INF2080

2. Regular Expressions and Nonregular languages

Daniel Lupp

Universitetet i Oslo

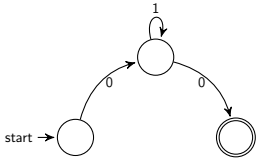
25th January 2018

Department of Informatics

University of Oslo

Last week

- Deterministic finite automata (DFA)



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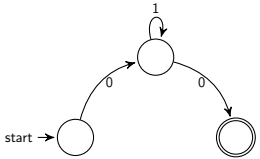

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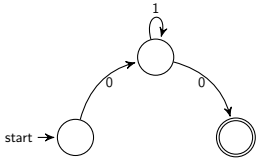

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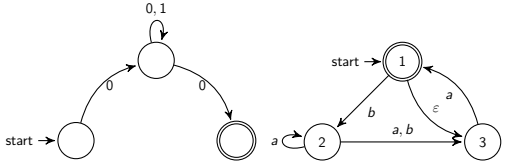

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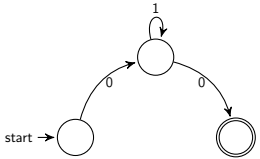

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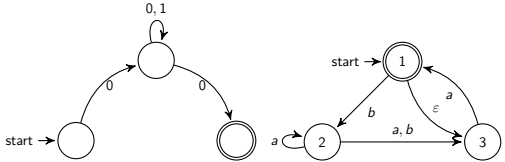

- DFA $\leftrightarrow$ NFA

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$\rightarrow$ Regular expressions represent languages!

Regular Expressions - Examples

What languages do the following regular expressions (RE) represent?

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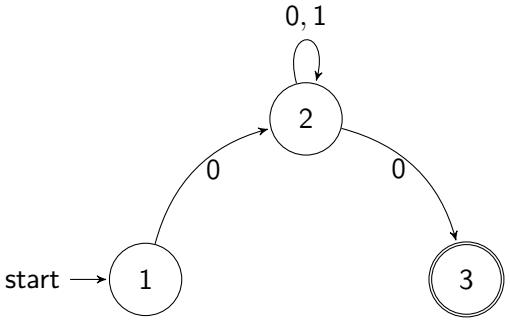

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Yes!

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Every language described by an RE is regular.

Proof based on inductive definition of RE!

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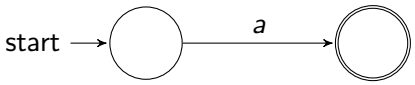

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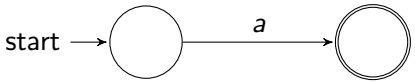

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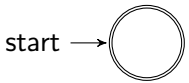

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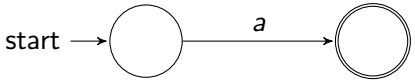

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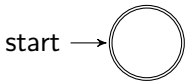

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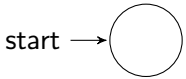

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(recall: the union/concatanation/Kleene star of regular languages is itself regular)

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Next:

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Every regular language can be described using a RE.

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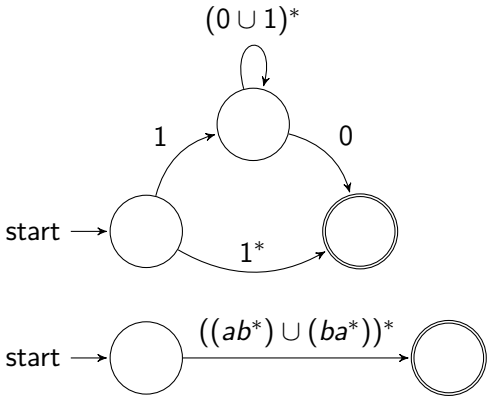

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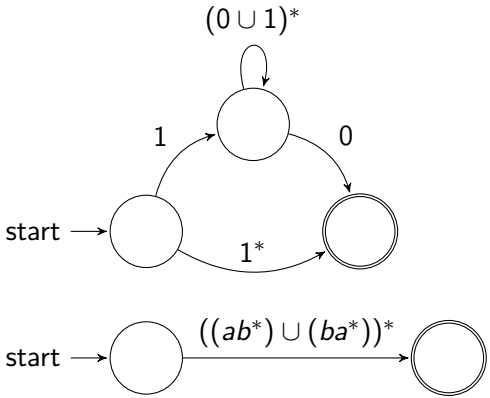

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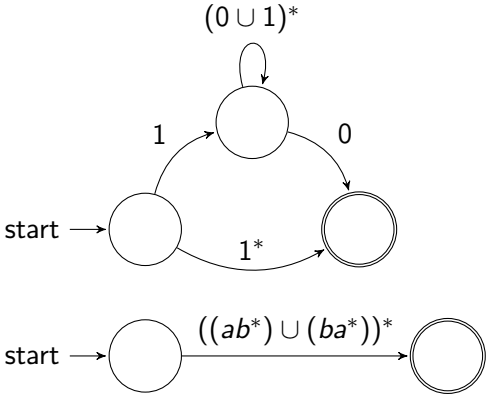

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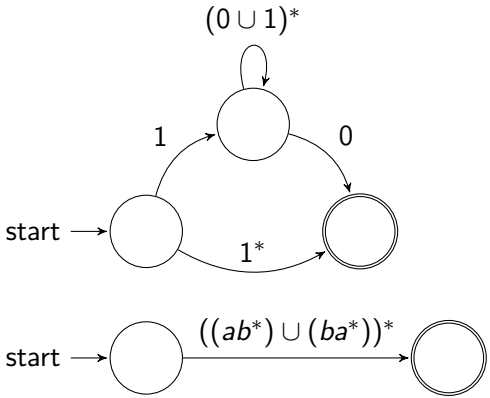

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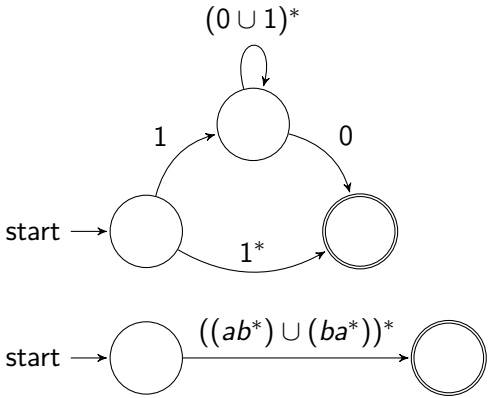

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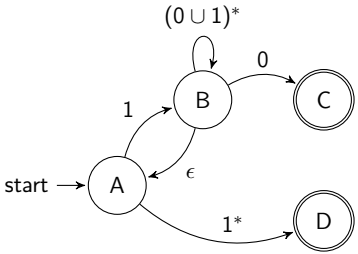

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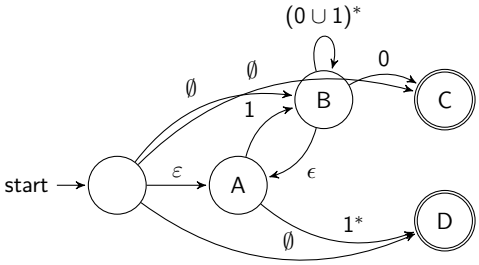

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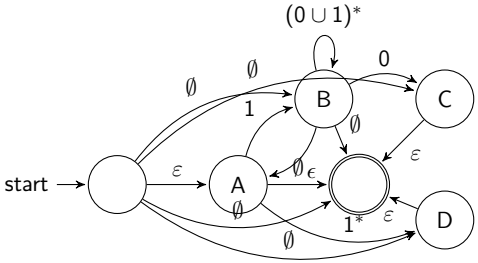

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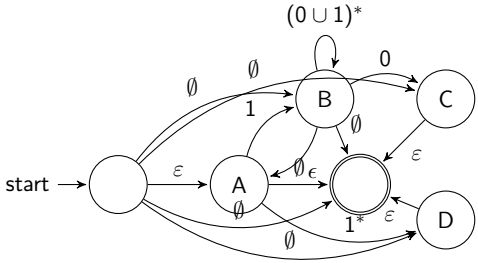

for the last point, add $\emptyset$ transitions between any two non-accepting/starting states that were not previously connected (e.g., (B, D))

Generalized Nondeterministic Finite Automata

Definition

A generalized nondeterministic finite automaton (GNFA) is a 5-tuple $(Q, \Sigma, \delta, q_{start}, q_{accept})$ where

1. Q is a finite set of states
2. Σ is the input alphabet
3. $\delta : (Q \setminus \{q_{accept}\}) \times (Q \setminus \{q_{start}\}) \to \mathcal{R}$ is the transition function, where $\mathcal{R}$ is the set of all RE's over Σ,
4. q_{start} is the start state, and
5. q_{accept} is the accept state.

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Proof idea: take DFA and transform into a GNFA that accepts the same language. Iteratively remove (non-starting and non-accepting) states so that the same language is accepted, until only the starting and accepting state remain. Then the RE along the transition between the two states describes the regular language.

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Proof:

- Given a DFA M, we construct an equivalent GNFA G by adding a new start state q_{start} with an ε transition to the old start state q_0, as well as a new accepting state q_{acept}, with ε transitions from all old accept states.
- add $\emptyset$ transitions for all state pairs that do not have a transition in M.

- Recall the "convenient" properties of GNFA:
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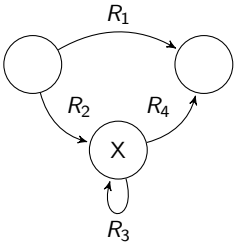

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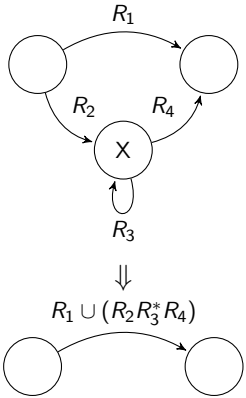

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$$\delta'(q_i, q_j) = R_1 \cup R_2 R_3^* R_4$$

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4. correctness still remains to be shown! See book for details! (Claim 1.65)

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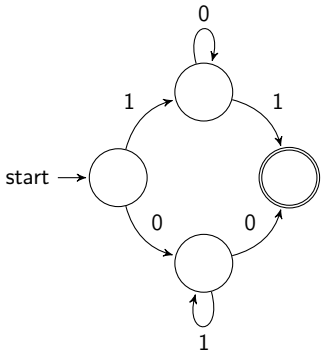

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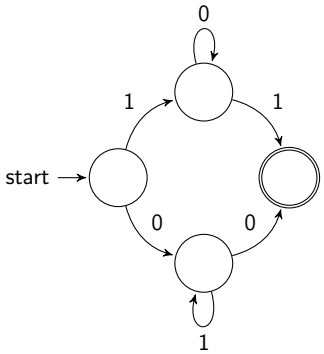

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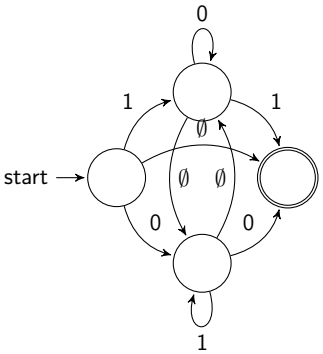

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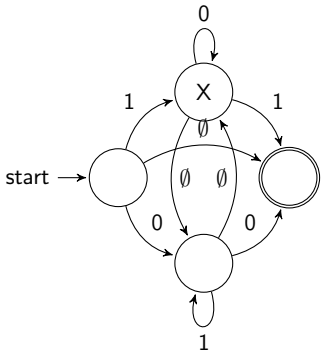

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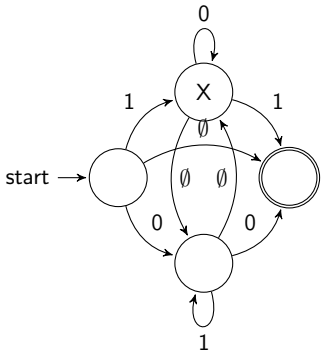

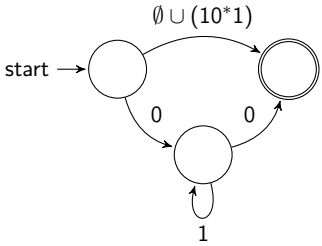

Proposition

Every regular language can be described using a RE.

Example:
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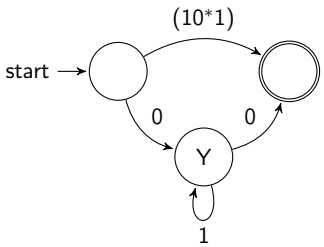

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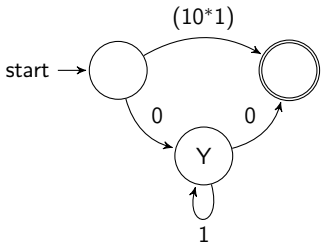

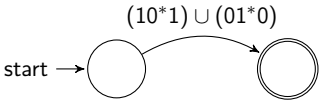

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But when is a language *nonregular*? How can we check?
$\Rightarrow$ Pumping Lemma!

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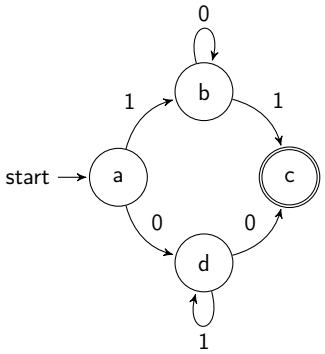
- Pumping lemma gives a *pumping length*: if a string is longer than the pumping length, it can be *pumped*, i.e., there is a substring that can be repeated arbitrarily often such that the string remains in the language
- If a DFA has p states, and a string has length $\geq p$, then the accepting path in the DFA must visit at least $p + 1$ states. In other words, at least one state appears twice. $\Rightarrow$ loop!
- This loop can be repeated while staying in the language.

Pumping Lemma - Example



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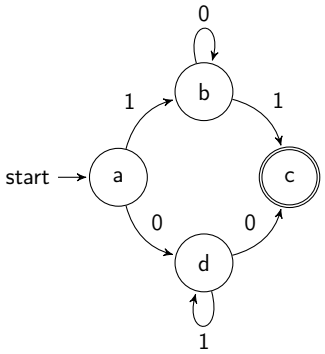

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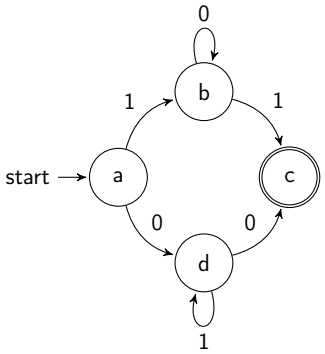

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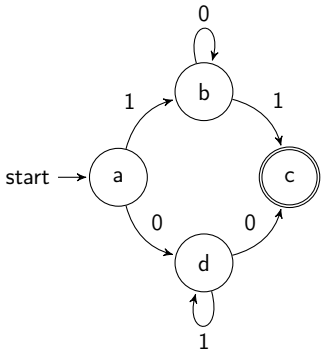

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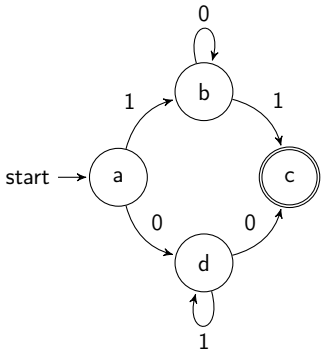

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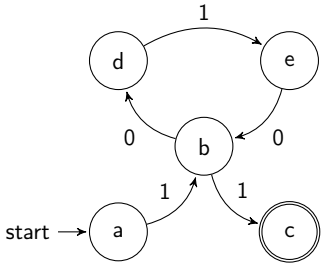

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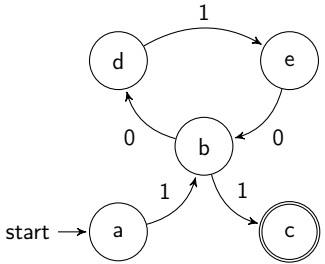

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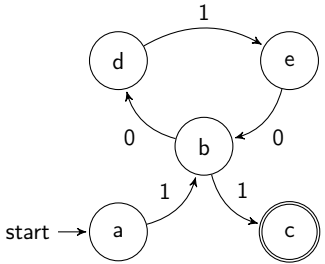

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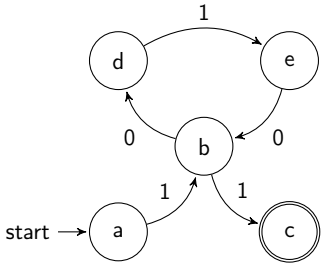

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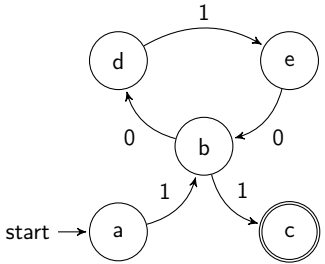

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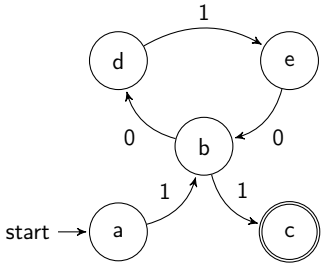

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- $\Rightarrow$ 10100101 is also a word!

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Lemma (Pumping Lemma)

If A is a regular language, then there is a number p, called the pumping length, where if w is a word in A of length $\geq p$ then w can be divided into three parts, $w = xyz$, such that

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- Is A regular?
- If it is, then the pumping lemma gives us a pumping length p.
- Let $s = 0^p1^p$.

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- Condition 3 tells us that y consists of only 0s.
- $\Rightarrow$ then xy^iz for $i \geq 2$ has more 0s than 1s. Contradiction! $\Rightarrow$ A is nonregular.

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- Even if a language is nonregular, it might contain strings for which the pumping lemma is true!

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- We have to be careful!

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- looks like it can be pumped, but are all conditions met?
- condition 3 $\Rightarrow$ y must contain only 0s, so it cannot be pumped$\Rightarrow$ B nonregular!

Pumping Lemma - Applied

- $A = \{0^n 1^n \mid n \geq 0\}$.
- $B = \{\omega \mid \omega$ contains an equal number of 0s and 1s$\}$
- Another way of showing B is nonregular is to reduce it to the nonregularity of A:

Pumping Lemma - Applied

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- useful for determining if a language is *nonregular*