# Cut elimination

Start – derivation using cut rule
Syntactic steps where cuts are eliminated
Termination – the process terminates
Estimate – height increase of derivation

# Cut rule

- From G,H and -H,F to G,F
- Variants
  - From G,H and -H,G to G
  - Using lemmas
  - Using auxiliary constructions
  - Using indirect proofs
- The process – think about eliminating lemmas

# Syntactic process

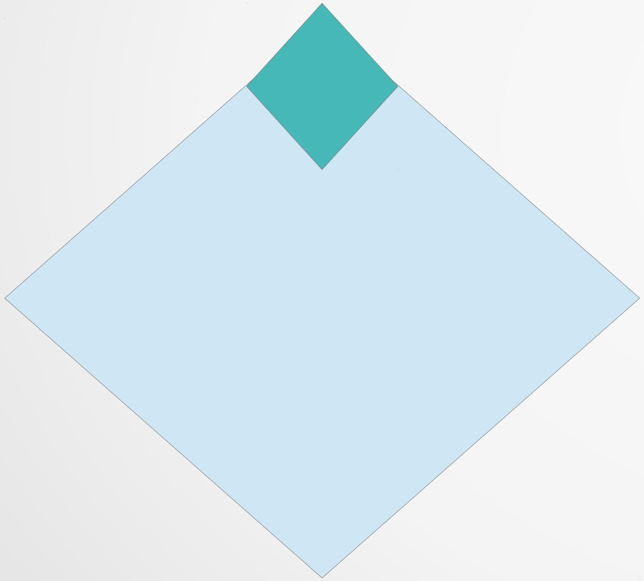
- Syntactic in X and Y
  - Transformation involving only X and Y and parts of X and Y (makes good sense if X and Y are say derivations, formulas, trees, ...)

- Measuring the process involving derivation D
  - Height – length of largest branch in D
  - Degree – length of largest cut formula

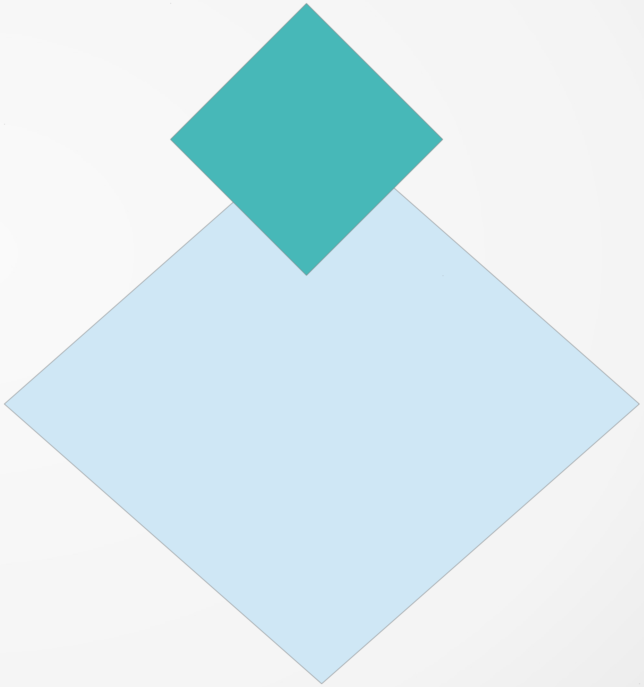
- Goal
  - From derivation F (h,d) get F(h*,0)
  - Estimate h*

# Process – picture – one step

- From D



- To D'



# Transformations

- Given derivation D
- Pick a node F in D with maximal degree and as high up in D as possible. Such a node is called critical .
- Check that the proposed transformations are syntactic
- Check that degree is not increased
- Check that the number of nodes in D with maximal degree is decreased

# Simple transformations

- Change names for new variables in forall
- Thinning : From G to G,H
- Conjunction : From G and H to G (or to H )
- Disjunction : From G or H to G,H
- All-quantifier : From all x.Fx to Fs

- Idea – change the formulas in the thread above the formula until you meet where the formula is introduced
- Neither height nor degree is increased
- No simple transformation for exists-quantifier

# Cut elimination - connectives

- Assume we have a connective cut
  - From F and G, H and -F or -G,H to H
  - Change this into two cuts – with F and with G
  - First F,H and -F,-G,H to H,-G,H ( = -G,H)
  - Then G,H and -G,H to H
  - Obtains smaller cut degree with only one extra step

# Cut elimination - quantifiers

- Assume we have a quantifier cut
  - From  all x.Fx , G  and  ex x.-Fx , G  to  G
  - Must trace  ex.-Fx  up to all the places where it is introduced. There we can use cuts with appropriate instantiations of  all x.Fx
  - We only know that the places are above the original quantifier cut.
  - In worst case the height above the original quantifier cut is doubled. We cannot say more than that.
  - We get rid of a large cut using a doubling of height.

# Process - termination

- We start with a derivation  D  of sequent  G
- We measure  D  with the pair  (height,degree)
- Pick a critical cut and eliminate it
- This decrease the number of nodes with maximal degree
- Repeat until we have eliminated all cuts of maximal degree
- Then repeat the process with a smaller maximal degree
- After passes for all degrees we get a derivation with no cuts
- The process terminates

# Process – estimate of height

- Assume we have a derivation D with pair (h,d)
- We have d passes of transformations
- In each pass we use syntactic transformations going from the top of the tree down to the root
- In worst case the transformation doubles the height above
- One pass - from (h,d) to $(2^h,d-1)$
- All passes - height a tower of 2's of height d and an h at the top. The parenthesis in the tower goes the awful way.
- From (16,3) to $2^{64k}$ - much larger than the number of atoms in the universe (about $2^{256}$)

# Example – notations for numbers

- Unary predicate N . We write 0:N 17:N x:N
- Constant 0:N
- Unary function $s:N \to N$ - successor
- Connectives, quantifiers, equality
- Other functions defined by primitive recursion
- Here the following is of special interest
  - $exy = 2^x+y$ - intended meaning
  - $e0y = sy$
  - $esxy = exexy$

# Process - problems

- Pro cuts
  - Cuts are needed to make derivations short
  - All texts use auxiliary notions and theorems . These can be faithfully represented as cuts
- Contra cuts
  - Must guess appropriate cut formulas
  - No automation of reasoning with cuts
  - Interactive reasoning – user interaction by user providing cuts
- Old discussion of direct versus indirect arguments

# Example - theory

- Basic theory
  - 0:N   and  all x:N . sx:N
- Equations for primitive recursive functions

- Problem: Can we derive  BASIC + EQUATIONS $\rightarrow$ t:N
- Answer:
  - Without cut  -  the height is at least the magnitude of t
  - With cut – much shorter derivations

# Example – auxiliaries - 1

- New notion for exy
  - $N_0 = N$
  - $x:N_{i+1} = \text{all } y:N_i \,.\, exy:N_i$
- New lemma
  - $0:N_i$ - for all i
- The axioms basic give the lemma for i=0 and i=1

# Example – auxiliaries - 2

- Let us prove the lemma for i+2
  - To prove all $y:N_{i+1}$ . $e0y:N_{i+1}$
  - Assume $y:N_{i+1}$. i.e. all $z:N_i$ . $eyz:N_i$
  - But then also all $z:N_i$ . $eyeyz:N_i$ i.e. $sy:N_{i+1}$
  - Conclude $0:N_{i+2}$
- A very short proof of the lemma

# Example - conclusion

- Number of atoms - eeeeee0000000
- No cut free proof of eeeeee0000000:N – not enough space
- Using lemma repeatedly we have a short proof
  - From $0{:}N_5$ and $0{:}N_6$ we get $e00{:}N_5$
  - From $0{:}N_4$ we get $ee000{:}N_4$
  - From $0{:}N_3$ we get $eee0000{:}N_3$
  - From $0{:}N_2$ we get $eeee00000{:}N_2$
  - From $0{:}N_1$ we get $eeeee000000{:}N_1$
  - From $0{:}N_0$ we get eeeeee0000000:N
- The rough estimates of height increase cannot be improved (using equality is not important)