INF3170 – Logikk

Forelesning 6: Description Logic 1

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Dagens plan

Introduction

Suppose you want to express set membership and subsumption, i.e., e.g,

- $a \in C \cap D$ and $C \subseteq D$.

You can do this is first-order logic (using predicates C and D)

- $C(a) \wedge D(a)$ and $\forall x.C(x) \supset D(x)$.

Now suppose you are afraid of variables. The only variable above is universally quantified, so you do not really need it. Instead, use a language that can express subsumption of **concepts** (which are intepreted as sets) in a more familiar way.

If C and D are concepts, then so is $C \sqcap D$. The first-order logic formulae above can then be expressed as follows:

- $C \sqcap D(a)$ and $C \sqsubseteq D$.

The concept language

$\mathcal{ALC}$ is the concept language constructed from

- atomic concepts (unary predicates),
- atomic roles (binary predicates),

and the following concept constructors:

- $\top$ universal concept
- $\bot$ bottom concept
- $\forall R.C$ value restriction
- $\exists R.C$ existential quantification
- $C \sqcup D$ union
- $C \sqcap D$ intersection
- $\neg C$ negation

where C and D are concepts, and R an atomic role.

Models

Recall that a **interpretation** (or **model**)

$$\mathfrak{A} = \langle \Delta, \cdot^{\mathfrak{A}} \rangle$$

over unary and binary predicates consists of

- a **domain** Δ that is a non-empty set, and
- an **interpretation function** $\mathfrak{A}$ that maps
  - individuals to the elements: $a^{\mathfrak{A}} \in \Delta$
  - atomic concepts to sets: $A^{\mathfrak{A}} \subseteq \Delta$
  - atomic roles to binary relations: $R^{\mathfrak{A}} \subseteq \Delta \times \Delta$

Semantics

The interpretation function is extended to complex concepts.

- $\top^{\mathfrak{A}} = \Delta$
- $\bot^{\mathfrak{A}} = \emptyset$
- $(\neg C)^{\mathfrak{A}} = \Delta \setminus C^{\mathfrak{A}}$
- $(C \sqcup D)^{\mathfrak{A}} = C^{\mathfrak{A}} \cup D^{\mathfrak{A}}$
- $(C \sqcap D)^{\mathfrak{A}} = C^{\mathfrak{A}} \cap D^{\mathfrak{A}}$
- $(\forall R.C)^{\mathfrak{A}} = \{x \in \Delta \mid R^{\mathfrak{A}}(x) \subseteq C^{\mathfrak{A}}\}$
- $(\exists R.C)^{\mathfrak{A}} = \{x \in \Delta \mid C^{\mathfrak{A}} \cap R^{\mathfrak{A}}(x) \neq \emptyset\}$

For a binary relation R, $R(x)$ denotes $\{y \mid xRy\}$.

Example

This will be our main example.

Example

We have

- *two atomic concepts* Man *and* Woman*, and*
- *an atomic role* hasChild*.*

Let $\mathfrak{A}$ *be the intepretation with*

- *the domain* $\Delta = \{\text{Maria}, \text{Jesus}\}$

such that our concepts and role have the following extensions:

- $\text{Woman}^{\mathfrak{A}} = \{\text{Maria}\}$,
- $\text{Man}^{\mathfrak{A}} = \{\text{Jesus}\}$,
- $\text{hasChild}^{\mathfrak{A}} = \{\langle \text{Maria}, \text{Jesus} \rangle\}$,

Example

Given this intperpretation,

- The extension of $\underline{\text{Man} \sqcap \text{Woman}}$ is empty: Nobody is both a man and a woman.

$$\begin{aligned}(\text{Man} \sqcap \text{Woman})^{\mathfrak{A}} &= \text{Man}^{\mathfrak{A}} \cap \text{Woman}^{\mathfrak{A}} \\ &= \{\text{Maria}\} \cap \{\text{Jesus}\} \\ &= \emptyset\end{aligned}$$

- The extension of $\underline{\text{Man} \sqcup \text{Woman}}$ equals the domain: Everyone is either a man or a woman.

$$\begin{aligned}(\text{Man} \sqcup \text{Woman})^{\mathfrak{A}} &= \text{Man}^{\mathfrak{A}} \cup \text{Woman}^{\mathfrak{A}} \\ &= \{\text{Maria}\} \cup \{\text{Jesus}\} \\ &= \Delta\end{aligned}$$

Example

Given this intperpretation,

- The extension of $\underline{\exists\text{hasChild.Man}}$ is $\{\text{Maria}\}$: She has a child, and that child is a man.

$$\begin{aligned}\exists\text{hasChild.Man}^{\mathfrak{A}} &= \{x \in \Delta \mid \text{Man}^{\mathfrak{A}} \cap \text{hasChild}^{\mathfrak{A}}(x) \neq \emptyset\} \\ &= \{\text{Maria}\},\end{aligned}$$

as $\text{hasChild}^{\mathfrak{A}}(\text{Maria}) = \{\text{Jesus}\}$.

- The extension of $\underline{\forall\text{hasChild.Man}}$ is $\{\text{Maria}, \text{Jesus}\}$: Every child Maria has is a man; for Jesus this holds vacuously, as he has no children.

$$\begin{aligned}\forall\text{hasChild.Man}^{\mathfrak{A}} &= \{x \in \Delta \mid \text{hasChild}^{\mathfrak{A}}(x) \subseteq \text{Man}^{\mathfrak{A}}\} \\ &= \{\text{Maria}, \text{Jesus}\},\end{aligned}$$

as $\text{hasChild}^{\mathfrak{A}}(\text{Jesus}) = \emptyset$.

Assertions

An **assertion** is of the form

- $C(a)$ for a concept C and individual a
- $R(a, b)$ for a role R and individuals a and b

In order to give semantics to an assertion, we must map each individual to the domain:

- $a^{\mathfrak{A}} \in \Delta$, such that
- $a^{\mathfrak{A}} \neq b^{\mathfrak{A}}$ if $a \neq b$ (**the unique name assumption** or **UNA**)

An interpretation $\mathfrak{A}$ **satisfies**

- $C(a)$ if $a^{\mathfrak{A}} \in C^{\mathfrak{A}}$
- $R(a, b)$ if $\langle a^{\mathfrak{A}}, b^{\mathfrak{A}} \rangle \in R^{\mathfrak{A}}$

Assertions

Let MARIA and JESUS be individuals, and map them as follows:

- $\mathsf{MARIA}^{\mathfrak{A}} = \mathrm{Maria}$
- $\mathsf{JESUS}^{\mathfrak{A}} = \mathrm{Jesus}$

The interpretation in our main example satisfies, e.g.,

- hasChild(MARIA, JESUS) as

$$\langle \mathsf{MARIA}^{\mathfrak{A}}, \mathsf{JESUS}^{\mathfrak{A}} \rangle \in \mathsf{hasChild}^{\mathfrak{A}};$$

- ∃hasChild.Man(MARIA), as

$$\mathsf{MARIA}^{\mathfrak{A}} \in \exists \mathsf{hasChild}.\mathsf{Man}^{\mathfrak{A}}.$$

Terminological axioms

A **terminological axiom** is of the form, for concepts C and D,

- $C \sqsubseteq D$ (inclusions)
- $C \equiv D$ (equalities)

An interpretation $\mathfrak{A}$ **satisfies**

- $C \sqsubseteq D$ if $C^{\mathfrak{A}} \subseteq D^{\mathfrak{A}}$
- $C \equiv D$ if $C^{\mathfrak{A}} = D^{\mathfrak{A}}$

The interpretation in our main example satisfies, e.g.,

- Woman ⊑ ∃hasChild.Man, and
- Man ≡ ¬Woman.

Knowledge bases

An $\mathcal{ALC}$ **knowledge base** is a pair consisting of:

- A **TBox**, a finite set of terminological axioms
- An **ABox**, a finite set of assertions

An interpretation $\mathfrak{A}$ **satisfies**

- an ABox if it satisfies every assertion in it
- a TBox if it satisfies every axiom in it
- a knowledge base if it satisfies both the ABox and the TBox

Example

An $\mathcal{ALC}$ knowledge base $\mathcal{K}_1 = \langle \mathcal{T}_1, \mathcal{A}_1 \rangle$:

Example

TBox $\mathcal{T}_1$

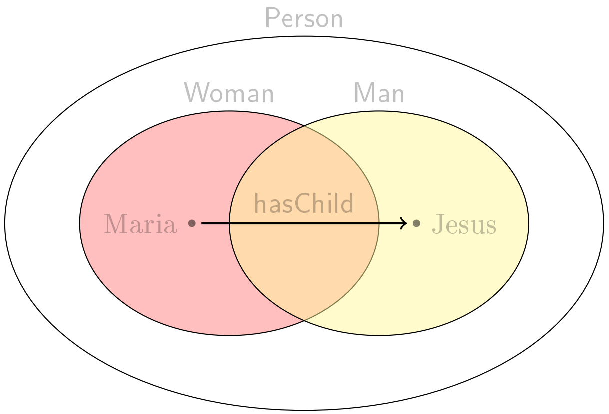
- Woman ⊑ Person
- Man ⊑ Person
- Mother ≡ Woman ⊓ ∃hasChild.Person

ABox $\mathcal{A}_1$

- Woman(MARIA)
- Man(JESUS)
- hasChild(MARIA, JESUS)

Example

Any model of the knowledge base $\langle \mathcal{T}_1, \mathcal{A}_1 \rangle$ must look more or less like this:



(The concept Mother should be a subset of Woman and include Maria)

Example

Our main example interpretation $\mathfrak{A}$ is a model of the ABox $\mathcal{A}_1$.

If we extend $\mathfrak{A}$ to interpret Person and Mother as follows,

- $\text{Person}^{\mathfrak{A}} = \Delta = \{\text{Maria}, \text{Jesus}\}$ and
- $\text{Mother}^{\mathfrak{A}} = \{\text{Maria}\}$,

it becomes a model of the TBox $\mathcal{T}_1$:

- $\text{Woman}^{\mathfrak{A}} \subseteq \text{Person}^{\mathfrak{A}}$
- $\text{Man}^{\mathfrak{A}} \subseteq \text{Person}^{\mathfrak{A}}$
- $\text{Mother}^{\mathfrak{A}} = \text{Woman}^{\mathfrak{A}} \cap \exists\text{hasChild.Person}^{\mathfrak{A}}$

Thus $\mathfrak{A}$ becomes a model of the knowledge base $\mathcal{K}_1 = \langle \mathcal{T}_1, \mathcal{A}_1 \rangle$.

Reasoning

We consider several reasoning problems, wrt. assertions:

- instance checking;

and concepts:

- satisfiability;
- subsumption;
- disjointness.

Instance checking

Definition (Instance checking)

$\mathcal{K} \models C(a)$, *meaning* $\mathfrak{A}$ *satisfies* $C(a)$ *for every model* $\mathfrak{A}$ *of* $\mathcal{K}$.

Example:

- $\mathcal{K}_1 \models \text{Mother}(\text{Maria})$
- $\mathcal{K}_1 \models \text{Person}(\text{Jesus})$

Concept satisfiability

Definition (Concept satisfiability)
A concept C is **satisfiable** *wrt. $\mathcal{T}$ if $C^{\mathfrak{A}} \neq \emptyset$ for some model $\mathfrak{A}$ of $\mathcal{T}$.*

Example: Man ⊓ ¬Person is

- satisfiable wrt. the empty TBox;
- unsatisfiable wrt. $\mathcal{T}_1$.

Example: Man ⊓ Woman is

- satisfiable wrt. $\mathcal{T}_1$.

Concept subsumption

Definition (Subsumption)
C is **subsumed** *by D wrt. $\mathcal{T}$ if $\mathfrak{A}$ satisfies $C \sqsubseteq D$ for every model $\mathfrak{A}$ of $\mathcal{T}$.*

We write $\mathcal{T} \models C \sqsubseteq D$ if C is subsumed by D wrt. $\mathcal{T}$.

Example:

- Mother is subsumed by Woman wrt. $\mathcal{T}_1$.
- Woman is subsumed by Woman ⊔ Man wrt. any TBox.

Concept disjointness

Definition (Disjointness)
C and D are **disjoint** *wrt. $\mathcal{T}$ if $C^{\mathfrak{A}} \cap D^{\mathfrak{A}} = \emptyset$ for every model $\mathfrak{A}$ of $\mathcal{T}$.*

Example: Wrt. $\mathcal{T}_1$,

- Man and Woman are **not** disjoint;
- Mother and ¬Woman are disjoint.

Reducing reasoning problems to each other

In $\mathcal{ALC}$, these problems can all be reduced to each other.

Subsumption

- C is unsatisfiable iff $\models C \sqsubseteq \bot$
- C and D are disjoint iff $\models C \sqcap D \sqsubseteq \bot$

Unsatisfiability

- $\models C \sqsubseteq D$ iff $C \sqcap \neg D$ is unsatisfiable
- C and D are disjoint iff $C \sqcap D$ is unsatisfiable

Disjointness

- $\models C \sqsubseteq D$ iff C and $\neg D$ are disjoint
- C is unsatisfiable iff C and $\top$ are disjoint

Properties

- "A person is either a man or a woman"

$$\text{Person} \sqsubseteq \text{Man} \sqcup \text{Woman}$$

- "No person is both a man and a woman"

$$\text{Man} \sqcap \text{Woman} \sqsubseteq \neg\text{Person}$$

- "Everyone (everything) is a person"

$$\top \sqsubseteq \text{Person}$$

- "Nobody (nothing) is a person"

$$\text{Person} \sqsubseteq \bot$$

Domain

Observe that:

- The domain of $R^{\mathfrak{A}}$ is $(\exists R.\top)^{\mathfrak{A}}$:

$$(\exists R.\top)^{\mathfrak{A}} = \{x \in \Delta \mid R^{\mathfrak{A}}(x) \neq \emptyset\}$$

In $\mathcal{ALC}$ you can express **domain restrictions**:

- "The domain of $\underline{R}$ is $\underline{C}$"
$$\exists R.\top \sqsubseteq C$$
- "The domain of having-a-brother is people"

$$\exists \text{hasBrother}.\top \sqsubseteq \text{Person}$$

Range

Observe that:

- The range of $R^{\mathfrak{A}}$ is $(\exists R^{-}.\top)^{\mathfrak{A}}$
- But R^{-} (the inverse of R) is not part of $\mathcal{ALC}$ (we will see logics with role inverse next time)

We can, however, express **range restrictions** in $\mathcal{ALC}$:

- "The range of $\underline{R}$ is $\underline{C}$"
$$\top \sqsubseteq \forall R.C$$
- "The range of having-a-brother is men"

$$\top \sqsubseteq \forall \text{hasBrother}.\text{Man}$$

The standard translation

Two functions π and μ map concepts to first-order formulae.

$$
\begin{aligned}
\pi(A) &= Ax & \mu(A) &= Ay \\
\pi(\neg C) &= \neg\pi(C) & \mu(\neg C) &= \neg\mu(C) \\
\pi(C \sqcup D) &= \pi(C) \vee \pi(D) & \mu(C \sqcup D) &= \mu(C) \vee \mu(D) \\
\pi(C \sqcap D) &= \pi(C) \wedge \pi(D) & \mu(C \sqcap D) &= \mu(C) \wedge \mu(D) \\
\pi(\exists R.C) &= \exists y(xRy \wedge \mu(C)) & \mu(\exists R.C) &= \exists x(yRx \wedge \pi(C)) \\
\pi(\forall R.C) &= \forall y(xRy \supset \mu(C)) & \mu(\forall R.C) &= \forall x(yRx \supset \pi(C))
\end{aligned}
$$

Proposition

$a^{\mathfrak{A}} \in C^{\mathfrak{A}}$ if and only if $\mathfrak{A} \models \pi(C)[x \mapsto a]$.

The guarded fragment

GF is the least set such that

- $\varphi \in$ GF if φ is atomic
- $\neg\varphi \in$ GF if $\varphi \in$ GF
- $\varphi \vee \psi \in$ GF and $\varphi \wedge \psi \in$ GF if $\varphi \in$ GF and $\psi \in$ GF
- $\exists x_1, \ldots, x_n(\varphi \wedge \psi) \in$ GF and $\forall x_1, \ldots, x_n(\varphi \supset \psi) \in$ GF if
  - φ is atomic,
  - $\psi \in$ GF, and
  - $\mathrm{FV}(\psi) \subseteq \mathrm{FV}(\varphi)$.

An example of a guarded formula is symmetry of a relation:

$$\forall xy(xRy \supset yRx)$$

Fragments of FOL

- First-order logic (FOL) is undecidable.
- But there are decidable fragments, such as propositional logic.
- Other decidable fragments include:
  - The two-variable fragment (**NEXPTIME**-complete)
  - The guarded fragment (**2EXPTIME**-complete)
  - The guarded fragment where the number of variables or the arity of relations is bounded (**EXPTIME**-complete)
- The standard translation maps concepts to the guarded two-variable fragment, e.g.,

$$\exists R.\exists R.\exists R.A \xrightarrow{\pi} \exists y(xRy \wedge \exists x(yRx \wedge \exists y(xRy \wedge Ay)))$$

- Hence satisfiability of $\mathcal{ALC}$ is in **EXPTIME**.

Tableaux

Any concept can be put on NNF using the following rewrite rules.

$$\neg\neg C \to C$$
$$\neg(C \sqcap D) \to \neg C \sqcup \neg D$$
$$\neg(C \sqcup D) \to \neg C \sqcap \neg D$$
$$\neg(\exists R.C) \to \forall R.\neg C$$
$$\neg(\forall R.C) \to \exists R.\neg C$$

If we go via FOL, it is easy to see that, e.g.,

- $\neg\forall y(xRy \supset Cy)$ is equivalent to $\exists y(xRy \wedge \neg Cy)$.

Tableaux

- The tableaux calculus operates on assertions on NNF.
- A branch is **closed** if both $C(a)$ and $\neg C(a)$ occurs for some concept C and individual a, e.g., the right branch below:

$D \sqcup (C \sqcap \neg C)(a)$

Left branch: $D(a)$

Right branch: $C \sqcap \neg C(a)$, $C(a)$, $\neg C(a)$, $\times$

- The rules are as follows; S denotes the branch.
- The preconditions ensure that the rules can be applied at most once to each assertion.

Tableaux

- If $C(a) \notin S$ or $D(a) \notin S$,

Intersection

$$\frac{(C \sqcap D)(a)}{\begin{array}{c} C(a) \\ D(a) \end{array}}$$

- If $C(a) \notin S$ and $D(a) \notin S$,

Union

$$\frac{(C \sqcup D)(a)}{C(a) \mid D(a)}$$

Tableaux

- If $C(b) \notin S$,

Value restriction

$$\frac{\begin{array}{c} \forall R.C(a) \\ R(a,b) \end{array}}{C(b)}$$

- If $C(b) \notin S$ or $R(a,b) \notin S$, for every b,

Existential quantification

$$\frac{\exists R.C(a)}{\begin{array}{c} C(b) \\ R(a,b) \end{array}}\ b \text{ fresh}$$

Example

- What is the relationship between the following?
  - $\exists R.C \sqcap \exists R.D$
  - $\exists R.(C \sqcap D)$
- Do they subsume each other?
- We first check if

$$\exists R.C \sqcap \exists R.D \sqsubseteq \exists R.(C \sqcap D),$$

- equivalently, if $\exists R.C \sqcap \exists R.D \sqcap \forall R.(\neg C \sqcup \neg D)$ is unsatisfiable.

Example

$\exists R.C \sqcap \exists R.D \sqcap \forall R.(\neg C \sqcup \neg D)(a)$

$\exists R.C(a)$

$\exists R.D(a)$

$\forall R.(\neg C \sqcup \neg D)(a)$

$C(b)$

$R(a,b)$

$D(c)$

$R(a,c)$

$\neg C \sqcup \neg D(b)$

$\neg C \sqcup \neg D(c)$

$\neg C(b)$ $\times$ | $\neg D(b)$

$\neg C(c)$ | $\neg D(c)$ $\times$

- No rules are applicable to this tableau.
- One branch is not closed.
- We can contruct a countermodel.
  - $\Delta = \{a, b, c\}$
  - $R^{\mathfrak{A}} = \{\langle a, b\rangle, \langle a, c\rangle\}$
  - $C^{\mathfrak{A}} = \{a\}$
  - $D^{\mathfrak{A}} = \{b\}$
- Then
  - $a \in (\exists R.C \sqcap \exists R.D)^{\mathfrak{A}}$
  - $a \notin (\exists R.(C \sqcap D))^{\mathfrak{A}}$

Example

- We have shown that it is **not** the case that

$$\exists R.C \sqcap \exists R.D \sqsubseteq \exists R.(C \sqcap D).$$

- Next we check if

$$\exists R.(C \sqcap D) \sqsubseteq \exists R.C \sqcap \exists R.D,$$

- equivalently, if $\exists R.(C \sqcap D) \sqcap (\forall R.\neg C \sqcup \forall R.\neg D)$ is unsatisfiable.

Example

$\exists R.(C \sqcap D) \sqcap (\forall R.\neg C \sqcup \forall R.\neg D)(a)$

$\exists R.(C \sqcap D)(a)$

$\forall R.\neg C \sqcup \forall R.\neg D(a)$

$C \sqcap D(b)$

$R(a,b)$

$C(b)$

$D(b)$

$\forall R.\neg C(a)$ | $\forall R.\neg D(a)$

$\neg C(b)$ | $\neg D(b)$

$\times$ | $\times$

- Every branch is closed.
- Hence $\exists R.(C \sqcap D) \sqcap (\forall R.\neg C \sqcup \forall R.\neg D)$ is unsatisfiable.
- Equivalently: $\exists R.(C \sqcap D)$ is subsumed by $\exists R.C \sqcap \exists R.D$.