INF3170 / INF4171

Predicate logic
Natural deduction and Sequent Calculus

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Notation

When using the symbol A to represent a formula for free variables, we use the same notation as if A was atomic:

$$A(x, y, z),$$

where A has at least x, y, z free. When substituting, we use the following notation:

$$A(x, y, z)[t/x] = A(t, y, z).$$

Natural Deduction ∀-rules

The rule for introducing ∀ is as follows:

$$\frac{A(y)}{\forall x A(x)}\,\forall I$$

where $x \notin FV(A(y))$. We also require that y does no occur as a free variable in any open assumption in the context of $A(y)$.

The rule for eliminating ∀ is as follows:

$$\frac{\forall x A(x)}{A(t)}\,\forall E$$

where t is any term.

Natural Deduction ∃-rules

The rue for introducing ∃ is:

$$\frac{A(t)}{\exists x A(x)}\ \exists I$$

where t is any term. The elimination rule looks a lot like $\vee E$:

$$\frac{\exists x A(x) \qquad \begin{array}{c}[A(y)]^1 \\ \vdots \\ C\end{array}}{C}\ \exists E_1$$

Examples

- $\forall x \forall y A(x, y) \vdash \forall y \forall x A(x, y)$
- $\exists x \exists y A(x, y) \vdash \exists y \exists x A(x, y)$
- $\exists x \forall y A(x, y) \vdash \forall y \exists x A(x, y)$

Sequent Calculus: $\forall L$

$$\frac{A(t), \forall x A(x), \Gamma \;\vdash\; \Delta}{\forall x A(x), \Gamma \;\vdash\; \Delta}\,\forall L$$

t can be any term. We argue that this is a good rule by showing that it preserves falsifiability upwards. Note that we now return to a familiar problem of having to keep a copy of the original formula.

Sequent Calculus: $\forall R$

$$\frac{\Gamma \vdash A(w), \Delta}{\Gamma \vdash \forall x A(x), \Delta} \forall R$$

where w is a parameter (a special type of variable) that does not occur in the conclusion of the rule. This gives us preservation of falisifiability. If we falsify $\forall x A(x)$, then there must be at least one w for which $A(w)$ is false. We cannot chose this w, however, so it is important that we do not have any assumptions about it.

Sequent Calculus: $\exists R$

$$\frac{\Gamma \ \vdash \ \exists x A(x), A(t), \Delta}{\Gamma \ \vdash \ \exists x A(x), \Delta} \exists R$$

where t is any term. This rule is the dual of $\forall L$, and the argument for its "correctness" is also dual.

Sequent Calculus: $\exists L$

$$\frac{A(w), \Gamma \ \vdash \ \Delta}{\exists x A(x), \Gamma \ \vdash \ \Delta} \exists L$$

where w is a parameter (a special type of variable) that does not occur in the conclusion of the rule. This rule is the dual of $\forall R$, and the argument for its "correctness" is also dual.

Examples

Prove the following sequents:

- $\forall x \forall y A(x, y) \vdash \forall y \forall x A(x, y)$
- $\exists x \exists y A(x, y) \vdash \exists y \exists x A(x, y)$
- $\exists x \forall y A(x, y) \vdash \forall y \exists x A(x, y)$

Use "failed" proof search to find counter models for the following sequents:

- $\exists x A(x) \vdash \forall x A(x)$
- $\forall x \exists y R(x, y) \vdash \exists x \forall y R(x, y)$