INF2080

Repetition

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Today

- wrap-up of last week Friday
- repetition of course so far

Wrap-up: Reducibility

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- The mathematician reduces the current problem ("boil water after a night camping") to a problem with a known solution. The mathematician (1) brings the pot back to the car, (2) goes back to camp. Thus the problem has been reduced to the problem solved the day before: how to get boiling water when the pot is in the car.

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We used this to show various decidability and undecidability results, e.g.,

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We formalized reducibility as follows:

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Recall: a function $f : \Sigma^* \rightarrow \Sigma^*$ is computable if there exists a Turing machine M that for every input w halts with just $f(w)$ on its tape.

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By this definition: $A \leq_m B \iff \overline{A} \leq_m \overline{B}$ (useful tool we will use soon).

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We can use this to show:

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The language $EQ_{TM} = \{\langle M_1, M_2\rangle \mid M_1, M_2$ are TMs with $L(M_1) = L(M_2)\}$ is neither Turing-recognizable nor co-Turing-recognizable.

Proof: Let's show that EQ_{TM} is not Turing recognizable. We show a mapping reduction from $\overline{A_{TM}}$, i.e., $\overline{A_{TM}} \leq_m EQ_{TM}$.

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- There exist languages that are neither recognizable nor co-recognizable, i.e., *no such computational model* can check membership or non-membership! (e.g., $\overline{EQ_{TM}}$)

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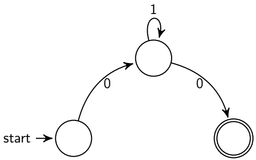

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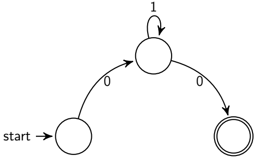

- contain a start state, possibly multiple accepting states. If after starting in the start state, parsing an input and following correct transitions the automaton ends in an accept state, the input is accepted
- The set of inputs accepted by a DFA is called a *regular language*

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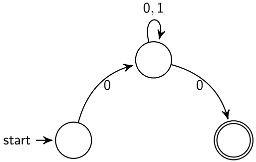

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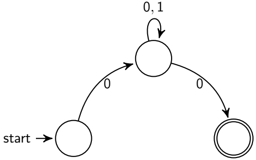

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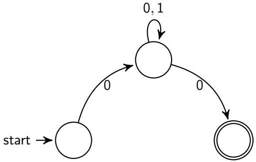

- NFA's accept the same languages as DFA's, i.e., a language is regular iff an NFA accepts it
- proof idea: Given an NFA N with state set Q, we define a DFA D with state set $P(Q)$, where the state $Q \in P(Q)$ in D represents that N could be in any state $q \in Q$.

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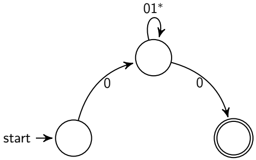
- The expressivity of regular languages is precisely that of DFA/NFA. To show this, we introduced GNFA (generalized finite automata), NFA's with RE's as labels instead of symbols.



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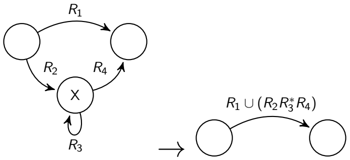

Pumping Lemma - Regular Languages

Lemma (Pumping Lemma)

If A is a regular language, then there is a number p, called the pumping length, where if w is a word in A of length $\geq p$ then w can be divided into three parts, $w = xyz$, such that

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- NOT useful for showing a language is regular:

$$\{ca^n b^n \mid n \geq 0\} \cup \{c^k w \mid k \neq 1, w \in \Sigma^* \text{ does not start with c}\}$$

- a language that is nonregular, yet every word can be pumped according to pumping lemma! $\rightarrow$ sometimes other tools are required (see, e.g., oblig 2)

Context-free languages

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$$A \to w$$

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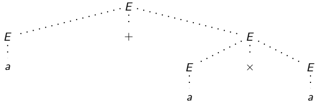
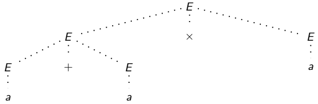


Intuitively corresponds to $a + (a \times a)$



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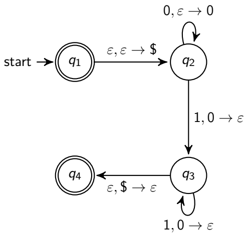

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- converting PDA to CFG: much more involved. General idea: For each pair of states p, q in PDA, add a variable A_{pq} to G that generates all strings that take the PDA from p to q with empty stacks (i.e., stack when arriving at p is equal to the stack when arriving at q). Add certain rules according to transition function δ.

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- So, CFG=PDA
- noteworthy: deterministic PDA (DPDA) is *not* equal to PDA, though we haven't covered this in the lecture

Context-free languages

Every CFG can be rewritten into a grammar in Chomsky normal form:

Definition

A grammar is in *Chomsky Normal Form* if every rule is of the form:

$$A \to BC$$
$$A \to a$$

where a is any terminal, A is any variable, B, C are any variables that are not the start variable. In addition the rule $S \to \varepsilon$ is permitted.

Pumping Lemma - CFL

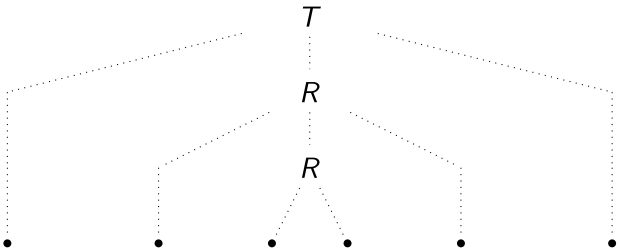
Lemma (Pumping Lemma for CFLs)

For every context-free language A there exists a number p (called the pumping length) where, if s is a word in A of length $\geq p$, then s can be divided into five parts, $s = uvxyz$, satisfying the following conditions:

1. $uv^ixy^iz \in A$ for all $i \geq 0$,
2. $|vy| > 0$,
3. $|vxy| \leq p$.

- similar to RL, we exploit the limited memory of CFL's
- If a word is "long enough", the smallest parse tree will contain two occurences of the same variable

Pumping Lemma - CFLs



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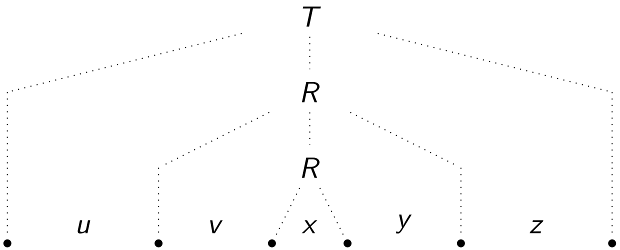

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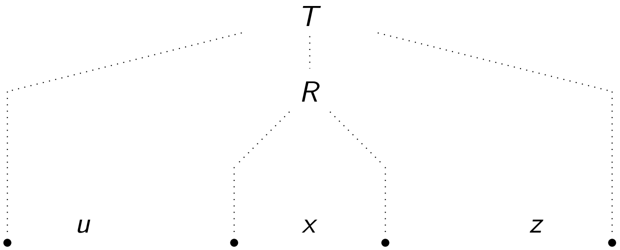

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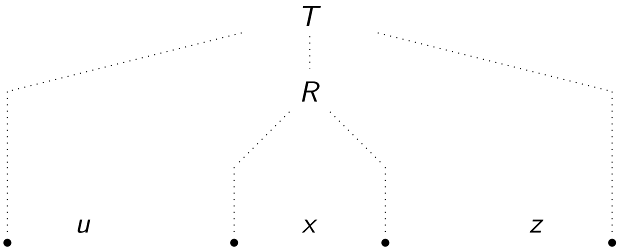

$$\rightarrow uv^0xy^0z = uxz$$

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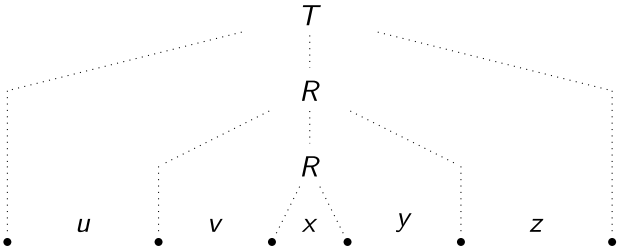

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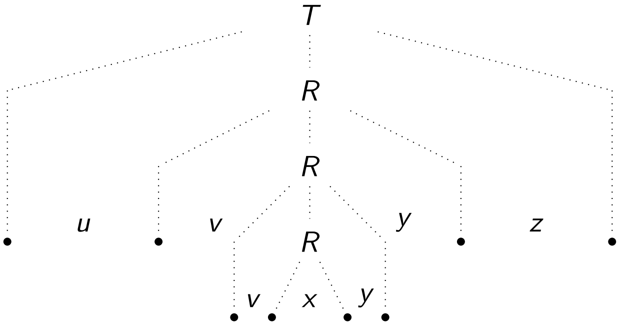

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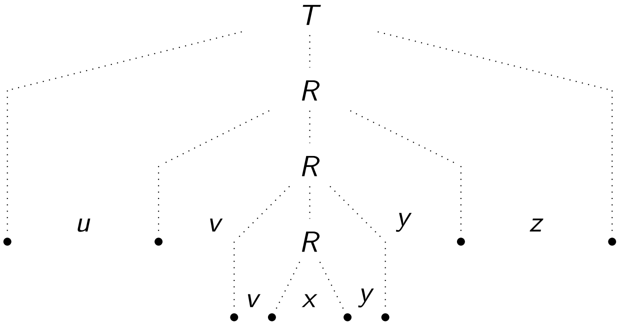

$\rightarrow uv^2xy^2z$, and so on

- all valid parse trees in G

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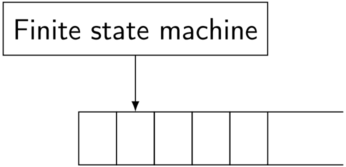
- Once again, useful tool for determining if a language is *not* context-free
- However, just like in the regular case, there exist languages that are not context-free that can be pumped. (analogous to the regular case)
- Thus, we have so far seen $\{\text{RL}\} \subsetneq \{\text{CFL}\}$, and that there exist non-context-free languages

Turing Machines

Defined Turing machines:



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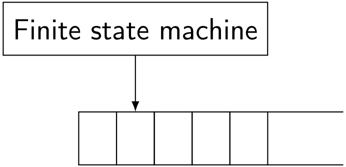

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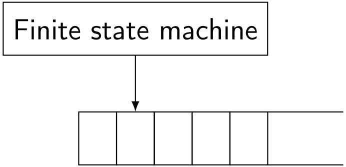

- a finite state machine with access to an infinite tape
- modelled by having a read/write head that can move left or right over the tape

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- each of the computational models we had seen so far were special cases of Turing machines
- different description levels of Turing machiens: high-level (“algorithmic” description, no fine-grained detail on tape operations), low-level (description of how the head operates on tape), implementation level (formal definition of the Turing machine)
- It is important to remember how high-level things can be implemented by tape manipulation, however formal definitions of Turing machines can be cumbersome

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DFA/PDA:

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- unrestricted access to infinite memory

Turing Machines

A language accepted by a Turing machine is called Turing-recognizable. If the machine halts on every input, then the language it recognizes is called decidable.

Turing Machines

Have looked at Turing machine variants, seen that they are equivalent:

- the LRS Turing machine (the head can move left, right, or stay put)
- the multitape Turing machine (multiple tapes, multiple heads)
- the nondeterministic Turing machine
- the enumerator
- NFA with two stacks
- ...

All computational models with unlimited access to infinite memory that can perform finite work in one step are equivalent to a Turing machine!

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- Church Turing thesis: each intuitive definition of algorithms can be described by decidable Turing machines

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Considered acceptance, emptiness, and equivalence problems for computational models, e.g.:

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We showed various decidability/undecidability results for languages:

	$x \in L$	$L = \emptyset$	$L = \Sigma^*$	$L = K$
regular	✓	✓	✓	✓
CFL	✓	✓	X	X
LBA	✓	X	X	X
decidable	✓	X	X	X
Turing-rec.	X	X	X	X

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→ decidability relates to more things than just Turing machines!

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- Type-2: context-free languages.
- Type-3: regular languages.

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- Type-3: regular languages.

We haven't gone through Type-1 (extra lecture at the end of the semester, if desired), however we have seen the computational model that accepts them: linear bounded automata (LBA) and seen that these are decidable.

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Complexity!

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- related to reducibility, computable functions
- *highly* relevant for anything within computer science, be it crypto/security, programming, theoretical work (AI, databases)