INF3170 / INF4171

Soundness and completeness of sequent calculus

Andreas Nakkerud

September 14, 2016

Formulas

We define the set of all propositional fomulas inductively. Let P_i denote the atomic formulas (propositional variables).
The set $\mathcal{F}$ of formulas are defined as follows:

- $P_i \in \mathcal{F}$ for all propositional variables P_i.
- If $F, G \in \mathcal{F}$, then
  - $(\neg F) \in \mathcal{F}$,
  - $(F \wedge G) \in \mathcal{F}$,
  - $(F \vee G) \in \mathcal{F}$, and
  - $(F \to G) \in \mathcal{F}$.

Assignments of truth values

An assignment of truth values is a function from atomic formulas (propositional vaiables) to truth values $\{0, 1\}$. Where 0 is used for false, and 1 for true.

Example

If the assignment v makes A true and B false, we write this as

$$v(A) = 1$$
$$v(B) = 0.$$

Valuations

A **valuation** $v : \mathcal{F} \to \{0, 1\}$ is a function from propositional formulas to truth values.

When restricted to atomic formulas, a valuation is an assignment of truth values. For non-atomic formulas, we define valuations recursively as follows:

Let $F, G \in \mathcal{F}$. We define v such that

- $v\big((\neg F)\big) = 1 - v(F)$,
- $v\big((F \wedge G)\big) = \min\{v(F), v(G)\}$,
- $v\big((F \vee G)\big) = \max\{v(F), v(G)\}$, and
- $v\big((F \to G)\big) = \begin{cases} 0, & \text{if } v(F) = 1 \text{ and } v(G) = 0 \\ 1, & \text{otherwise.} \end{cases}$

Sequent

A sequent if an object on the form

$$\Gamma \vdash \Delta,$$

where Γ and Δ are (possibly empty) collections of formulas. Γ is called the antecedent, and Δ is called the succedent.
A sequent is *valid* if any valuation satisfying each formula in Γ satisfies at least one formula in Δ.
If a sequent is not valid, then it is falisifiable, and it is falsified by those valuation that make all formulas in Γ true and all formulas in Δ false.

Inference rules

We define the following inference rules for the connectives. The sequents above the line of an inference rule are called premisses, the sequent below the line is the conclusion. The inference rules are designed so that whenever the premisses are valid, the conclusion is valid.

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ R\neg$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ L\neg$$

In the above rules $\neg A$ is the active formula in the conclusion, and A in the premiss. Γ and Δ contain possible inactive formulas.

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ R\wedge$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ L\wedge$$

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ R\vee$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ L\vee$$

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta} \; R\to$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \; L\to$$

The rules with one premiss are called α rules. The rules with multiple premisses are called β rules.

Theorem

The inference rules $R\neg$, $L\neg$, $R\wedge$, $L\wedge$, $R\vee$, $L\vee$, $R\rightarrow$, and $L\rightarrow$ each have the property that the conclusion is valid whenever the premisses are valid.

The contrapositive of the above theorem is that whenever the conclusion is falsifiable, at least one of the premisses must be falsifiable. In fact:

Theorem

If v falsifies the conclusion of an inference rule, it must also falsify one of its premisses. If v falsifies a premiss of an inference rule, if falsifies the conclusion.

Proof.

Left as an exercise. □

Axioms

Axioms are sequents of the form

$$\Gamma, A \vdash A, \Delta,$$

where A is an atomic formula.

Theorem

Axioms are valid sequents.

Proof.

If v satisfies each formula in the antecedent of an axiom, then it must satisfy each atomic formula in that antecedent. Since one of these also uccur in the succedent, the sequent is valid. □

Derivations

A derivation is a tree build from sequents. The inference rules define which such trees are derivations. We define derivations as follows:

- Let $\Gamma \vdash \Delta$ be a sequent. This sequent is also a derivation with $\Gamma \vdash \Delta$ as the root, and as the only leaf.
- Let D be a derivation with a leaf sequent $\Gamma \vdash \Delta$. And suppose there is a rule of inference with $\Gamma \vdash \Delta$ as conclusion and $\Gamma_1 \vdash \Delta_1, \ldots, \Gamma_n \vdash \Delta_n$ as premisses. We can extend D by extending the tree upwards from $\Gamma \vdash \Delta$. The leaf $\Gamma \vdash \Delta$ is replaced by the leaves $\Gamma_1 \vdash \Delta_1, \ldots, \Gamma_n \vdash \Delta_n$.

We usually draw derivations as follows:

$$\dfrac{\dfrac{A \vdash A, B}{\vdash A, B, \neg A}\,R\neg \qquad \vdash A, B, \neg B}{\dfrac{\vdash A, B, \neg A \wedge \neg B}{\neg(\neg A \wedge \neg B) \vdash A, B}\,L\neg}\,R\wedge$$

In this derivation, the root is $\neg(\neg A \wedge \neg B) \vdash A, B$, and the leaves are $A \vdash A, B$ and $\vdash A, B, \neg B$. Note that one of the roots is an axiom.

Proofs in sequent calculus

Definition

A derivation is a proof if all its leaves are axioms.

Theorem (Soundness)

The root sequent of a proof is valid.

Theorem (Completeness)

If a sequent is valid, then there is a proof with that sequent as root.

Soundness

In order to prove that sequent calculus is sound, we are going to use a proof by contradiction. *Assume we have a proof with a falsifiable root.*

- Since the root is falsifiable, at least one of its premisses is falsifiable.
- We can prove by structural induction on derivations that any derivation with a falsifiable root must have at least one falsifiable leaf. (Exercise.)
- Thus, any derivation with a falsifiable root will have at least one leaf that is not an axiom.
- A derivation with non-axiom leaves is not a proof.
- We have a contradiction, and conclude that the root of a proof must be valid.

Completeness

Lemma

If a sequent contains only atomic formulas, then it is valid if and only if it is an axiom.

Lemma

Any derivation can be extended to a maximal derivation where the leaves contain only atomic formulas.

Lemma

Any derivation with a valid root has only valid leaves.

Proof of completeness

Let $\Gamma \vdash \Delta$ be a valid sequent. Extend the root-only derivation

$$\Gamma \vdash \Delta$$

to a maximal derivation. All leaves of this derivation are valid and have only atomic formulas in them. It follows that all leaves of the derivation are axioms, and that the derivation is a proof.

Model existence theorem

Theorem

If $\Gamma \vdash \Delta$ is not provable, then there is a valuation that falsifies it.

Proof.

Create a maximal derivation of $\Gamma \vdash \Delta$. Since the sequent is not provable, there must be a non-axiom leaf $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ where $A_i \neq B_j$ for all i, j. The valuation defined so that $v(A_i) = 1$ and $v(B_j) = 0$ falsifies our leaf, and therefore also the root. □

The model existence theorem guarantees counter models for non-provable sequents. The contrapositive to the model existence theorem is completeness.