INF2080

Turing Machines

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Last week, we saw the pumping lemma for context-free languages
Saw some examples of languages that PDA's do not cover!

Turing Machiens

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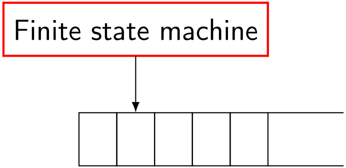

A Turing machine is a finite state machine that has access to an infinite tape

FA vs. TM

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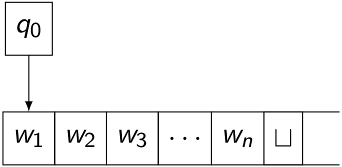

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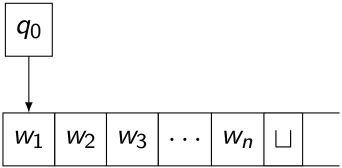

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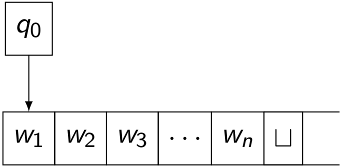

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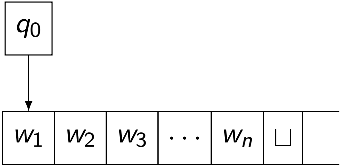

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- $\sqcup$ is the *blank symbol*: represents that the cell on tape does not contain any value

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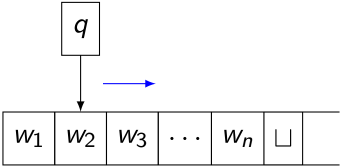

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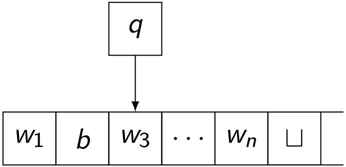

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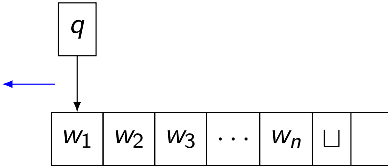

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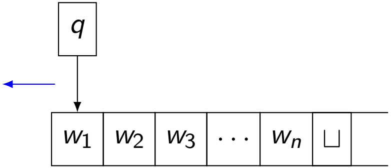

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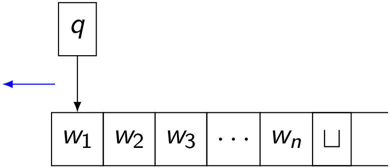

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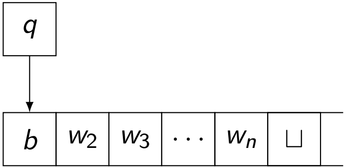

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Example

$L = \{w\#w \mid w \in \{0,1\}^*\}$:
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- If no non-crossed off symbols after $\#$ remain or next symbol is not the same, reject. Else, cross off if equal to the last crossed off symbol

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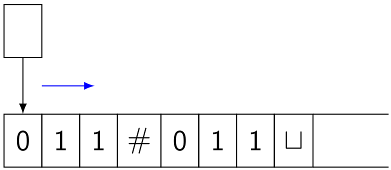

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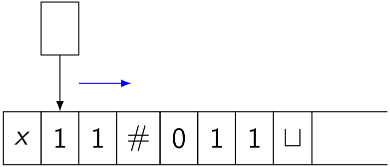

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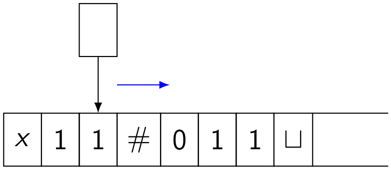

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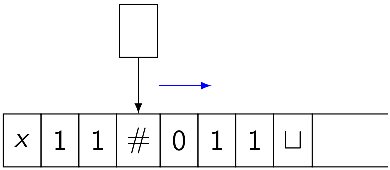

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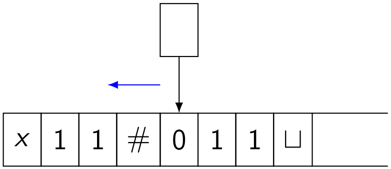

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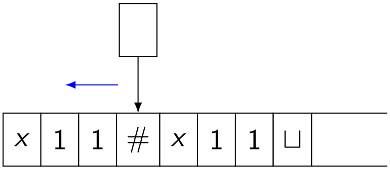

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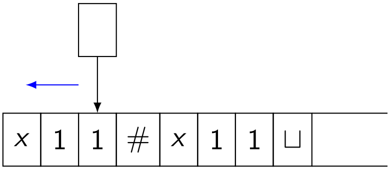

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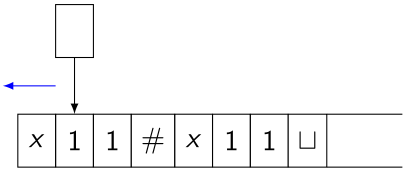

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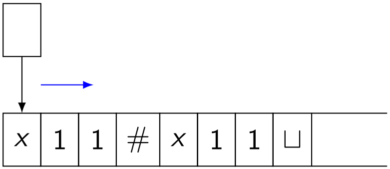

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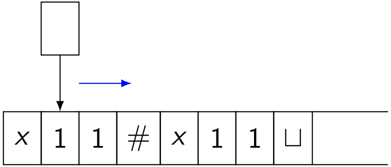

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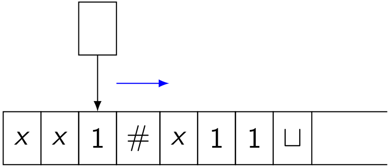

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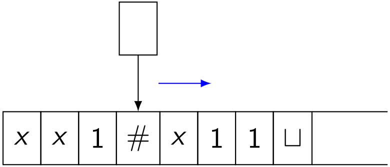

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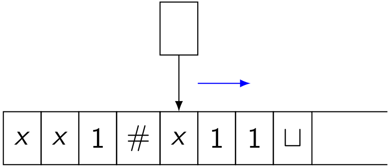

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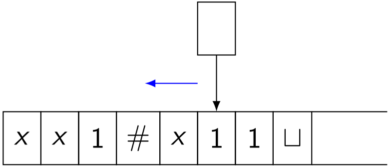

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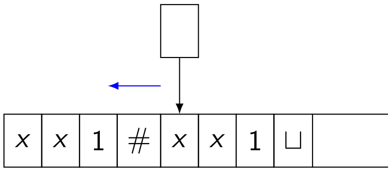

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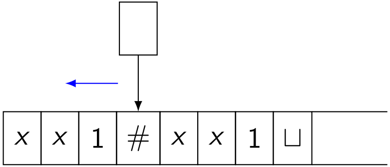

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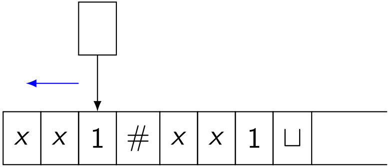

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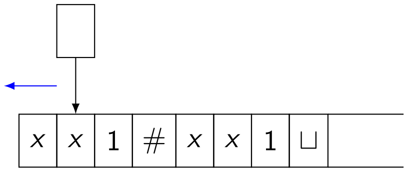

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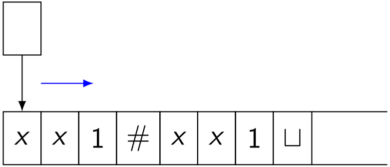

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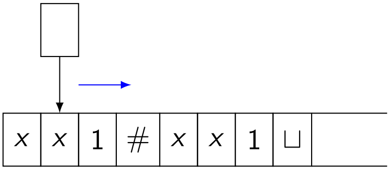

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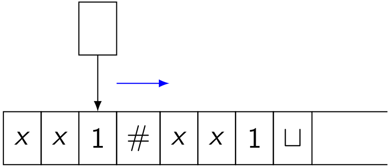

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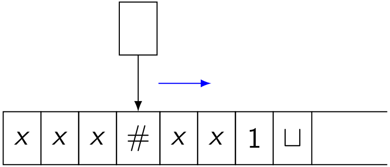

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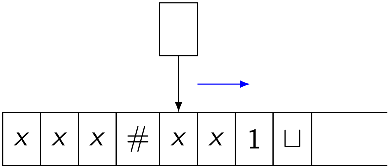

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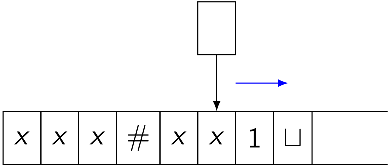

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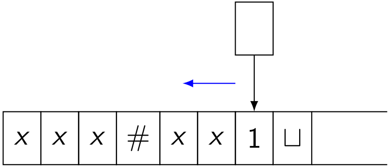

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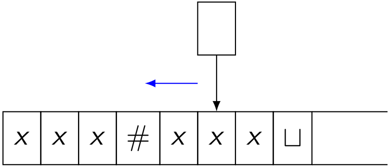

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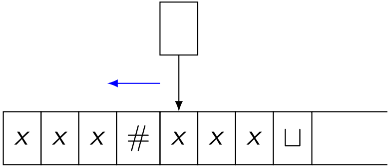

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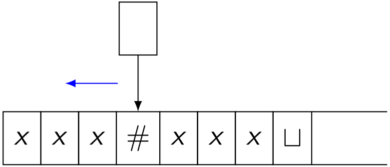

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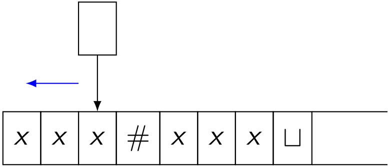

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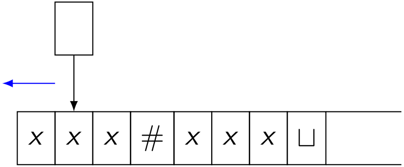

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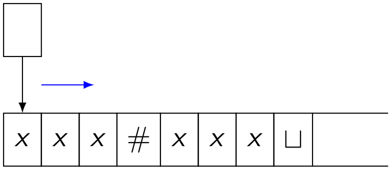

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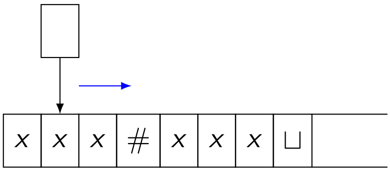

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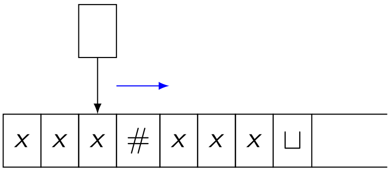

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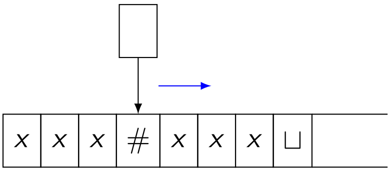

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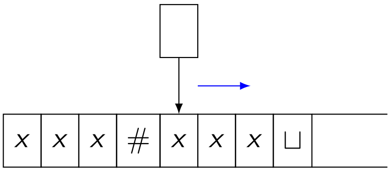

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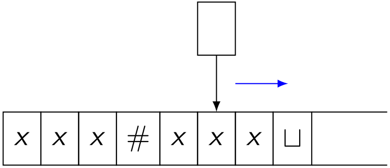

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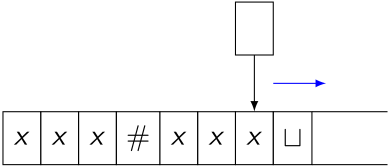

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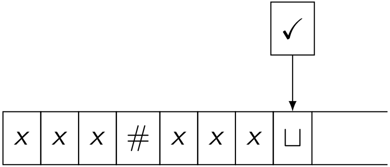

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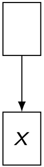

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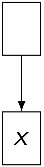

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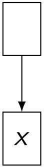

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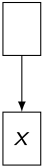

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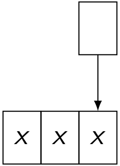

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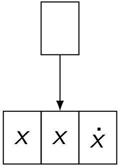

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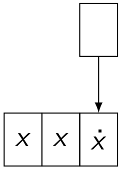

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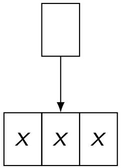

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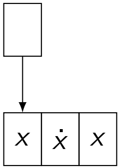

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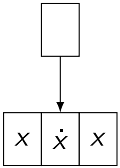

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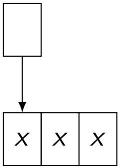

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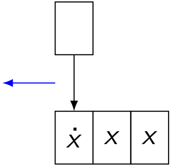

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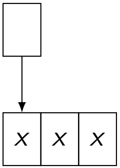

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- High-level: "Compare words before and after # and accept if equal, reject if not equal"
- Implementation level: giving the formal definition of a Turing machine, with all states and transitions (we'll go through some examples tomorrow)

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5. $q_0 \in Q$ is the start state,
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Configurations

During each computation step, a TM is in a specific *configuration*:

- current tape contents uv;
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Notation: uqv encodes that the machine is in state q, pointing at the first symbol in v.
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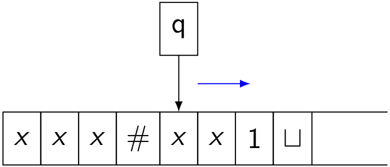

Notation: $xxx\#qxx1$

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A configuration C_1 *yields* C_2 if TM can legally go from C_1 to C_2 in a single step:

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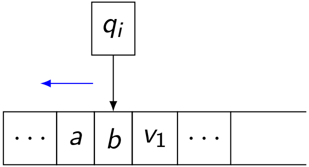

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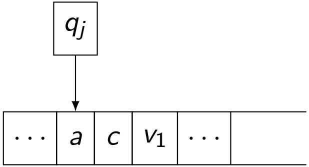

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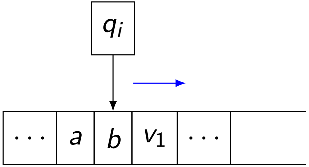

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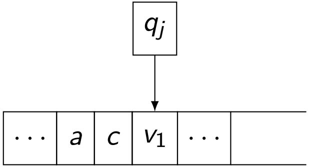

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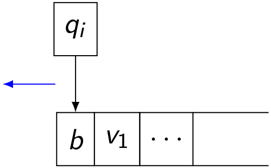

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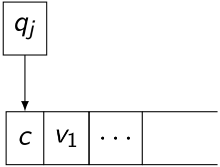

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  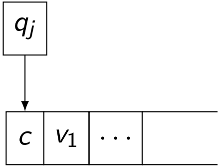

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  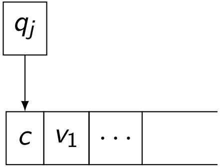

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  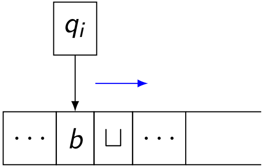

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  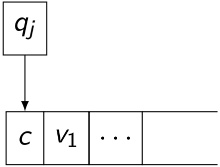

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  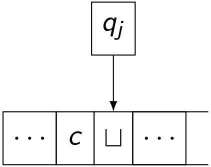

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A language is *Turing-decidable* [decidable, recursive] if there is a Turing machine that decides it.

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Previous example of language $\{w\#w \mid w \in \{0,1\}^*\}$ was recognized by machine:
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It was, in fact, decided by M!

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- all words except 0 have length divisible by 0.
- idea: if length=1, accept (if input symbol is correct), otherwise divide length by 2. Repeat

How to divide length by 2?

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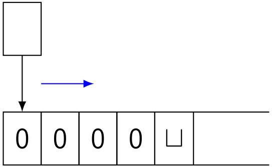

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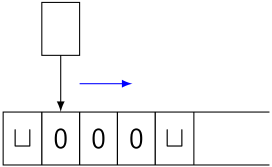

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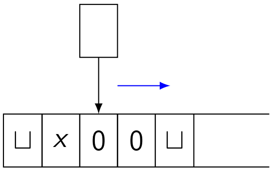

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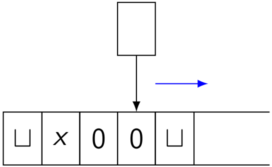

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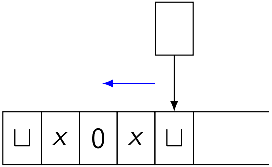

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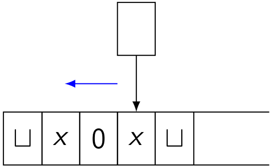

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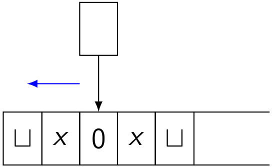

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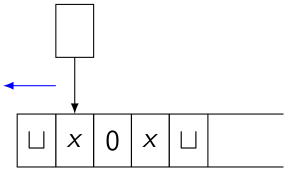

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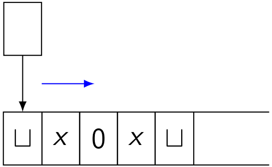

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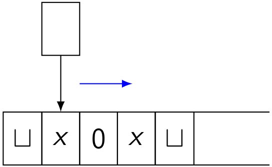

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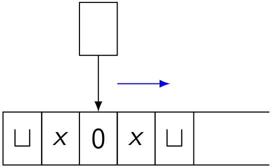

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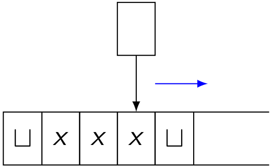

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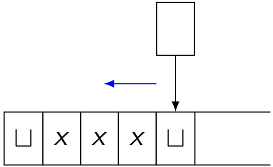

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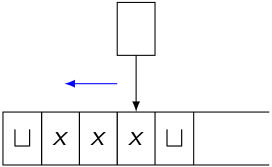

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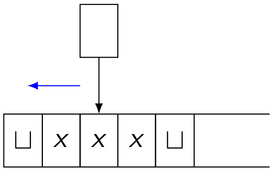

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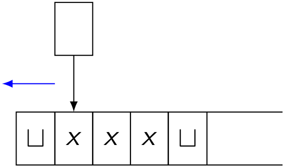

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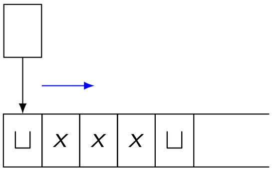

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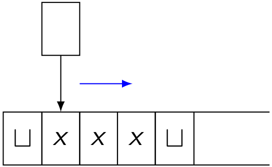

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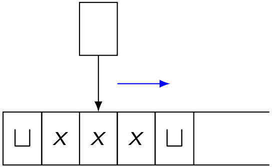

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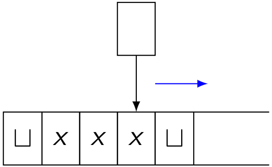

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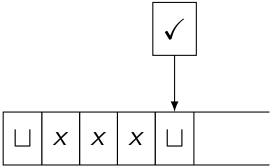

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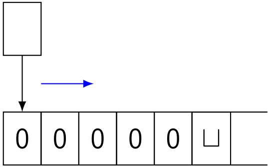

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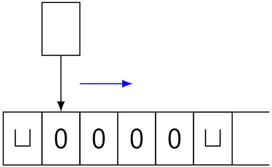

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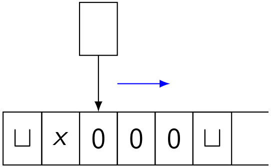

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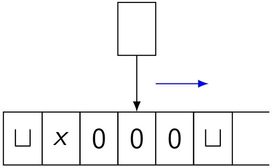

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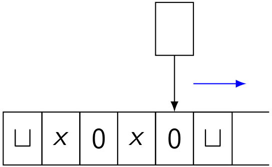

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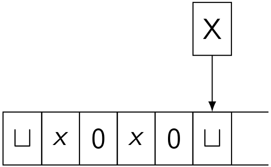

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A *LRS Turing machine* is a Turing machine with a transition function is defined as

$$\delta : Q \setminus \{q_{accept}, q_{reject}\} \times \Gamma \to Q \times \Gamma \times \{L, R, S\}$$

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So, equally expressive as a normal TM!

Alternative Turing machines

- Can we make it *even more expressive* by, e.g., adding more tapes?

Definition

A *multitape Turing machine* is a turing machine with $k \geq 2$ tapes, i.e., its transition function is defined as

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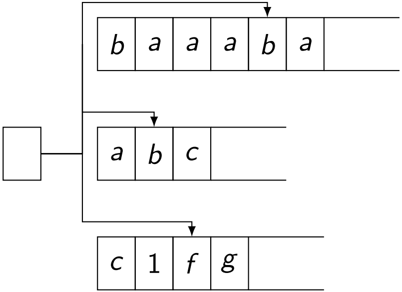
Multitape Turing machines



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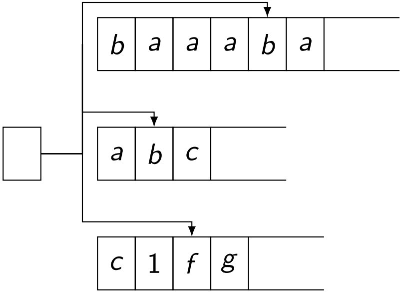

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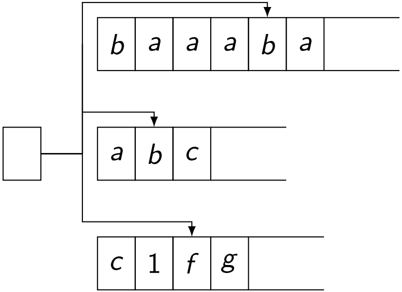

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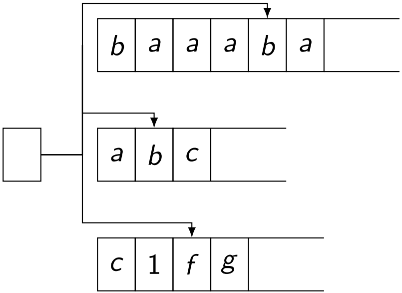

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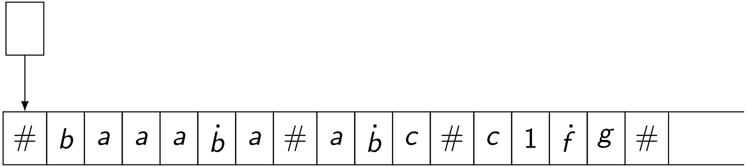
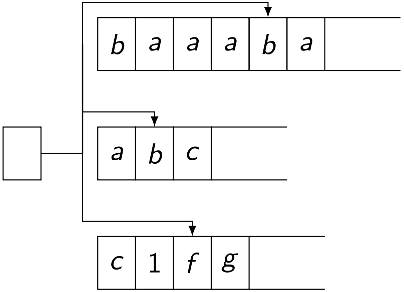
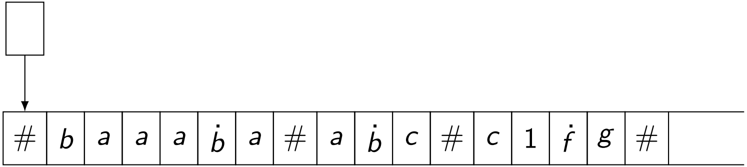


Multitape Turing machines



- The machine has access to multiple tapes
- Can this be modelled on a single tape? Yes!
- Idea: encode content of each tape on one tape, using a symbol (#) to separate the different tapes
- in each tape block (space between two #), one symbol is marked (represents where the head is on that tape)



Multitape Turing Machines

More formally:

S= on input w:

1. Put S into the following format to model k tapes:

$$\#\dot{w}_1 \ldots w_n\#\dot{\sqcup}\#\dot{\sqcup}\# \ldots \#$$

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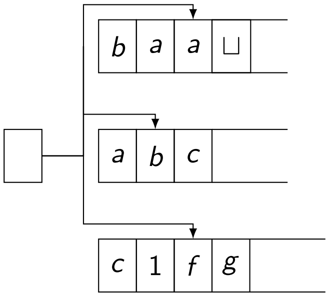

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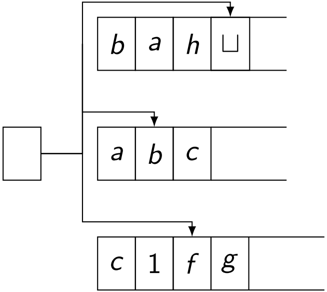

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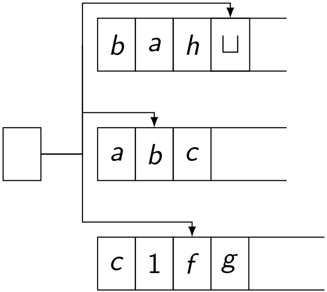

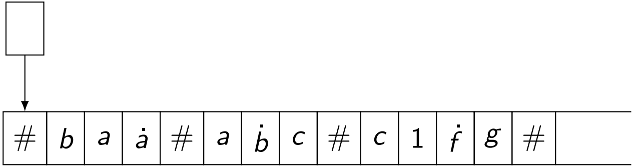

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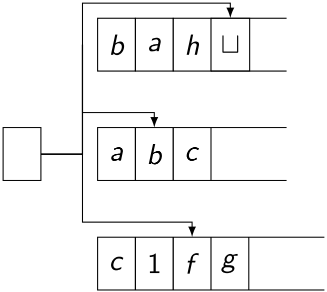
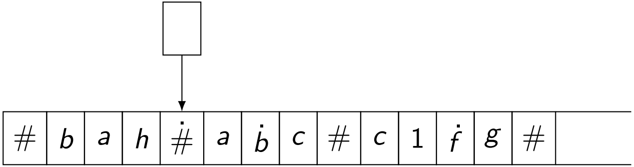
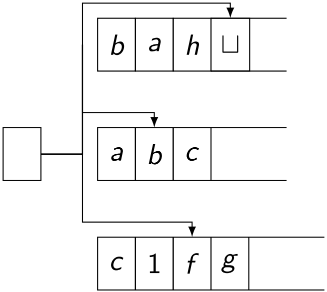
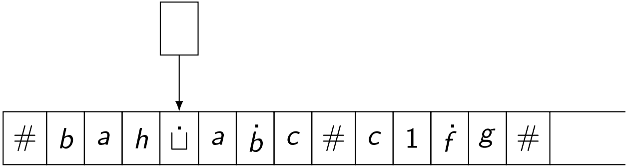
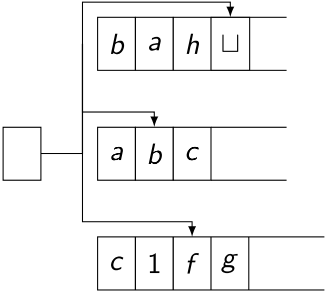





b a h ⊔
a b c
c 1 f g


b a h ⊔̇ a ḃ c # c 1 ḟ g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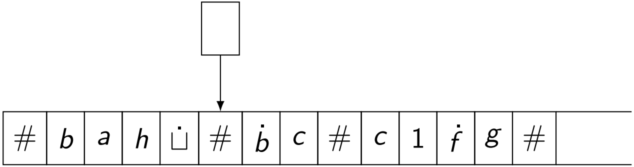
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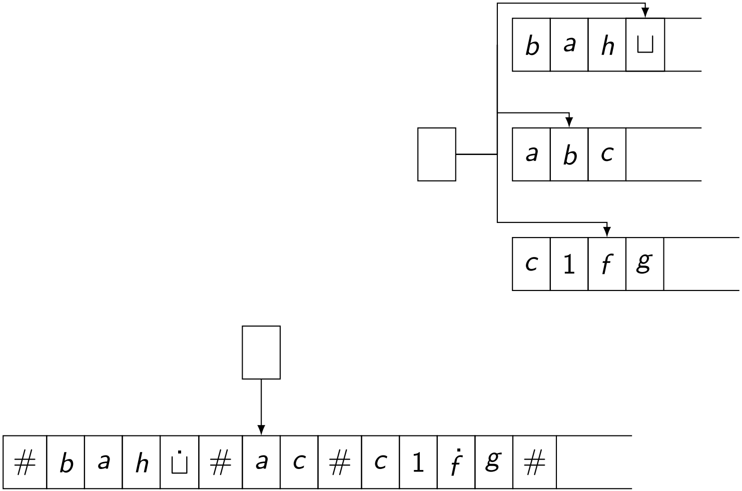

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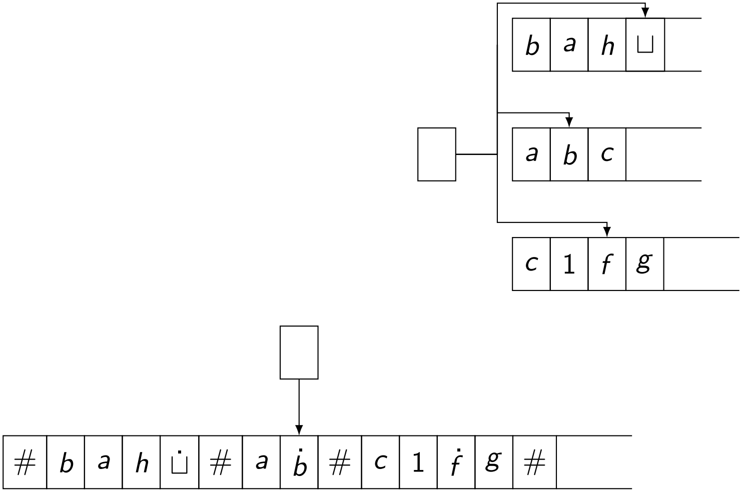

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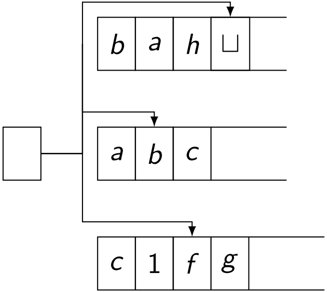

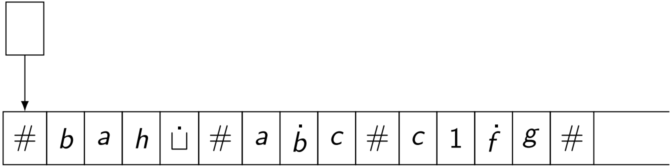

Multitape Turing machiens

Theorem

A language is Turing recognizable [decidable] if and only if some multitape Turing machine recognizes [decides] it.

Turing Machine alternatives

OK...what about *nondeterministic* Turing machines? Do they let us do anything a deterministic TM cannot?

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Definition

A nondeterministic Turing machine has a transition function of the form

$$\delta : Q \setminus \{q_{accept}, q_{reject}\} \times \Gamma \to P(Q \times \Gamma \times \{L, R\})$$

Nondeterministic TM

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- simulation tape: "where the magic happens"...tape simulating current computational branch in N;
- address tape: tells us which choices to make in the computational tree of N

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- To ensure that we do not loop before reaching a possible accept configuration, we parse the computational tree with *breadth first search*:
- we order the addresses (strings over $\{1, \dots b\}$) as follows (lexicographic ordering): $1, 2, \dots, b, 11, 12, 13, \dots, 1b, 21, 22, \dots, bb, 111, \dots$

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Given NTM N, we construct a 3-tape TM M as follows:
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4. Replace address tape's contents with next address according to our ordering. Go to 2.

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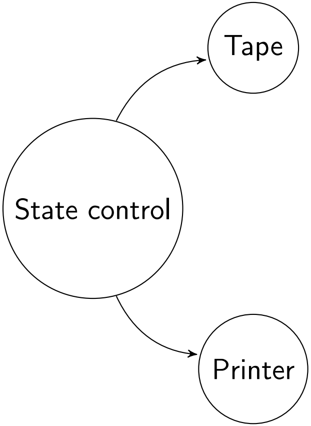
A little extra work needed for decidability result! $\rightarrow$ Exercise 3.3 in book!

Enumerators



- An enumerator is a slightly altered Turing machine:
- It has a working tape and an attached printer
- initializes with an empty working tape, taking no input
- throughout its computation, it can output strings using the printer
- If the enumerator does not halt, it can potentially output infinitely many strings

Enumerators

An enumerator *enumerates* a word iff at some point it prints it.

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$A =$ On input w

1. Run E. Every time E prints a string, compare to w.
2. If w appears in the output of E, *accept*.

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- Le Σ be the alphabet of $L(A)$. Then we can order all strings in Σ^* (first list all strings of length 1, then of length 2, etc). Label them $s_1, s_2, s_3, \ldots$. Then we can construct an enumerator:

$E =$ Ingore input

1. Repeat for $i = 1, 2, 3, \ldots$.
2. Run A on $s_1, \ldots, s_i$ for i steps.
3. If any computation accepts, print corresponding s_j.