INF3170 / INF4171

Predicate logic
Natural deduction

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September 26, 2016

Individuals and Domains

In Predicate logic we express statements about some collection of objects or individuals.

- Individuals: $a, b, c, a_1, a_2, a_3, \ldots$.
- Variables: $x, y, z, x_1, x_2, x_3, \ldots$.

A set of individuals is called a *domain*, and is denoted $\mathcal{D}$.

Note: To make a more thurough distinction between representation and meaning, we sometimes use different symbols for the individuals depending on the context. When refering to domain individual a in a formula, one would use the modified symbol $\overline{a}$ to make the distiction.

Atmoic formulas: Properties and Relations

The atomic formulas or Predicate Logic are assertions about properties and relations.

- $P(a)$: Individual a has property P.
- $Q(a, b)$: Individuals a and b are related through relation Q.

If we let P represent the property "prime number", then we can claim that 4 and 17 are primes as follows:

$$P(4) \qquad P(17).$$

Note: If we use separate symbols for meaning and representation, then in a formula we would write $\overline{P}(\overline{4})$ and $\overline{P}(\overline{17})$.

If a is a person, b is a car, and Q represents ownership, then we can say that a owns b as follows:

$$Q(a, b).$$

Using 3-place relations, we can also express more complex relationships. For example

$$C(\text{E16}, \text{Oslo}, \text{Bergen})$$

can represent the statement that E16 connects Oslo and Bergen.

When given a domain $\mathcal{D}$, we can build and interpretation $\mathcal{I}$ that determines the truth of assertions about properties and relationships. (We'll get back to this in detail.)

Predicates

Predicates are an essential part of Predicate Logic.

We informally define a predicate as an atomic formula (property or relation expression) with at least one variable in place of an individual. The predicates are important types of atmoic formulas.

- $P(x)$: x is a prime number.
- $Q(a, z)$: a owns z.
- $C(x, \text{Oslo}, y)$: x connects Oslo to y.

Even with an interpretation of all the properties and relations, the above formulas can be either true of false, depending on what the variables represent.

Atomic Formulas

Properties can be seen as special unary relations. Relations that relate two objects are call binary. Relations that relate n objects are called n-ary.

Definition

Atmoic formula Let P be an n-ary property (or predicate symbol), $\mathcal{C}$ a set of constants, $\mathcal{V}$ a set of variables, and $d_1, \ldots, d_n$ be a list of n symbols from $\mathcal{C} \cup \mathcal{V}$ (with possible repetitions). Then

$$P(d_1, \ldots, d_n)$$

is an atomic formula.

Models

A model consists of a domain $\mathcal{D}$ and an interpretation function $\mathcal{I}$. The function $\mathcal{I}$ must map every constant $c \in \mathcal{C}$ to some individual in the domain. $\mathcal{I}$ must also contain information about when a relation assertion is true.

Consider the formula $C(a, b, c)$ and the domain $\mathcal{D} = \{\text{Bergen}, \text{E6}, \text{E16}, \text{Oslo}, \text{Trondheim}\}$. Define $\mathcal{I}_1$ so that

$$\mathcal{I}_1(a) = \text{E16}, \quad \mathcal{I}_1(b) = \text{Oslo}, \quad \mathcal{I}_1(c) = \text{Bergen}$$

and $\mathcal{I}_2$ so that

$$\mathcal{I}_2(a) = \text{E16}, \quad \mathcal{I}_2(b) = \text{Oslo}, \quad \mathcal{I}_2(c) = \text{Trondheim}.$$

If $\mathcal{I}$ iterprets C according to the Norwegian road network, we have that $\mathcal{I}_1 \models C(a, b, c)$, while $\mathcal{I}_2 \not\models C(a, b, c)$.

How do we specify how $\mathcal{I}$ interprets relations?

Modelling Properties

The interpretation of a unary relation P defines for which domain elements d that $P(d)$ is true. Let $P^{\mathcal{I}} \subseteq \mathcal{D}$. We say that $\mathcal{I} \models P(c)$ if and only if $c^{\mathcal{I}} \in P^{\mathcal{I}}$, where $c^{\mathcal{I}} = \mathcal{I}(c)$.

For binary relations Q we let $Q^{\mathcal{I}} \subseteq \mathcal{D} \times \mathcal{D}$. That is, $Q^{\mathcal{I}}$ is a set of pairs over $\mathcal{D}$. Now $\mathcal{I} \models Q(a, b)$ if and only if $\langle a^{\mathcal{I}}, b^{\mathcal{I}} \rangle \in Q^{\mathcal{I}}$.

Returning to the road example, $\mathcal{I}_1$ can be written as

$$\begin{aligned} a^{\mathcal{I}_1} &= \mathsf{E16} \\ b^{\mathcal{I}_1} &= \mathsf{Oslo} \\ c^{\mathcal{I}_1} &= \mathsf{Bergen} \\ C^{\mathcal{I}_1} &= \{\langle \mathsf{E16}, \mathsf{Oslo}, \mathsf{Bergen}\rangle, \\ &\quad \langle \mathsf{E16}, \mathsf{Bergen}, \mathsf{Oslo}\rangle \\ &\quad \langle \mathsf{E6}, \mathsf{Oslo}, \mathsf{Trondheim}\rangle \\ &\quad \langle \mathsf{E6}, \mathsf{Trondheim}, \mathsf{Oslo}\rangle\}. \end{aligned}$$

We have that $\mathcal{I}_1 \models C(a, b, c)$. We also have

$$\mathcal{I}_1 \models C(\overline{\mathsf{E6}}, \overline{\mathsf{Trondheim}}, \overline{\mathsf{Oslo}}).$$

In the last formula, we have introduced symbols for domain elements we didn't have any constants for.

Interpretation

Definition (Interpretation)

Let $\mathcal{D}$ be a domain, $\mathcal{C}$ a set of constants, and $\mathcal{P}$ a set of relation (predicate) symbols. An interpretation $\mathcal{I}$ is an interpretation function defined so that

- $c^{\mathcal{I}} \in \mathcal{D}$ for every $c \in \mathcal{C}$, and
- $P^{\mathcal{I}} \subseteq \mathcal{D}^n$ for every n-ary $P \in \mathcal{P}$.

If $P \in \mathcal{P}$ is an n-ary relation symbol and $c_1, \ldots, c_n \in \mathcal{C}$ are constants, then $\mathcal{I} \models P(c_1, \ldots, c_n)$ if and only if $\langle c_1, \ldots, c_n \rangle \in P^{\mathcal{I}}$.

Interpretations determine the truth of variable-free atomic formulas. If $c \in \mathcal{D}$, then we require $\overline{c}^{\mathcal{I}} = c$ for all $\mathcal{I}$. That is, all interpretations must respect the dedicated domain constants.

Universal Quantification

Sometimes we want to express universal truths about individuals. For example, we may want to express that if x is prime, that is $P(x)$, then x is also odd, $O(x)$.

We can write $P(x) \to O(x)$, but when does this hold? In order to state that the formula holds whatever x is, we use the quantifier $\forall$ followed by the variable the quantifier is applied to:

$$\forall x \, [P(x) \to Q(x)] \, .$$

Existential Quantification

It is usefull to express that some individual has a property without having to specify the individual. For example, we may want to say that at least one number x i even, $E(x)$.

For this, we use the quantifier $\exists$:

$$\exists x E(x).$$

Substitutions

The set of *terms* is the set $\mathcal{T} = \mathcal{C} \cup \mathcal{V}$ of constants and variables. When we introduce functions, the set of terms will become larger.

Definition (Substitution)

A *substitution* is a partial function $\sigma : \mathcal{V} \to \mathcal{T}$. If σ maps variables $x_1, \ldots, x_n$ to terms $t_1, \ldots, t_n$, we often write it as $[t_1/x_1, \ldots, t_n/x_n]$.

Note: A substitution σ can be extended to a total function by letting it map all variables it does not allread map to a term to the variables themselves.

We can apply a subsitution σ to a formula. The result is a new formula, where we simultaneously replace each variable x with $\sigma(x)$. Constants are not changed by substitutions.

Example

We define a substitution $\sigma = [a/x, b/y, x/z]$. We now have

$$P(x)\sigma = P(a)$$
$$Q(x, b)\sigma = Q(a, b)$$
$$Q(x, z)\sigma = Q(a, x)$$
$$R(x, y, z)\sigma = R(a, b, x).$$

Syntax of Predicate Logic

Definition (Formula)

The set *FORM* of formulas is the smallest set such that

- $P(t_1, \ldots, t_n) \in FORM$, where P is an n-ary predicate symbol and t_i are terms,

and ff $F, G \in FORM$ and x is a variable, then

- $(\neg F), (F \wedge G), (F \vee G), (F \rightarrow G) \in FORM$, and
- $(\forall x F), (\exists x F) \in FORM$.

We define the semantics of Predicate Logic in terms of interpretations, but we need some tools before we are ready.

Free variables

Definition

We define the function $FV : FORM \to \mathcal{P}(\mathcal{V})$ recursively as follows:

- $FV(P(t_1, \ldots, t_n))$ is the set of all variables occurring in the terms t_i,
- $FV(\neg\phi) = FV(\phi)$,
- $FV(\phi_1 \Box \phi_2) = FV(\phi_1) \cup FV(\phi_2)$, where $\Box \in \{\wedge, \vee, \to\}$,
- $FV(\forall x \phi) = FV(\phi) \setminus \{x\}$, and
- $FV(\exists x \phi) = FV(\phi) \setminus \{x\}$.

Note: If a variable is not free, it is often said to be bound.

Definition

A formula with no free variables is a *sentence*.

Substitutions revisited

We can now define the actions of substitutions on non-atomic formulas. We restrict ourselves to the case where the substitution maps only one variable: $[t/x]$. The definition can be generalised with a bit of notation juggling, but we will not need the general definition.

Definition

The result of applying the substitution $[t/x]$ to a non-atomic formula is as follows ($\mathcal{Q} \in \{\forall, \exists\}$):

- $(\neg\phi)[t/x] = (\neg\phi[t/x])$,
- $(\phi_1 \Box \phi_2)[t/x] = (\phi_1[t/x] \Box \phi_2[t/x])$,
- $(\mathcal{Q}y\phi)[t/x] = \begin{cases} (\mathcal{Q}y\phi), & \text{if } x = y, \\ (\mathcal{Q}y\phi[t/x]), & \text{otherwise.} \end{cases}$

Interpretations revisited

Definition (Truth and satisfaction)

Let $\mathcal{I}$ be an interpretation with domain $\mathcal{D}$, and ϕ be an atomic formula. We have defined the conditions for $\mathcal{I} \models \phi$. We now extend this definition to all sentences (variable-free formulas):

- $\mathcal{I} \models (\neg\phi)$ iff $\mathcal{I} \not\models \phi$,
- $\mathcal{I} \models (\phi_1 \wedge \phi_2)$ iff $\mathcal{I} \models \phi_1$ and $\mathcal{I} \models \phi_2$,
- $\mathcal{I} \models (\phi_1 \vee \phi_2)$ iff $\mathcal{I} \models \phi_1$ or $\mathcal{I} \models \phi_2$,
- $\mathcal{I} \models (\phi_1 \rightarrow \phi_2)$ iff $\mathcal{I} \models \phi_1$ implies that $\mathcal{I} \models \phi_2$,
- $\mathcal{I} \models (\forall x\phi)$ iff $\mathcal{I} \models \phi[\overline{c}/x]$ for any $c \in \mathcal{D}$, and
- $\mathcal{I} \models (\exists x\phi)$ iff $\mathcal{I} \models \phi[\overline{c}/x]$ for some $c \in \mathcal{D}$

Note: There are several ways of extending interpretations to formulas with variables if needed.

A simple tautology

Prove that $\exists x \forall y R(x, y) \rightarrow \forall y \exists x R(x, y)$ is a tautalogy (is valid). That is, it is true under any interpretation.

Proof.

From the definition, we know that $\mathcal{I} \models \phi_1 \rightarrow \phi_2$ if and only if $\mathcal{I} \models \phi_1$ implies $\mathcal{I} \models \phi_2$. We therefore assume that $\mathcal{I} \models \exists x \forall y R(x, y)$. Then, there must be some $d \in \mathcal{D}$ so that $\mathcal{I} \models \forall y R(\overline{d}, y)$. It follow that for any $e \in \mathcal{D}$, $\mathcal{I} \models R(\overline{d}, \overline{e})$. Now, $\mathcal{I} \models \exists x R(x, \overline{e})$ for any $e \in \mathcal{D}$. An so, $\mathcal{I} \models \forall y \exists x R(x, y)$. □

This proof illustrates that our definitions of $\mathcal{I}$ and $\models$ match our intuition about truth and quantification, at least in some cases.

Counter models

The formula $\forall x \exists y R(x,y) \rightarrow \exists y \forall x R(x,y)$ is not valid. We show this with a counter model.

Let $\mathcal{D} = \{a, b\}$, and $R^{\mathcal{I}} = \{\langle a, b\rangle, \langle b, a\rangle\}$. $\mathcal{I} \models \forall x \exists y R(x,y)$. There are two similar cases for $\forall$, we look at the first: $\mathcal{I} \models \exists y R(\bar{a}, y)$. This is true, since $\mathcal{I} \models R(\bar{a}, \bar{b})$.

On the other hand, $\mathcal{I} \not\models \exists y \forall x R(x,y)$. Again, we must show both cases since we must show that no choise for y will work. We look at the first option: $\mathcal{I} \not\models \forall x R(x, \bar{a})$. This is true, since $\mathcal{I} \not\models R(\bar{a}, \bar{a})$.