INF2080

Context-Free Langugaes: Pushdown Automata

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- What computational model accepts context-free languages? $\rightarrow$ pushdown automata!

Pushdown Automata (PDA)

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- we now add a (very limited) form of infinite memory: a stack (LIFO principle: last in, first out)
- So, essentially, PDA's are NFA's with an additional stack

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$$\text{NFA} \xrightarrow{\text{read}} \text{input}$$

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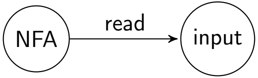

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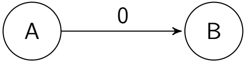

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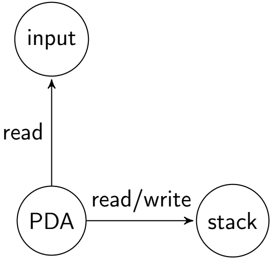

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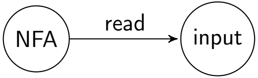

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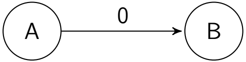

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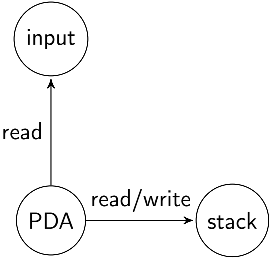

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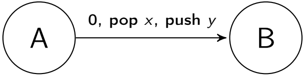

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For notational ease, we sketch these transition as follows:

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- $\varepsilon, x \rightarrow y$: read no input, perform stack operations as described above (x and/or y may be ε)

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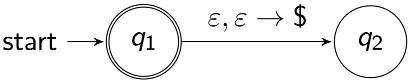

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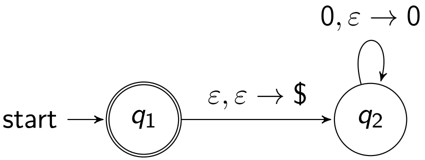

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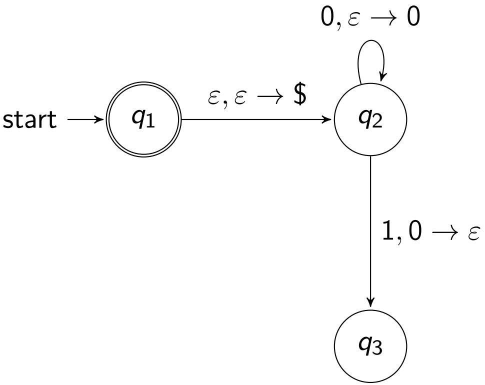

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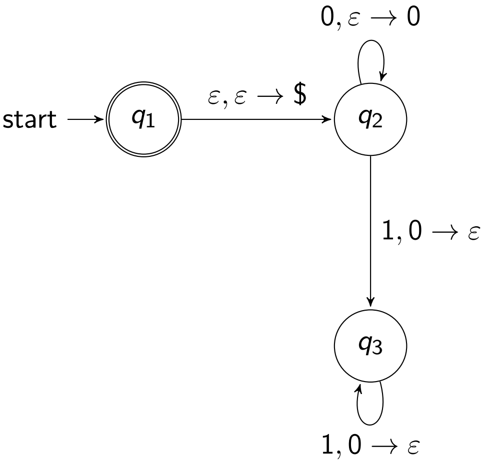

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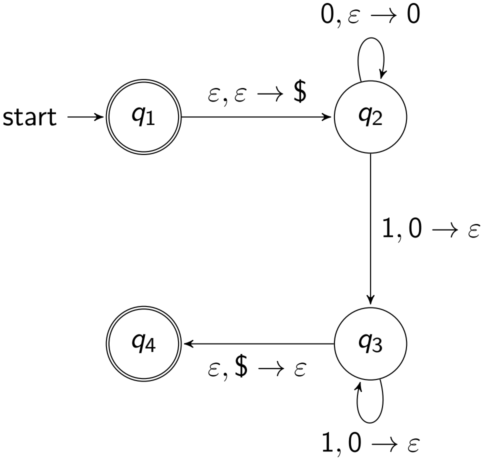

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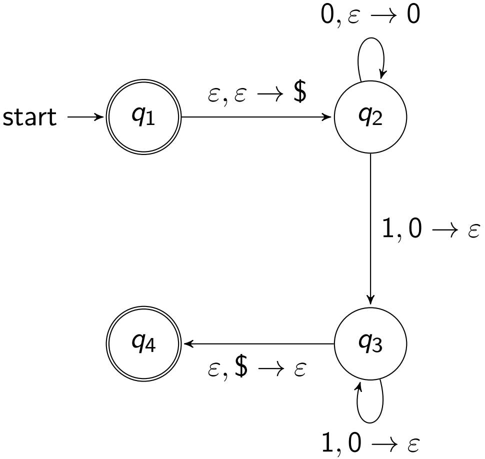

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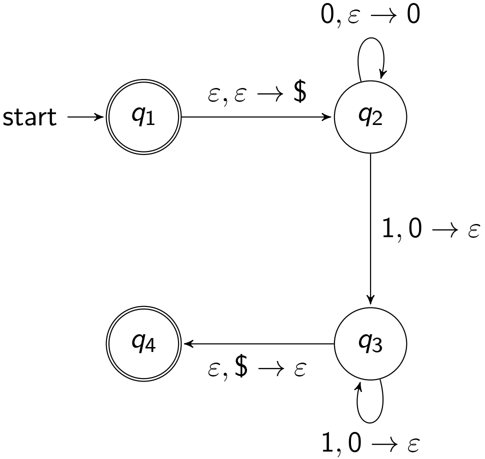

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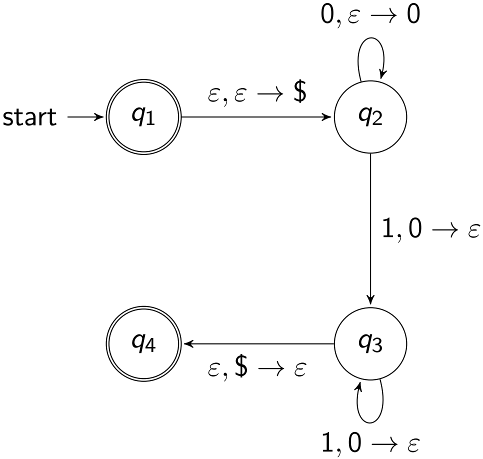

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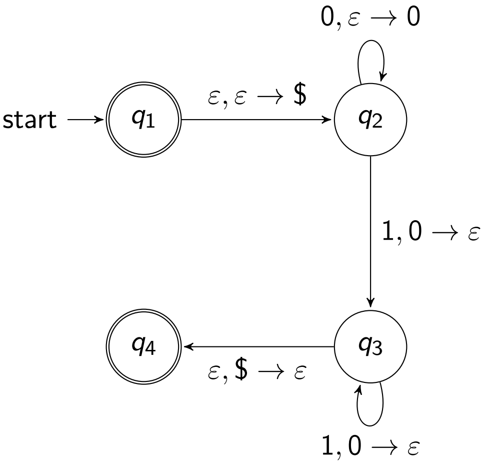

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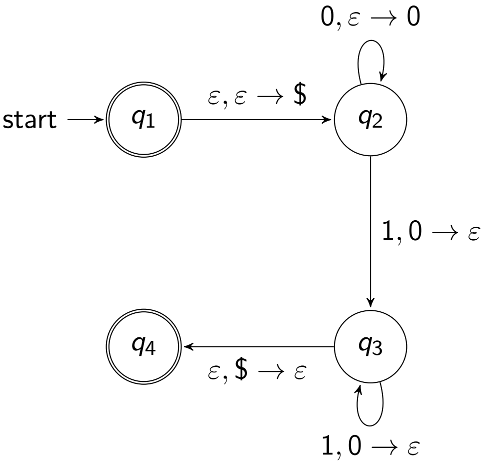

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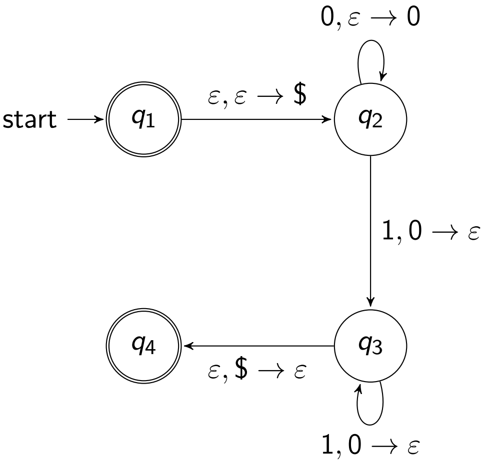

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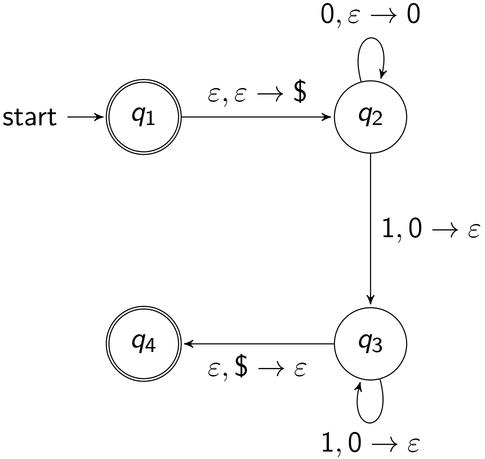

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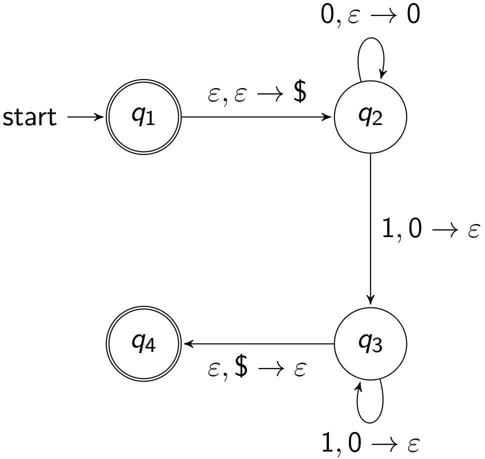

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- How to design a PDA that accepts L_2?

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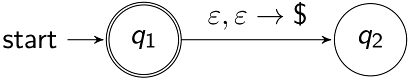

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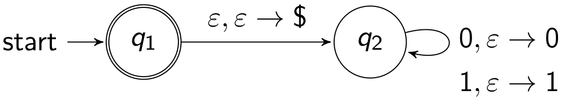

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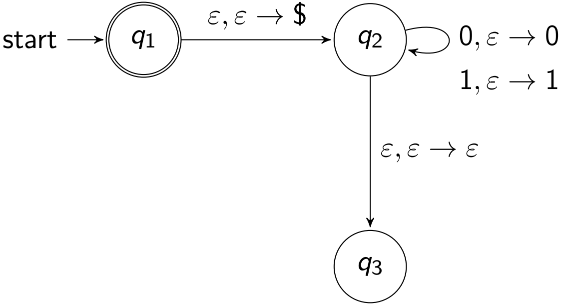

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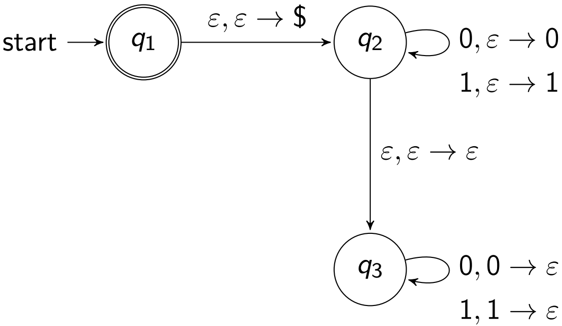

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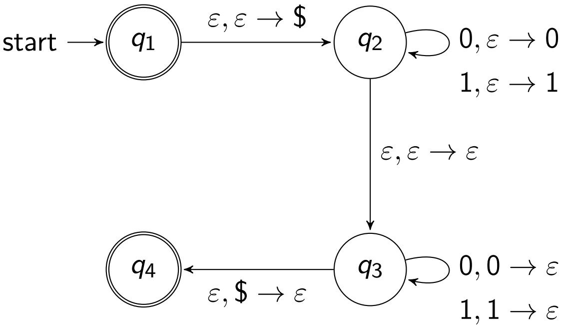

PDA

Definition (PDA)

A PDA is a tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q, Σ, Γ, F are finite states and

1. Q is a set of states,
2. Σ is the input alphabet,
3. Γ is the stack alphabet,
4. $\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to \mathcal{P}(Q \times \Gamma_\varepsilon)$ is the transition function
5. $q_0 \in Q$ is the start state, and
6. $F \subseteq Q$ is the set of accepting states.

This lecture

We want to show:

Theorem

A language is context-free if and only if some pushdown automaton recognizes it.

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Step by step.... first we show:

Lemma

If a language is context-free then there exists a pushdown automaton that recognizes it.

In other words: given a CFG G, construct a PDA that recognizes $L(G)$!

CFG->PDA

Lemma

If a language is context-free then there exists a pushdown automaton that recognizes it.

Proof idea:

- construct a PDA that for input w determines if it has a valid derivation.

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- Use stack to store *how far we've gotten in our derivation*, i.e., the intermediate strings

$$\color{blue}{S} \rightsquigarrow a\color{blue}{S}b \rightsquigarrow aa\color{blue}{S}bb \rightsquigarrow aa\varepsilon bb \rightsquigarrow aabb$$

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- if top of stack is variable, we pop the variable and replace with the righthand side of a rule

CFG→PDA

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If a language is context-free then there exists a pushdown automaton that recognizes it.

Proof idea: Intuitively, the PDA works as follows:

- Place special symbol \$ on stack
- Repeat the following:

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1. **input:** *aabb*
   Stack: \$

CFG→PDA

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   Stack: Sbb\$
7. input: $aa\overset{\downarrow}{b}b$
   Stack: bb\$
8. compare all terminals in stack to input

CFG→PDA

Note how in the previous slide we added multiple symbols the stack in one step!

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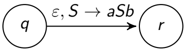

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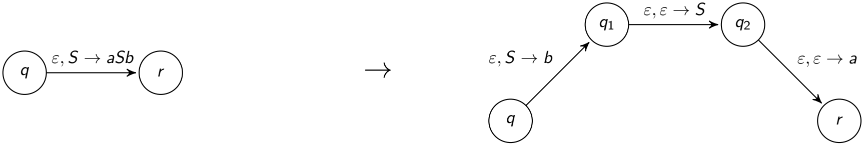

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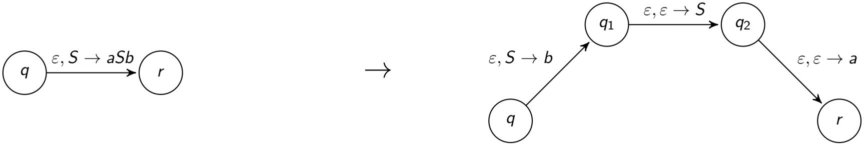

We will use this shorthand notation during the proof.

CFG→PDA: Proof

Proof:

- Given CFG $G = (V, \Sigma, R, S)$, we want to construct a PDA $(Q, \Sigma, \Gamma, \delta, q_0, F)$ that accepts $L(G)$.

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- Let $Q = \{q_0, q_{\mathrm{loop}}, q_{\mathrm{accept}}\} \cup E$, where E is the set of help states needed for our shorthand notation (previous slide).
- Before anything else, push \$ and start variable on to stack: $\delta(q_0, \varepsilon, \varepsilon) = \{(q_{\mathrm{loop}}, S\$)\}$.

CFG$\rightarrow$PDA: Proof

Recall intuition:

- if the top of stack is a variable A, nondeterministically choose a rule $A \rightarrow w$, pop A and push w
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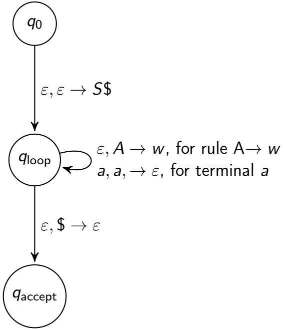
CFG→ PDA

State diagram (without help states from shorthand notation):



PDA→CFG

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PDA→CFG

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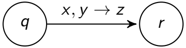

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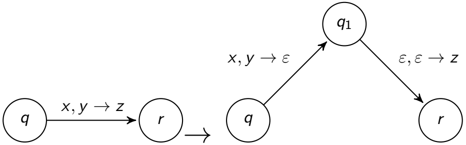

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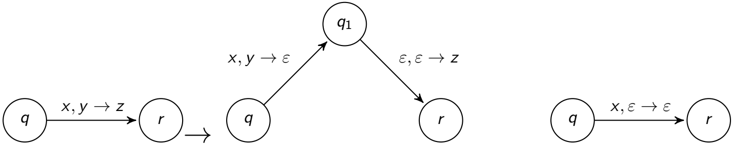

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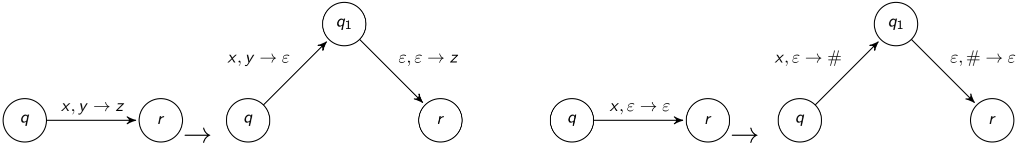

PDA→CFG

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$\rightsquigarrow$ input s takes P from p to q with empty stacks if:

- starting in p with an empty stack, after parsing the input s the PDA P ends in state q.
- when P arrives at q, the stack is once again empty

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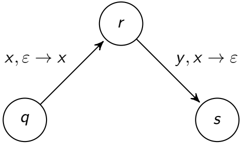
Then $A_{q_0 q_{\text{accept}}}$ will generate precisely the words the *PDA* accepts!

PDA→CFG

Example:



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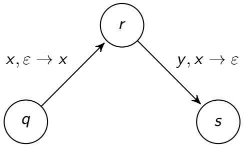

- string x does not take P from q to r with empty stacks, since the stack ends with an extra x

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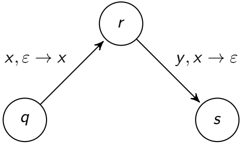
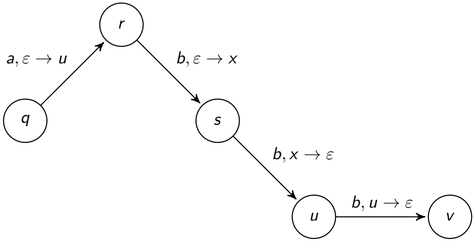


- string x does not take P from q to r with empty stacks, since the stack ends with an extra x
- string xy takes P from q to s with empty stacks, since the x that was pushed gets popped in the last transition to s.

PDA→CFG

Two possibilities:



Input *abbb* takes P from q to v with empty stacks. Here the stack is never emptied in between.

PDA→CFG

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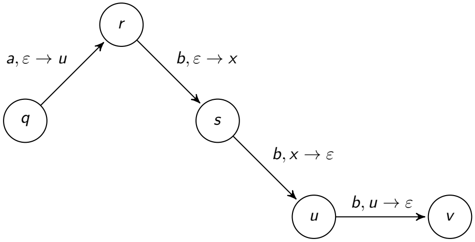

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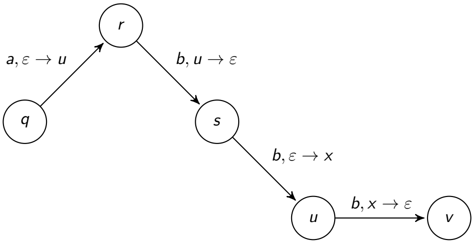

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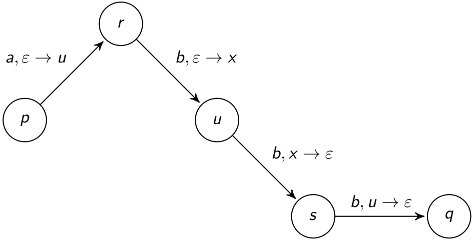
PDA→CFG

Stack not emptied in between:



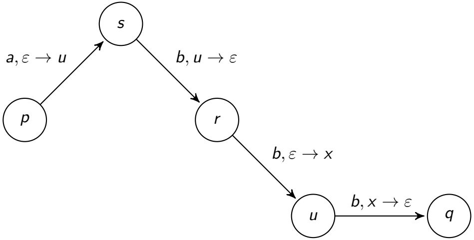
To address this case, add

$$A_{pq} \to aA_{rs}b$$

for every $p, q, r, s \in Q$, $u \in \Gamma$ and $a, b \in \Sigma_\varepsilon$ such that $\delta(p, a, \varepsilon)$ contains (r, u) and $\delta(s, b, u)$ contains (q, ε).

PDA→CFG

Stack is emptied in between:



To address this case, add rule

$$A_{pq} \to A_{pr} A_{rq}$$

for all $p, q, r \in Q$

PDA→CFG

Finally, add rules $A_{pp} \to \varepsilon$ for all $p \in Q$.

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Summarized:

- $A_{pq} \to aA_{rs}b$ for every $p, q, r, s \in Q$, $u \in \Gamma$ and $a, b \in \Sigma_\varepsilon$ such that $\delta(p, a, \varepsilon)$ contains (r, u) and $\delta(s, b, u)$ contains (q, ε),

PDA$\rightarrow$CFG

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- $A_{pq} \to A_{pr}A_{rq}$ for every $p, q, r \in Q$,
- $A_{pp} \to \varepsilon$ for all $p \in Q$
- start variable: $A_{q_0 q_{\text{accept}}}$

PDA→CFG

We are done if we can show:

Lemma

If A_{pq} generates x then x brings P from p to q with empty stacks.

Lemma

If x can bring P from p to q with empty stacks, A_{pq} generates x.

PDA→CFG

- $A_{pq} \to aA_{rs}b$ for every $p, q, r, s \in Q$, $u \in \Gamma$ and $a, b \in \Sigma_\varepsilon$ such that $\delta(p, a, \varepsilon)$ contains (r, u) and $\delta(s, b, u)$ contains (q, ε),
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- A derivation with 1 step uses a rule with no variables on the righthand side.
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Proof by induction over number of derivation steps:
Step: assume true for derivations of length at most $k \geq 1$.

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If A_{pq} generates x then x brings P from p to q with empty stacks.

Proof by induction over number of derivation steps:

Step: assume true for derivations of length at most $k \geq 1$.

- Assume A_{pq} derives x in $k + 1$ steps.

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Proof by induction over number of derivation steps:

Step: assume true for derivations of length at most $k \geq 1$.

- Assume A_{pq} derives x in $k+1$ steps. Then the first rule applications is either $A_{pq} \to aA_{rs}b$ or $A_{pq} \to A_{pr}A_{rq}$

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- A_{rs} derives y in at most k steps, hence the induction hypothesis tells us that y takes P from r to s with empty stacks.

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- If $A_{pq} \to aA_{rs}b$ was used, let y be generated by A_{rs} such that $x = ayb$.
- A_{rs} derives y in at most k steps, hence the induction hypothesis tells us that y takes P from r to s with empty stacks.
- $\to$ a brings P from p to r by pushing u, y brings P from r to s on empty stacks, b brings PP from s to q by popping u.

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- A_{pr} (resp. A_{rq}) generates y (resp. z) in at most k steps.

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Proof by induction over computation steps:
Base: computation has 0 steps.

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Proof by induction over computation steps:
Base: computation has 0 steps.

- A computation with 0 steps stays in the same state p and cannot read any input. Since $A_{pp} \to \varepsilon$ is a rule in G, base is proved.

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Proof by induction over computation steps:
Step: Assume true for computations of length at most k.

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If x can bring P from p to q with empty stacks, A_{pq} generates x.

Proof by induction over computation steps:
Step: Assume true for computations of length at most k.

- Assume P has a computation where input x takes it from state p to q with empty stacks that takes $k + 1$ steps.

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If x can bring P from p to q with empty stacks, A_{pq} generates x.

Proof by induction over computation steps:
Step: Assume true for computations of length at most k.

- Assume P has a computation where input x takes it from state p to q with empty stacks that takes $k + 1$ steps.
- Two cases: either the stack is only empty at the beginning and end, or it is emptied in betwen.

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If x can bring P from p to q with empty stacks, A_{pq} generates x.

- First case (empty only at beginning and end):
- Symbol u pushed in the first step must be the same symbol popped in the last step

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- Let a denote the input read in step 1, b the input in the last step, r the state after the first move, s the state before the last move.
- Then $A_{pq} \to aA_{rs}b$ is in G.

PDA→CFG

- $A_{pq} \to aA_{rs}b$ for every $p, q, r, s \in Q$, $u \in \Gamma$ and $a, b \in \Sigma_\varepsilon$ such that $\delta(p, a, \varepsilon)$ contains (r, u) and $\delta(s, b, u)$ contains (q, ε),
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- Then $A_{pq} \to aA_{rs}b$ is in G.
- Let $x = ayb$. y takes P from r to s with empty stacks in $k - 1$ steps

PDA→CFG

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- $\Rightarrow A_{pq}$ generates x.

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If x can bring P from p to q with empty stacks, A_{pq} generates x.

- Second case (empty inbetween):

PDA→CFG

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- start variable: $A_{q_\mathbf{0} q_\mathbf{accept}}$

Lemma

If x can bring P from p to q with empty stacks, A_{pq} generates x.

- Second case (empty inbetween):
- Let r be a state where the stack empties other than at the beginning and end of the computation of x.

PDA$\rightarrow$CFG

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Lemma

If x can bring P from p to q with empty stacks, A_{pq} generates x.

- Second case (empty inbetween):
- Let r be a state where the stack empties other than at the beginning and end of the computation of x.
- Then the computations from p to r (call its parsed input y) and from r to q (call its parsed input z) contain at most k steps.

PDA→CFG

- $A_{pq} \to aA_{rs}b$ for every $p, q, r, s \in Q$, $u \in \Gamma$ and $a, b \in \Sigma_\varepsilon$ such that $\delta(p, a, \varepsilon)$ contains (r, u) and $\delta(s, b, u)$ contains (q, ε),
- $A_{pq} \to A_{pr}A_{rq}$ for every $p, q, r \in Q$,
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- Then A_{pq} generates $x = yz$. $\square$