept state is reached, *accept*; else, *reject*.

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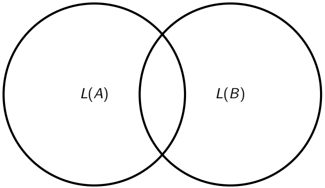

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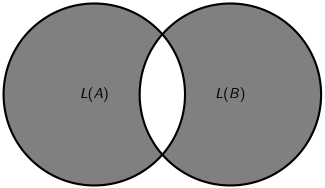

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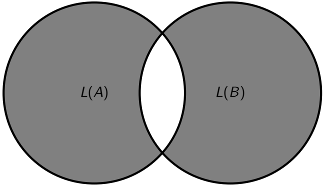

- Two sets are equal if and only if their symmetric difference is empty!

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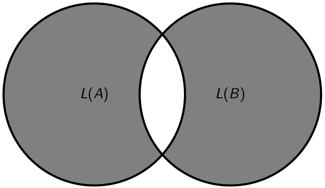

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Recall closure properties of regular languages:

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Recall closure properties of regular languages:

- closed under union, intersection, and complement (among other things)
- have seen procedures for constructing the DFA for unions/intersections/complements of regular languages.
- Using these, we can construct a DFA that accepts the symmetric difference of two regular languages.

Equality problem - Regular languages

Theorem

The language EQ_{DFA} is decidable.

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Theorem

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Proof:

S_{DFA} = On input $\langle A, B \rangle$

1. Construct C, the DFA of the symmetric difference of $L(A)$ and $L(B)$.
2. Run N_{DFA} on C. (checks whether $L(C)$ is empty)
3. If N_{DFA} accepts, *accept*; if N_{DFA} rejects, *reject*.

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Summary - Regular languages

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- the acceptance problem (does A accept w?) is decidable, independent of the computational model in which we chose to describe regular languages;
- the emptiness problem (is $L(A)$ empty?) is decidable;
- the equality problem (are $L(A)$ and $L(B)$ equal?) is decidable.
- in each case: we reduced the question to checking membership in a language.

Decision problems - CFLs

What about the decision problems for context-free languages?

Decision problems - CFLs

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Are the languages

$$
\begin{aligned}
A_{CFG} =& \{\langle G, w\rangle \mid G \text{ is a CFG that generates } w\} \\
E_{CFG} =& \{\langle G\rangle \mid G \text{ is a CFG and } L(G) = \emptyset\} \\
EQ_{CFG} =& \{\langle G, H\rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}
\end{aligned}
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decidable?

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Theorem

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Proof:

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The language A_{CFG} is decidable.

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- Instead, we use the fact that every CFG can be converted to a grammar in Chomsky Normal Form.

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- One can show (Problem 2.38 in Sipser) that if a grammar is CNF, then every derivation of w has length $2n - 1$, where n is the length of w.

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- Instead, we use the fact that every CFG can be converted to a grammar in Chomsky Normal Form.
- One can show (Problem 2.38 in Sipser) that if a grammar is CNF, then every derivation of w has length $2n - 1$, where n is the length of w.
- That way we only need to check all derivations of length $2n - 1$ to see if any generates w!

Acceptance problem - CFLs

Theorem

The language A_{CFG} is decidable.

Proof:

M_{CFG} = On input $\langle G, w \rangle$

1. Convert G to a CFG in Chomsky Normal Form.
2. If $n = 0$, where n is the length of w, list all derivations with 1 step.
   Else, list all derivations with $2n - 1$ steps.
3. If any of the derivations generate w *accept*; otherwise, *reject*.

Decidability of CFLs

As in the regular language case, we can use this last result to show:

Corollary

Every context-free language is decidable.

Decidability of CFLs

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Corollary

Every context-free language is decidable.

Proof: completely analogous to the DFA/regular case:

M_L = On input w

1. Simulate M_{CFG} on $\langle B, w \rangle$.
2. If M_{CFG} accepts, *accept*,
   if it rejects, *reject*.

Emptiness problem - CFLs

Theorem

The language $E_{CFG} = \{\langle G \rangle \mid G \text{ is a CFG and } L(G) = \emptyset\}$ *is decidable.*

Proof idea:

- In the DFA case, we checked reachability of accept states from the start state through a marking procedure.

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- Can we do the same here?

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Proof idea:

- In the DFA case, we checked reachability of accept states from the start state through a marking procedure.
- Can we do the same here?
- Yes! but slightly differently.
- Consider the grammar consisting of only $S \rightarrow S$. If we were to start with S and iteratively generate all derivations, we would never terminate.

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- We're interested in finding out whether a string of terminals can be generated from S. So why not first mark terminals, then mark a variable A if there is a rule $A \to s$ where s consists of marked symbols?

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- We're interested in finding out whether a string of terminals can be generated from S. So why not first mark terminals, then mark a variable A if there is a rule $A \rightarrow s$ where s consists of marked symbols?$\rightarrow$ go through derivations "backwards". If S is marked, then a string of terminals can be generated.

Emptiness problem - CFLs

Theorem

The language $E_{CFG} = \{\langle G \rangle \mid G \text{ is a CFG and } L(G) = \emptyset\}$ *is decidable.*

Example: Grammar

$$S \to ARB$$
$$A \to a$$
$$B \to b$$
$$R \to aRb \mid \varepsilon$$

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$$A \rightarrow \dot{a}$$
$$B \rightarrow \dot{b}$$
$$R \rightarrow \dot{a}R\dot{b} \mid \dot{\varepsilon}$$

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$$\dot{R} \to \dot{a}\dot{R}\dot{b} \mid \dot{\varepsilon}$$

Emptiness problem - CFLs

Theorem

The language $E_{CFG} = \{\langle G \rangle \mid G \text{ is a CFG and } L(G) = \emptyset\}$ *is decidable.*

Example: Grammar

$$\dot{S} \to \dot{A}\dot{R}\dot{B}$$
$$\dot{A} \to \dot{a}$$
$$\dot{B} \to \dot{b}$$
$$\dot{R} \to \dot{a}\dot{R}\dot{b} \mid \dot{\varepsilon}$$

$\to$ S is marked, so language is not empty!

Emptiness problem - CFLs

Theorem

The language $E_{CFG} = \{\langle G\rangle \mid G \text{ is a CFG and } L(G) = \emptyset\}$ *is decidable.*

Proof:

N_{CFG} = On input $\langle G\rangle$

1. Mark all terminal symbols in G.
2. Repeat 3. until no new variables are marked:
3. Mark any variable A where G has a rule $A \to U_1 \dots U_k$ and each symbol U_i has been marked.
4. If the start variable is not marked, *accept*. otherwise, *reject*.

Equality problem - CFLs

- So what about $EQ_{CFG} = \{\langle G, H \rangle \mid G$ and H are CFGs and $L(G) = L(H)\}$? Is it decidable?
- Before we used the symmetric difference $(L(A) \cap \overline{L(B)}) \cup (\overline{L(A)} \cap L(B))$ to use the emptiness decider.

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Equality problem - CFLs

- So what about $EQ_{CFG} = \{\langle G, H\rangle \mid G$ and H are CFGs and $L(G) = L(H)\}$? Is it decidable?
- Before we used the symmetric difference $(L(A) \cap \overline{L(B)}) \cup (\overline{L(A)} \cap L(B))$ to use the emptiness decider.
- But context-free languages *are not closed under complementation or intersection!*
- in fact, EQ_{CFG} is *not* decidable. Tomorrow we'll see techniques to show this.

Summary- CFLs

- the acceptance and emptiness decision problems are decidable for context-free languages
- hence, each context-free language is decidable.
- checking equivalence of two grammars (in the sense of languages generated) is *not* decidable!

Acceptance problems - TMs

- What about Turing-recognizable languages? Are they *also* decidable?

Acceptance problems - TMs

- What about Turing-recognizable languages? Are they *also* decidable?
- If they were, every Turing machine could be converted into an equivalent TM that is guaranteed to halt on every input!

Acceptance problem - TMs

First things first...

Theorem

The language $A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts } w\}$ *is Turing-recognizable.*

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Proof:

$U =$ On input $\langle M, w\rangle$

1. Simulate M on w.
2. If M ever enters its accept state, *accept*; if M ever enters its reject state, *reject*.

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The language $A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts } w\}$ *is Turing-recognizable.*

Proof:

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1. Simulate M on w.
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U is an example of a *universal Turing machine*!

Acceptance problem - TMs

- So what about decidability?

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- Let's assume there exists a decider H for A_{TM}, i.e., $H(\langle M, w \rangle) = \textit{accept}$ if M accepts w, and $H(\langle M, w \rangle) = \textit{reject}$ if M does not accept w.

Acceptance problem - TMs

- So what about decidability?
- Let's assume there exists a decider H for A_{TM}, i.e., $H(\langle M, w \rangle) = accept$ if M accepts w, and $H(\langle M, w \rangle) = reject$ if M does not accept w.
- We are going to use a standard mathematical trick in order to create a contradiction

Diagonalization method

Let's write all Turing machines into the following table:

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\ldots$
M_1	*accept*		*accept*	
M_2			*accept*	
M_3	*accept*	*accept*	*accept*	
$\vdots$				

- Each cell (i, j) represents whether M_i accepts the string $\langle M_j \rangle$ (the string representation of machine M_j).
- *accept* means that it accepts, blank means it loops or rejects.

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- *accept* means that it accepts, blank means it loops or rejects.
- The decider H let's us fill out the blank cells with *reject*.

Diagonalization method

Result of H with input $M_i, \langle M_j \rangle$:

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\dots$
M_1	*accept*	*reject*	*accept*	
M_2	*reject*	*reject*	*accept*	
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- Each cell (i, j) represents $H(\langle M_i, \langle M_j \rangle \rangle)$

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Result of H with input $M_i, \langle M_j \rangle$:

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\dots$
M_1	*reject*	*reject*	*accept*	
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Diagonalization method

But D must occure in the table too!

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\ldots$	$\langle D \rangle$	$\ldots$
M_1	*accept*	*reject*	*accept*			
M_2	*reject*	*reject*	*accept*			
M_3	*accept*	*accept*	*accept*			
$\vdots$						
D					?	
$\vdots$						

- What does D do with input $\langle D \rangle$?
- if $D(\langle D \rangle) = accept$, $H(\langle D, \langle D \rangle \rangle) = reject$.

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	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\dots$	$\langle D \rangle$	$\dots$
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- if $D(\langle D) = reject$, $H(\langle D, \langle D \rangle \rangle) = accept$. This means D does accept $D(\langle D \rangle)$, i.e., $D(\langle D \rangle) = accept$.

Acceptance problem - TMs

Let's formalize this:

Theorem

The language A_{TM} is not decidable.

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Proof:

- Assume it is decidable. Then there exists a decider H that decides A_{TM}. So $H(\langle M, w\rangle) = accept$ iff M accepts w and $H(\langle M, w\rangle) = reject$ iff M fails to accept w.

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- We define a decider D that on input $\langle M\rangle$ flips the result of $H(\langle M, \langle M\rangle\rangle)$.

$D =$ On input $\langle M\rangle$

1. Simulate H on $\langle M, \langle M\rangle\rangle$.
2. If H accepts, *reject*; if H rejects, *accept*.

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- if H accepts, D rejects and if H rejects, D accepts.

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- So what is the result of $D(\langle D \rangle)$? Remember, $H(\langle M, w \rangle)$ accepts iff $M(w) = accept$.

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- If $D(\langle D \rangle) = accept$, then $H(\langle D, \langle D \rangle \rangle) = reject$, i.e., $D(\langle D \rangle) = reject$. Contradiction!
- Hence neither D nor H can exist! $\rightarrow A_{TM}$ is undecidable!

Turing unrecognizability

- So we've seen there exists an undecidable language: A_{TM}
- Do there exist *non-Turing-recognizable* languages?

Turing unrecognizability

Theorem

A language A is decidable iff it is Turing-recognizable and co-Turing-recognizable, i.e., if A and its complement $\overline{A}$ are Turing-recognizable.

Turing unrecognizability

Theorem

A language A is decidable iff it is Turing-recognizable and co-Turing-recognizable, i.e., if A and its complement $\overline{A}$ are Turing-recognizable.

Proof:

- If A is decidable, then it is Turing-recognizable. Since decidable langauges are closed under complementation, this means $\overline{A}$ is decidable, in particular Turing-recognizable.

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- Now assume A and $\overline{A}$ are Turing recognizable. Then there exist recognizers M_A and $M_{\overline{A}}$ that accept w if it is in A or $\overline{A}$, respectively.

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Proof:

- Now assume A and $\overline{A}$ are Turing recognizable. Then there exist recognizers M_A and $M_{\overline{A}}$ that accept w if it is in A or $\overline{A}$, respectively.
- we combine these to a machine M:

$M =$ On input w

1. Run both M_A and $M_{\overline{A}}$ in parallel on input w
2. If M_A accepts, *accept*; if $M_{\overline{A}}$ accepts, *reject*.

□

Turing-unrecongizability

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We can use this result to show that $\overline{A_{TM}}$ is not Turing-recognizable:

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We can use this result to show that $\overline{A_{TM}}$ is not Turing-recognizable:

- A_{TM} is Turing-recognizable
- If $\overline{A_{TM}}$ were Turing-recognizable, then by the last theorem A_{TM} must be decidable
- But we just saw that A_{TM} is undecidable.
- Hence $\overline{A_{TM}}$ must be Turing-unrecognizable