V

Intuitionistic Logic

1 Intuitionism and Constructivism

During the past century, a major debate in the philosophy of mathematics has centered on the question of how to regard noneffective or nonconstructive proofs in mathematics. Is it legitimate to claim to have proven the existence of a number with some property without actually being able, even in principle, to produce one? Is it legitimate to claim to have proven the existence of a function without providing any way to calculate it? L. E. J. Brouwer is perhaps the best known early proponent of an extreme constructivist point of view. He rejected much of early twentieth century mathematics on the grounds that it did not provide acceptable existence proofs. He held that a proof of $p \vee q$ must consist of either a proof of p or one of q and that a proof of $\exists x P(x)$ must contain a construction of a witness c and a proof that $P(c)$ is true. At the heart of most nonconstructive proofs lies the law of the excluded middle: For every sentence A, $A \vee \neg A$ is true. Based on this law of classical logic one can prove that $\exists x P(x)$ by showing that its negation leads to a contradiction without providing any hint as to how to find an x satisfying P. Similarly, one can prove $p \vee q$ by proving $\neg(\neg p \wedge \neg q)$ without knowing which of p and q is true.

Example 1.1: We wish to prove that there are two irrational numbers a and b such that a^b is rational. Let $c = \sqrt{2}^{\sqrt{2}}$. If c is rational, then we may take $a = \sqrt{2} = b$. On the other hand, if c is not rational, then $c^{\sqrt{2}} = 2$ is rational and we may take $a = c$ and $b = \sqrt{2}$. Thus, in either case, we have two irrational numbers a and b such that a^b is rational. This proof depends on the law of the excluded middle in that we assume that either c is rational or it is not. It gives us no clue as to which of the two pairs contains the desired numbers.

Example 1.2: Consider the proof of König's lemma (Theorem I.1.4). We defined the infinite path by induction. At each step we knew by induction that one of the finitely many immediate successors had infinitely many nodes below it. We then "picked" one such successor as the next node in our path. We had proved by induction that a disjunction is true and then simply continued the argument "by cases". As we had not in any way established which successor had infinitely many nodes below it, we had no actual construction of (no algorithm for defining) the infinite path that we proved to exist. Similar considerations apply to our proofs of completeness, compactness and other theorems.

A formal logic that attempts to capture Brouwer's philosophical position was developed by his student Heyting. This logic is called intuitionistic logic. It is an important attempt at capturing constructive reasoning. In particular, the law of the excluded middle is not valid in intuitionistic logic.

A number of paradigms have been suggested for explaining Brouwer's views. Each one can provide models or semantics for intuitionistic logic. One paradigm considers mathematical statements as assertions about our (or someone's) knowledge or possession of proofs. A sentence is true only when we know it to be so or only after we have proven it. At any moment we cannot know what new facts will be discovered or proven later. This interpretation fits well with a number of situations in computer science involving both databases and program verification. In terms of databases, one caveat is necessary. We view our knowledge as always increasing so new facts may be added but no old ones removed or contradicted. This is a plausible view of the advance of mathematical knowledge but in many other situations it is not accurate. Much of the time this model can still be used by simply attaching time stamps to facts. Thus the database records what we knew and when we knew it. The intuitionistic model is then a good one for dealing with deductions from such a database.

In terms of program verification, intuitionistic logic has played a basic role in the development of constructive proof checkers and reasoning systems. A key idea here is that, in accordance with Brouwer's ideas, the proof of an existential statement entails the construction of a witness. Similarly, the proof that for every x there is a y such that $P(x, y)$ entails the construction of an algorithm for computing a value of y from one for x. The appeal of such a logical system is obvious. On a practical level, there are now implementations of large–scale systems that (interactively) provide intuitionistic proofs of such assertions. The systems can then actually extract the algorithm computing the intended function. One then has a verified algorithm since the proof of existence is in fact a proof that the algorithm specified actually runs correctly. One such system is NUPRL developed at Cornell University by R. Constable [1986, 5.6] and others.

In this chapter, we present the basics of intuitionistic logic including a semantics developed by Kripke that reflects the "state of knowledge" interpretation of Heyting's formalism. In addition to the intuitive considerations, the claim that this choice of semantics adequately reflects constructivist reasoning is confirmed by the fact that the following *disjunction* and *existence properties* hold:

Theorem 2.20: *If* $(\varphi \vee \psi)$ *is intuitionistically valid, then either* φ *or* ψ *is intuitionistically valid.*

Theorem 2.21: *If $\exists x\varphi(x)$ is intuitionistically valid, then so is $\varphi(c)$ for some constant c.*

We then develop an intuitionistic proof theory based on a tableau method like that for classical logic and prove the appropriate soundness and completeness theorems. Of course, the completeness theorem converts the above theorems into ones about provability. We can (intuitionistically) prove $\varphi \vee \psi$ only if we can prove one of them. We can prove $\exists x\varphi(x)$ only if we can prove $\varphi(c)$ for some explicit constant c.

The presentation in this chapter is designed to be independent of Chapter IV. Thus there is some overlap of material. For those readers who have read Chapter IV, we supply a guide comparing classical, modal and intuitionistic logics in §6.

2 Frames and Forcing

Our notion of a language is the same as that for classical predicate logic in Chapter II except that we make one modification and two restrictions that simplify the technical details in the development of forcing. The modification is that we formally omit the logical connective $\leftrightarrow$ from our language. We instead view $\varphi \leftrightarrow \psi$ as an abbreviation for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. Our restrictions are on the nonlogical components of our language. We assume throughout this chapter that every language $\mathcal{L}$ has at least one constant symbol but no function symbols other than constants.

We now present a semantics for intuitionistic logic that formalizes the "state of knowledge" interpretation.

Definition 2.1: Let $\mathcal{C} = (R, \leq, \{\mathcal{C}(p)\}_{p \in R})$ consist of a partially ordered set $(R, \leq)$ together with an assignment, to each p in R, of a structure $\mathcal{C}(p)$ for $\mathcal{L}$ (in the sense of Definition II.4.1). To simplify the notation, we write $\mathcal{C} = (R, \leq, \mathcal{C}(p))$ instead of the more formally precise version, $\mathcal{C} = (R, \leq, \{\mathcal{C}(p)\}_{p \in R})$. As usual, we let $C(p)$ denote the domain of the structure $\mathcal{C}(p)$. We also let $\mathcal{L}(p)$ denote the extension of $\mathcal{L}$ gotten by adding on a name c_a for each element a of $C(p)$ in the style of the definition of truth in II.4. $A(p)$ denotes the set of atomic formulas of $\mathcal{L}(p)$ true in $\mathcal{C}(p)$. We say that $\mathcal{C}$ is a *frame for the language* $\mathcal{L}$, or simply an *$\mathcal{L}$-frame* if, for every p and q in R, $p \leq q$ implies that $C(p) \subseteq C(q)$, the interpretations of the constants in $\mathcal{L}(p) \subseteq \mathcal{L}(q)$ are the same in $\mathcal{C}(p)$ as in $\mathcal{C}(q)$ and $A(p) \subseteq A(q)$.

Often $p \leq q$ is read "*q extends p*", or "*q is a future of p*". The elements of R are called *forcing conditions, possible worlds or states of knowledge.*

We now define the forcing relation for frames.

Definition 2.2 (Forcing for frames): Let $\mathcal{C} = (R, \leq, \mathcal{C}(p))$ be a frame for a language $\mathcal{L}$, p be in R and φ be a sentence of the language $\mathcal{L}(p)$. We give a definition of *p forces φ* , written $p \Vdash \varphi$ by induction on sentences φ.

(i) For atomic sentences φ, $p \Vdash \varphi \Leftrightarrow \varphi$ is in $A(p)$.

(ii) $p \Vdash (\varphi \to \psi) \Leftrightarrow$ for all $q \geq p$, $q \Vdash \varphi$ implies $q \Vdash \psi$.

(iii) $p \Vdash \neg\varphi \Leftrightarrow$ for all $q \geq p$, q does not force φ .

(iv) $p \Vdash (\forall x)\varphi(x) \Leftrightarrow$ for every $q \geq p$ and for every constant c in $\mathcal{L}(q)$, $q \Vdash \varphi(c)$.

(v) $p \Vdash (\exists x)\varphi(x) \Leftrightarrow$ there is a constant c in $\mathcal{L}(p)$ such that $p \Vdash \varphi(c)$.

(vi) $p \Vdash (\varphi \wedge \psi) \Leftrightarrow p \Vdash \varphi$ and $p \Vdash \psi$.

(vii) $p \Vdash (\varphi \vee \psi) \Leftrightarrow p \Vdash \varphi$ or $p \Vdash \psi$.

If we need to make the frame explicit, we say that *p forces φ in $\mathcal{C}$* and write $p \Vdash_{\mathcal{C}} \varphi$.

Definition 2.3: Let φ be a sentence of the language $\mathcal{L}$. We say that *φ is forced in the $\mathcal{L}$-frame $\mathcal{C}$* if every p in R forces φ. We say *φ is intuitionistically valid* if it is forced in every $\mathcal{L}$-frame.

Clauses (ii), (iii) and (iv) defining $p \Vdash \varphi \to \psi$, $p \Vdash \neg\varphi$ and $p \Vdash (\forall x)\varphi(x)$, respectively, each have a quantifier ranging over elements of the partial ordering, namely, "for all q, if $q \geq p$, then ...". Clause (ii) says that p forces an implication $\varphi \to \psi$ only if any greater state of knowledge q which forces the antecedent φ also forces the consequent ψ. This is a sort of permanence of implication in the face of more knowledge. Clause (iii) says p forces the negation of φ when no greater state of knowledge forces φ. This says that $\neg\varphi$ is forced if φ cannot be forced by supplying more knowledge than p supplies. Clause (iv) says p forces a universally quantified sentence only if in all greater states of knowledge all instances of the sentence are forced. This is a permanence of forcing universal sentences in the face of any new knowledge beyond that supplied by p. Another aspect of the permanence of forcing that says the past does not count in forcing, only the future, is given by the following lemma. (Note that the logic of our metalanguage remains classical throughout this chapter. Thus, for example, in Clause (ii) "implies" has the same meaning it had in Chapter I.)

Lemma 2.4: (Restriction lemma) *Let $\mathcal{C} = (R, \leq, \{\mathcal{C}(p)\}_{p\in R})$ be a frame, let q be in R and let $R_q = \{r \in R \mid r \geq q\}$. Then*

$$\mathcal{C}_q = (R_q\,,\ \leq\,,\ \mathcal{C}(p))$$

is a frame, where $\leq$ and the function $\mathcal{C}(p)$ are restricted to R_q. Moreover, for r in R_q, r forces φ in $\mathcal{C}$ iff r forces φ in $\mathcal{C}_q$.

Proof: By an induction on the length of formulas which we leave as Exercise 7. □

Consider the classical structures $\mathcal{C}(p)$ in an $\mathcal{L}$–frame $\mathcal{C}$. As we go from p to a $q > p$, we go from the classical structure $\mathcal{C}(p)$ associated with p to a (possibly) larger one $\mathcal{C}(q)$ associated with q with more atomic sentences classically true, and therefore fewer atomic sentences classically false. Clauses (i), (v), (vi) and (vii) for the cases of atomic sentences, "and", "or" and "there exists", respectively, are exactly as in the definition of truth in $\mathcal{C}(p)$ given in II.4.3. The other clauses have a new flavor and indeed the classical truth of φ in $\mathcal{C}(p)$ and p's forcing φ do not in general coincide. They do, however, in an important special case.

Lemma 2.5: (Degeneracy lemma) *Let $\mathcal{C}$ be a frame for a language $\mathcal{L}$ and φ a sentence of $\mathcal{L}$. If p is a maximal element of the partial ordering R associated with $\mathcal{C}$, then φ is classically true in $\mathcal{C}(p)$, i.e., $\mathcal{C}(p) \models \varphi$, if and only if $p \Vdash \varphi$. In particular, if there is only one state of knowledge p in R, then $\mathcal{C}(p) \models \varphi$ if and only if $p \Vdash \varphi$.*

Proof: The proof proceeds by an induction on formulas. For a maximal element p of R the clauses in the definition of $p \Vdash \varphi$ coincide with those in II.4.3 for $\mathcal{C}(p) \models \varphi$. In Clauses (ii), (iii) and (iv) the dependence on future states of knowledge reduces simply to the classical situation at p. Consider, for example, the case of Clause (ii): $p \Vdash (\varphi \rightarrow \psi) \Leftrightarrow (\forall q \geq p)(q \Vdash \varphi$ implies $q \Vdash \psi)$. Since p is maximal in R, $q \geq p$ is the same as $q = p$. Thus Clause (ii) reduces to ($p \Vdash \varphi \rightarrow \psi$ iff $p \Vdash \varphi$ implies $p \Vdash \psi$) which is the analog as the corresponding clause, II.4.3(v), for classical implication. We leave the verification that all the other clauses are also equivalent as Exercise 8. □

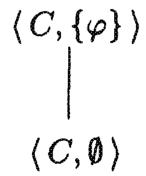
Theorem 2.6: *Any intuitionistically valid sentence is classically valid.*

Proof: By the degeneracy lemma (Lemma 2.5), every classical model is a frame model with a one–element partially ordered set in which forcing and classical truth are equivalent. As a sentence is classically valid if true in all classical models, it is valid if forced in every frame. □

It remains to see which classically valid sentences are intuitionistically valid and which are not. We show how to verify that some classically valid sentences are not intuitionistically valid by constructing frame counterexamples. Before presenting the examples, we want to establish some notational conventions for displaying frames. All the examples below have orderings that are suborderings of the full binary tree. We can therefore view the associated frames as labeled binary trees with the label of a node p being the structure $\mathcal{C}(p)$, or equivalently, the pair consisting of $C(p)$ and $A(p)$. We thus draw frames as labeled binary trees in our usual style and display the labels in the form $\langle C(p), A(p) \rangle$. The theoretical development of tableaux and the proof of their completeness requires somewhat more general trees but we leave that for the next section.

In the examples below of sentences that are not intuitionistically valid (2.7–2.11), φ and ψ denote atomic formulas of $\mathcal{L}$ with no free variables or only x free as displayed. In each of these examples, $\mathcal{C}(\emptyset)$, the structure associated with the bottom node $\emptyset$ of our partial ordering, is C with all the constants of $\mathcal{L}$ interpreted as c. We begin with the archetypal classically valid sentence which is not intuitionistically valid.

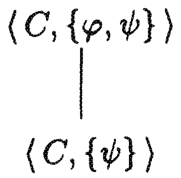
Example 2.7: As expected, the sentence $\varphi \vee \neg\varphi$ (an instance of the law of the excluded middle) is not intuitionistically valid. Let the frame $\mathcal{C}$ be



(Thus we have taken C as the domain at both nodes, $\emptyset$ and 0, of the frame.) At the bottom node, no atomic facts are true, i.e., $A(\emptyset)$ is empty. At the upper node 0, we have made the single atomic fact φ true by setting $A(0) = \{\varphi\}$.

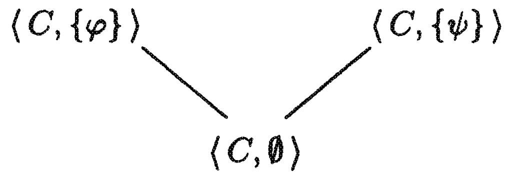
Consider now whether or not $\emptyset \Vdash \varphi \vee \neg\varphi$. Certainly $\emptyset$ does not force φ since φ is atomic and not true in $\mathcal{C}(\emptyset)$, i.e., not in $A(\emptyset)$. On the other hand, $0 \Vdash \varphi$ since $\varphi \in A(0)$. Thus $\emptyset$ does not force $\neg\varphi$ since it has an extension 0 forcing φ. So by definition, $\emptyset$ does not force $\varphi \vee \neg\varphi$ and this sentence is not intuitionistically valid.

Example 2.8: The sentence $(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$ is not intuitionistically valid. Let the frame $\mathcal{C}$ be



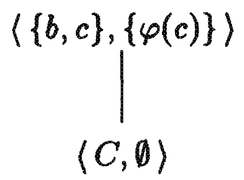
Suppose, for the sake of a contradiction, that $\emptyset \Vdash (\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$. Then $\emptyset \Vdash (\neg\varphi \rightarrow \neg\psi)$ would imply $\emptyset \Vdash (\psi \rightarrow \varphi)$ by Clause (ii) of the definition of forcing (Definition 2.2). Now by Clause (iii) of the definition, neither $\emptyset$ nor 0 forces $\neg\varphi$ since φ is in $A(0)$ and so forced at 0. Thus we see that $\emptyset$ does in fact force $(\neg\varphi \rightarrow \neg\psi)$ by applying Clause (ii) again and the fact that $\emptyset$ and 0 are the only elements $\geq \emptyset$. On the other hand, $\emptyset$ does not force $(\psi \rightarrow \varphi)$ because $\emptyset$ forces ψ but not φ and so we have our desired contradiction.

Example 2.9: The sentence $(\varphi \to \psi) \vee (\psi \to \varphi)$ is not intuitionistically valid. Let the frame $\mathcal{C}$ be



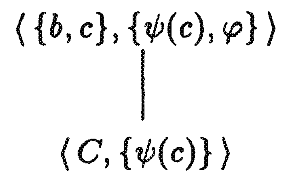
In this frame, $\emptyset$ forces neither φ nor ψ, 0 forces φ but not ψ and 1 forces ψ but not φ. Since there is a node above $\emptyset$, namely 0, which forces φ but not ψ, $\emptyset$ does not force $\varphi \to \psi$. Similarly, $\emptyset$ does not force $\psi \to \varphi$. So $\emptyset$ does not force $(\varphi \to \psi) \vee (\psi \to \varphi)$.

Example 2.10: The sentence $\neg(\forall x)\varphi(x) \to (\exists x)\neg\varphi(x)$ is not intuitionistically valid. Let b be anything other than the sole element c of C. Let the frame $\mathcal{C}$ be



Now by Clause (iv) of Definition 2.2, neither $\emptyset$ nor 0 forces $(\forall x)\varphi(x)$ since $b \in C(0)$ but 0 does not force $\varphi(b)$. Thus $\emptyset \Vdash \neg(\forall x)\varphi(x)$. If $\emptyset \Vdash \neg(\forall x)\varphi(x) \to (\exists x)\neg\varphi(x)$, as it would were our given sentence valid, then $\emptyset$ would also force $(\exists x)\neg\varphi(x)$. By Clause (v) of the definition this can happen only if there is a $c \in C$ such that $\emptyset \Vdash \neg\varphi(c)$. As c is the only element of C and $0 \Vdash \varphi(c)$, $\emptyset$ does not force $(\exists x)\neg\varphi(x)$.

Example 2.11: The sentence $(\forall x)(\varphi \vee \psi(x)) \to \varphi \vee (\forall x)\psi(x)$ is not intuitionistically valid. The required frame is



We first claim that $\emptyset \Vdash (\forall x)(\varphi \vee \psi(x))$. As $\emptyset \Vdash \psi(c)$ and $0 \Vdash \varphi$, combining the clauses for disjunction (vii) and universal quantification (iv) we see that $\emptyset \Vdash (\forall x)(\varphi \vee \psi(x))$ as claimed. Suppose now for the sake of a contradiction that $\emptyset \Vdash (\forall x)(\varphi \vee \psi(x)) \rightarrow \varphi \vee (\forall x)\psi(x)$. We would then have that $\emptyset \Vdash \varphi \vee (\forall x)\psi(x)$. However, $\emptyset$ does not force φ and, as 0 does not force $\psi(b)$, $\emptyset$ does not force $(\forall x)\psi(x)$ either. Thus $\emptyset$ does not force the disjunction $\varphi \vee (\forall x)\psi(x)$, so we have the desired contradiction.

We now give some examples of intuitionistically valid sentences whose validity can be verified directly using the definition of forcing. Before presenting the examples, we prove a few basic facts about the forcing relation that are useful for these verifications as well as future arguments. The first is perhaps the single most useful fact about forcing. It expresses the stability of forcing as one moves up in the partial ordering.

Lemma 2.12: (Monotonicity lemma) *For every sentence φ of $\mathcal{L}$ and every $p, q \in R$, if $p \Vdash \varphi$ and $q \geq p$, then $q \Vdash \varphi$.*

Proof: We prove the lemma by induction on the logical complexity of φ. The inductive hypothesis is not needed to verify the conclusion that $q \Vdash \varphi$ for Clauses (i), (ii), (iii) and (iv). The first follows immediately from the definition of a frame and Clause (i) itself which defines forcing for atomic sentences. The other clauses define the meaning of (intuitionistic) implication, negation and universal quantification precisely so as to make this lemma work. We use the induction hypothesis in the verifications of Clauses (v), (vi) and (vii) which define forcing for the existential quantifier, conjunction and disjunction, respectively.

(i) If φ is atomic and $p \Vdash \varphi$, then φ is in $A(p)$. By the definition of a frame, however, $A(p) \subseteq A(q)$, and so φ is in $A(q)$. Thus, by definition, $q \Vdash \varphi$.

(ii) Suppose $p \Vdash \varphi \rightarrow \psi$ and $q \geq p$. We show that $q \Vdash \varphi \rightarrow \psi$ by showing that if $r \geq q$ and $r \Vdash \varphi$, then $r \Vdash \psi$. Now $r \geq p$ by transitivity and so our assumptions that $p \Vdash \varphi \rightarrow \psi$ and $r \Vdash \varphi$ imply that $r \Vdash \psi$, as required.

(iii) Suppose $p \Vdash \neg\varphi$ and $q \geq p$. We show that $q \Vdash \neg\varphi$ by showing that if $r \geq q$, then r does not force φ. Again by transitivity, $r \geq p$. The definition of $p \Vdash \neg\varphi$ then implies that r does not force φ.

(iv) Suppose $p \Vdash (\forall x)\varphi(x)$ and $q \geq p$. We show that $q \Vdash (\forall x)\varphi(x)$ by showing that, for any $r \geq q$ and any $c \in C(r)$, $r \Vdash \varphi(c)$. Again, $r \geq p$ by transitivity. The definition of $p \Vdash (\forall x)\varphi(x)$ then implies that for any c in $C(r)$, $r \Vdash \varphi(c)$.

(v) Suppose $p \Vdash (\exists x)A(x)$ and $q \geq p$. Then by the definition of forcing there is a c in $C(p)$ such that $p \Vdash \varphi(c)$. By the inductive hypothesis, $q \geq p$ and $p \Vdash \varphi(c)$ imply that $q \Vdash \varphi(c)$. Thus $q \Vdash (\exists x)\varphi(x)$.

(vi) Suppose $p \Vdash (\varphi \wedge \psi)$ and $q \geq p$. Then by the definition of forcing $p \Vdash \varphi$ and $p \Vdash \psi$. By the inductive hypothesis, $q \Vdash \varphi$ and $q \Vdash \psi$. Thus $q \Vdash (\varphi \wedge \psi)$.

(vii) Suppose $p \Vdash (\varphi \vee \psi)$, and $q \geq p$. Then by the definition of forcing either $p \Vdash \varphi$ or $p \Vdash \psi$. By the inductive hypothesis, we get that either $q \Vdash \varphi$ or $q \Vdash \psi$. By the definition of forcing a disjunction, this says that $q \Vdash (\varphi \vee \psi)$.

□

Monotonicity says that the addition of new atomic sentences at later states of knowledge q will not change forcing at earlier states of knowledge. This monotone character distinguishes "truth" in an intuitionistic frame from "truth" in "nonmonotonic logics", as discussed in III.7. In those logics, sentences forced at state of knowledge p need not be forced at states of knowledge $q > p$. In frames, as time evolves, we learn new "facts" but never discover that old ones are false.

Lemma 2.13: (Double negation lemma) *$p \Vdash \neg\neg\varphi$ if and only if for any $q \geq p$ there is an $r \geq q$ such that $r \Vdash \varphi$.*

Proof: $p \Vdash \neg\neg\varphi$ if and only if every $q \geq p$ fails to force $\neg\varphi$, or equivalently, if and only if every $q \geq p$ has an $r \geq q$ forcing φ. □

Lemma 2.14: (Weak quantifier lemma)

(i) $p \Vdash \neg(\exists x)\neg\varphi(x)$ if and only if for all $q \geq p$ and for all $c \in C(q)$, there is an $r \geq q$ such that $r \Vdash \varphi(c)$.

(ii) $p \Vdash \neg(\forall x)\neg\varphi(x)$ if and only if for all $q \geq p$, there exists an $s \geq q$ and a $c \in C(s)$ such that $s \Vdash \varphi(c)$.

Proof:

(i) This claim follows immediately from the definition.

(ii) $q \Vdash (\forall x)\neg\varphi(x)$ if and only if for all $r \geq q$ and all $c \in C(r)$ there is no $s \geq r$ such that $s \Vdash \varphi(c)$. Thus q does not force $(\forall x)\neg\varphi(x)$ if and only if there is an $r \geq q$ and a $c \in C(r)$ such that for some $s \geq r$, $s \Vdash \varphi(c)$. So $p \Vdash \neg(\forall x)\neg\varphi(x)$ if and only if for all $q \geq p$, there is an $r \geq q$ and a $c \in C(r)$ such that for some $s \geq r$, $s \Vdash \varphi(c)$. By transitivity $s \geq q$ and c is in $C(s)$ as required in the claim.

□

We now produce the promised examples of intuitionistic validity. In the following examples (2.15–2.19) φ are ψ are arbitrary sentences.

Example 2.15: $\varphi \to \neg\neg\varphi$ is intuitionistically valid. To see that any p forces $\varphi \to \neg\neg\varphi$ we assume that $q \geq p$ and $q \Vdash \varphi$. We must show that $q \Vdash \neg\neg\varphi$. By the double negation lemma, it suffices to show that for every $r \geq q$ there is an $s \geq r$ such that $s \Vdash \varphi$. By the monotonicity lemma $r \Vdash \varphi$, and so r is the required s.

Example 2.16: $\neg(\varphi \wedge \neg\varphi)$ is intuitionistically valid. To show that any p forces $\neg(\varphi \wedge \neg\varphi)$ we need to show that no $q \geq p$ forces $\varphi \wedge \neg\varphi$, or equivalently no $q \geq p$ forces both φ and $\neg\varphi$. Suppose then that q forces both φ and $\neg\varphi$. Now $q \Vdash \neg\varphi$ means no $r \geq q$ forces φ. Since $q \geq q$, we have both q forces φ and q does not force φ for the desired contradiction.

Example 2.17: $(\exists x)\neg\varphi(x) \to \neg(\forall x)\varphi(x)$ is intuitionistically valid. To see that any p forces $(\exists x)\neg\varphi(x) \to \neg(\forall x)\varphi(x)$, we need to show that if $q \geq p$ and $q \Vdash (\exists x)\neg\varphi(x)$, then $q \Vdash \neg(\forall x)\varphi(x)$. Now $q \Vdash (\exists x)\neg\varphi(x)$ says there is a c in $C(q)$ such that $q \Vdash \neg\varphi(c)$. By monotonicity, any $r \geq q$ forces $\neg\varphi(c)$ as well, so no such r forces $(\forall x)\varphi(x)$, thus $q \Vdash \neg(\forall x)\varphi(x)$. This example should be compared with its contrapositive (Example 2.10) which is classically but not intuitionistically valid.

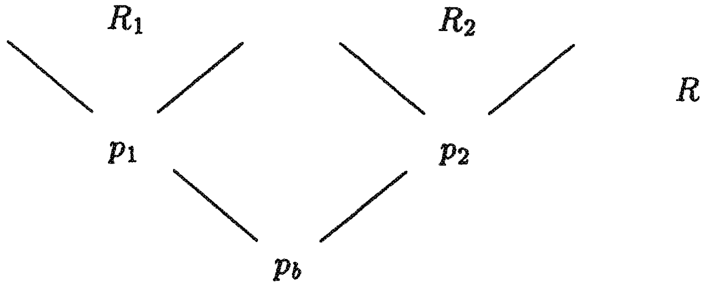
Example 2.18: $\neg(\exists x)\varphi(x) \to (\forall x)\neg\varphi(x)$ is intuitionistically valid. To see that any p forces $\neg(\exists x)\varphi(x) \to (\forall x)\neg\varphi(x)$ we have to show that for any $q \geq p$, if $q \Vdash \neg(\exists x)\varphi(x)$, then $q \Vdash (\forall x)\neg\varphi(x)$. Now $q \Vdash \neg(\exists x)\varphi(x)$ says that, for every $r \geq q$ and every c in $C(r)$, r does not force $\varphi(c)$. By transitivity $s \geq r$ implies $s \geq q$. So for every $r \geq q$ and every c in $C(r)$, no $s \geq r$ forces $\varphi(c)$. This says $q \Vdash (\forall x)\neg\varphi(x)$.

Example 2.19: If x is not free in φ, then $\varphi \vee (\forall x)\psi(x) \to (\forall x)(\varphi \vee \psi(x))$ is intuitionistically valid. To see that any p forces $\varphi \vee (\forall x)\psi(x) \to (\forall x)(\varphi \vee \psi(x))$ we must show that, for any $q \geq p$, if $q \Vdash \varphi$ or $q \Vdash (\forall x)\psi(x)$, then $q \Vdash (\forall x)(\varphi \vee \psi(x))$. There are two cases. If $q \Vdash \varphi$, then for any $r \geq q$ and any c in $C(r)$, $r \Vdash \varphi \vee \psi(c)$, so $q \Vdash (\forall x)(\varphi \vee \psi(x))$. If $q \Vdash (\forall x)\psi(x)$, then for all $r \geq q$ and all c in $C(r)$, $r \Vdash \psi(c)$, so $r \Vdash \varphi \vee \psi(c)$. This says that $q \Vdash (\forall x)(\varphi \vee \psi(x))$. This example should be compared with Example 2.11.

The frame definition of intuitionistic validity makes it remarkably simple to prove two important properties of intuitionistic logic that embody its constructivity: the disjunction and existence properties. The first says that, if a disjunction is valid, then one of its disjuncts is valid. The second says that, if an existential sentence of $\mathcal{L}$ is valid, then one of its instances via a constant from $\mathcal{L}$ is also valid. When we combine this with the completeness theorem for intuitionistic logic (Theorem 4.10), we see that this means that if we can prove an existential sentence, we can in fact prove some particular instance. Similarly, if we can prove a disjunction, then we can prove one of the disjuncts.

Theorem 2.20: (Disjunction property) *If* $(\varphi_1 \vee \varphi_2)$ *is intuitionistically valid, then one of* φ_1, φ_2 *is intuitionistically valid.*

Proof: We prove the theorem by establishing its contrapositive. So suppose neither φ_1 nor φ_2 is intuitionistically valid. Thus there are, for $i = 1, 2$, frames $\mathcal{C}_i$ and elements p_i of the associated partial orderings R_i such that φ_1 is not forced by p_1 in $\mathcal{C}_1$ and φ_2 is not forced by p_2 in $\mathcal{C}_2$. By the restriction lemma (2.4), we may assume that p_i is the least element of R_i. By Exercise 11 we may assume that no two distinct constants of the language $\mathcal{L}$ are interpreted as the same element in any one of the structures in either frame $\mathcal{C}_i(p)$. Now simply by relabeling the elements of $\mathcal{C}_i(p)$ (and so of all the other structures in the frames $\mathcal{C}_i$) and R_i we may assume that the interpretation of each constant c of $\mathcal{L}$ is the same in the two structures $\mathcal{C}_i(p_i)$ and that the R_i are disjoint. Let R be the union of R_1, R_2, and $\{p_b\}$, with p_b not in either R_i. Make R into a partial order by ordering R_1 and R_2 as before and putting p_b below p_1 and p_2.



We define a frame $\mathcal{C}$ with this ordering on R by setting $\mathcal{C}(p)$ equal to $\mathcal{C}_i(p)$ for $p \in R_i$ and $\mathcal{C}(p_b)$ equal to the structure defined on the set of the interpretations of the constants of $\mathcal{L}$ in $\mathcal{C}_i(p_i)$ (remember these interpretations are the same for $i = 1, 2$) by setting $A(p_b) = \emptyset$. (Our standing assumption that $\mathcal{L}$ has at least one constant guarantees that this structure is nonempty.) In this frame $\mathcal{C}$, p_1 does not force φ_1 by the restriction lemma (2.4). Thus, p_b does not force φ_1 by the monotonicity lemma (2.12). Similarly, p_b does not force φ_2 as p_2 does not. Thus, p_b does not force $\varphi_1 \vee \varphi_2$; hence $\varphi_1 \vee \varphi_2$ is not intuitionistically valid: contradiction. □

Theorem 2.21: (Existence property) *If $(\exists x)\varphi(x)$ is an intuitionistically valid sentence of a language $\mathcal{L}$, then for some constant c in $\mathcal{L}$, $\varphi(c)$ is also intuitionistically valid. (Remember that, by convention, $\mathcal{L}$ has at least one constant.)*

Proof: Suppose that, for each constant a in $\mathcal{L}$, $\varphi(a)$ is not intuitionistically valid. Then, for each such constant, there is an $\mathcal{L}$–frame $\mathcal{C}_a$ with a partially ordered set R_a containing an element p_a that does not force $\varphi(a)$. As in the previous proof, we may, without loss of generality, assume that p_a is the least element of R_a and all the R_a's are pairwise disjoint. We also assume that the interpretation of some fixed constant c of $\mathcal{L}$ is the same element d in every $\mathcal{C}(p_a)$. We now form a new partial ordering R by taking the union of all R_a and the union of the partial orders and adding on a new bottom element p_b under all the p_a. We next define an $\mathcal{L}$–frame associated with R, as in the previous proof, by letting $\mathcal{C}(p_b) = \{d\}$, $A(p_b) = \emptyset$ and $\mathcal{C}(p) = \mathcal{C}_a(p)$ for every $p \in R_a$ and every constant

a of $\mathcal{L}$. We can now imitate the argument in Theorem 2.20. As we are assuming that $\exists x\varphi(x)$ is intuitionistically valid, we must have $p_b \Vdash_{\mathcal{C}} \exists x\varphi(x)$. Then, by definition, $p_b \Vdash \varphi(a)$ for some constant a in $\mathcal{L}$. Applying first the monotonicity lemma and then the restriction lemma we would have p_a forcing $\varphi(a)$ first in $\mathcal{C}$ and then in $\mathcal{C}_a$; this contradicts our initial hypothesis that p_a and $\mathcal{C}_a$ show that $\varphi(a)$ is not intuitionistically valid. □

Exercises

Sentences (1)–(6) below are classically valid. Verify that they are intuitionistically valid by direct arguments with frames. Remember that $\varphi \leftrightarrow \psi$ is an abbreviation for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

1. $\neg\varphi \leftrightarrow \neg\neg\neg\varphi$
2. $(\varphi \wedge \neg\psi) \rightarrow \neg(\varphi \rightarrow \psi)$
3. $(\varphi \rightarrow \psi) \rightarrow (\neg\neg\varphi \rightarrow \neg\neg\psi)$
4. $(\neg\neg(\varphi \rightarrow \psi) \leftrightarrow (\neg\neg\varphi \rightarrow \neg\neg\psi))$
5. $\neg\neg(\varphi \wedge \psi) \leftrightarrow (\neg\neg\varphi \wedge \neg\neg\psi)$
6. $\neg\neg(\forall x)\varphi(x) \rightarrow (\forall x)\neg\neg\varphi(x)$
7. Supply the proof for Lemma 2.4.
8. Supply the proofs for the remaining cases of Lemma 2.5.
9. Let K be the set of constants occurring in $(\exists x)\varphi(x)$ and suppose that $(\exists x)\varphi(x)$ is intuitionistically valid. Show that, if K is nonempty, then for some c in K, $\varphi(c)$ is intuitionistically valid. (Hint: Define the restriction of a given frame $\mathcal{C}$ for a language $\mathcal{L}$ to one $\mathcal{C}'$ for a given restriction $\mathcal{L}'$ of $\mathcal{L}$. Now prove that, for any sentence φ of $\mathcal{L}'$ and any element p of the appropriate partial ordering, $p \Vdash_{\mathcal{C}} \varphi$ if and only if $p \Vdash_{\mathcal{C}'} \varphi$.)
10. In case K is empty in the previous exercise, show that $\varphi(c)$ is intuitionistically valid for any constant c. (Hint: For any constants a and c of $\mathcal{L}$ define a map Θ on formulas of $\mathcal{L}$ and on the frames for $\mathcal{L}$ that interchanges a and c. Prove that, for every $\mathcal{C}$ and p, $p \Vdash_{\mathcal{C}} \varphi$ if and only if $p \Vdash_{\Theta(\mathcal{C})} \Theta(\varphi)$.)
11. Suppose that $\mathcal{A}$ is a structure for a language $\mathcal{L}$ containing a constant symbol c. Let $\mathcal{A}'$ be the expansion of $\mathcal{A}$ defined by setting $A' = A \cup \{b_i | i \in \mathcal{N}\}$ for distinct elements b_i not in A, expanding the language to $\mathcal{L}'$ by adding on new constants d_i naming the b_i and declaring that, for $\vec{d}$ any sequence of d_i and ψ any atomic formula, $\mathcal{A}' \vDash \psi(\vec{d})$ if and only if $\mathcal{A} \vDash \psi(\vec{c})$ where

every element of the sequence $\vec{c}$ is just c. Prove that, for any sequence $\vec{d}$ of d_i and $\gamma(\vec{x})$ any formula of $\mathcal{L}$ with free variables $\vec{x}$, $\mathcal{A}' \models \gamma(\vec{d})$ if and only if $\mathcal{A} \models \gamma(\vec{c})$ where again each element of $\vec{c}$ is c.

3 Intuitionistic Tableaux

We describe a proof procedure for intuitionistic logic based on a tableau–style system like that used for classical logic in II.6. In classical logic, the idea of a tableau proof is to systematically search for a structure agreeing with the starting signed sentence. We either get such a structure or see that each possible analysis leads to a contradiction. When we begin with a signed sentence $F\varphi$, we thus either find a structure in which φ fails or decide that we have a proof of φ. For intuitionistic logic we instead begin with a *signed forcing assertion* $Tp \Vdash \varphi$ or $Fp \Vdash \varphi$ (φ is again a sentence) and try to either build a frame agreeing with the assertion or decide that any such attempt leads to a contradiction. If we begin with $Fp \Vdash \varphi$, we either find a frame in which p does not force φ or decide that we have an intuitionistic proof of φ.

There are many possible variants on the tableau method suitable for intuitionistic propositional and predicate logic due to Kripke, Hughes and Cresswell, Fitting, and others. The one we choose is designed to precisely match our definition of frame so that the systematic tableau represents a systematic search for a frame agreeing with the starting signed forcing assertion. It is a variant of Fitting's [1983, 4.1] prefixed tableau.

The definitions of tableau and tableau proof for intuitionistic logic are formally very much like those of II.6 for classical logic. *Intuitionistic tableaux* and *tableau proofs* are labeled binary trees. The labels (again called the *entries of the tableau*) are now *signed forcing assertions*, i.e., labels of the form $Tp \Vdash \varphi$ or $Fp \Vdash \varphi$ for φ a sentence of any appropriate language. We read $Tp \Vdash \varphi$ as p forces φ and $Fp \Vdash \varphi$ as p does not force φ.

In classical logic, the elements of the structure we built by developing a tableau were the constant symbols appearing on some path of the tableau. We are now attempting to build an entire frame. The p's and q's appearing in the entries of some path P through our intuitionistic tableau constitute the elements of the partial ordering for the frame. The ordering on them is also specified as part of the development of the tableau. As in the classical case, we always build a tableau based on a language expanded from the one for the starting signed assertion by adding on new constants $c_0, c_1, \ldots$. The constants appearing in the sentences φ of entries on P of the form $Tq \Vdash \varphi$ or $Fq \Vdash \varphi$ for $q \leq p$ are the elements of the required domains $C(p)$. (We use the entries with $q \leq p$ so as to ensure the monotonicity required for domains in the definition of a frame.)

With this motivation in mind, we can specify the *atomic intuitionistic tableaux.*

Definition 3.1: We begin by fixing a language $\mathcal{L}$ and an expansion $\mathcal{L}_C$ given by adding new constant symbols c_i for $i \in \mathcal{N}$. We list in Figure 55 the atomic intuitionistic tableaux (for the language $\mathcal{L}$). In the tableaux in this list, φ and ψ, if unquantified, are any sentences in the language $\mathcal{L}_C$. If quantified, they are formulas in which only x is free.

Formally, the precise meaning of "new c" and "new p" is defined along with the definition of intuitionistic tableau. The intention for the constants is essentially as in the classical case: When we develop $T\forall x\varphi(x)$, we can put in any c for x and add on $T\varphi(c)$ but, when we develop $\exists x\varphi(x)$ by adding $T\varphi(c)$ on to the tableau, we can only use a c for which no previous commitments have been made. One warning is necessary here. When we say "any appropriate c" we mean any c in the appropriate language. In the classical case that meant any c in $\mathcal{L}_C$. Here, in developing $Tp \Vdash \forall x\varphi(x)$ as in (T$\forall$) above, it means any c in $\mathcal{L}$ or appearing on the path so far in a forcing assertion about a $q \leq p$. These restrictions correspond to our intention to define $C(p)$ in accordance with the requirement in the definition of frame that $C(q) \subseteq C(p)$ for $q \leq p$. Technically, similar considerations could be applied to the use of a new c as in (T$\exists$) although as a practical matter we can always choose c from among the c_i in $\mathcal{L}_C$ which have not yet appeared anywhere in the tableau. We do in fact incorporate such a choice into our formal definition.

The restrictions on the elements p introduced into the ordering should also be understood in terms of the definition of frames. In (TAt), for example, we follow the requirement in the definition of a frame that $A(p) \subseteq A(p')$ if $p \leq p'$. The reader should also keep in mind that we are determining the elements p of the partial ordering for our frame as well as defining the ordering itself as we develop the tableau. Thus, for example, when developing $Tp \Vdash \neg\varphi$ we can, in accordance with the definition of forcing a negation, add on $Fp' \Vdash \varphi$ for any $p' \geq p$ that appears on the path so far. On the other hand, if we wish to assert that p does not force $\neg\varphi$, i.e., $Fp \Vdash \neg\varphi$, then the definition of forcing tells us that there must be some $p' \geq p$ that does force φ. As with putting in a new constant, we cannot use a p' for which other commitments have already been made. Thus we can develop $Fp \Vdash \neg\varphi$ as in (F$\neg$) by adding on $Tp' \Vdash \varphi$ for a new element p' of the ordering of which we can only say that it is bigger than p. Thus, we require that p' is larger than p (and so by the requirement of transitivity bigger than any $q \leq p$) but that p' is incomparable with all other elements of the ordering introduced so far. (Again, technically, we only need to worry about the relation between p' and the q appearing on the branch so far. It is simpler to just take an entirely new p', i.e., one not yet appearing anywhere in the tableau. It is then automatically true that $p \leq q$ only if p and q are on the same path through the tableau.)

The formal definitions of tableaux and tableau proof for intuitionistic logic could perhaps even be left as an exercise. As it would be an exercise with many pitfalls for the unwary we give them in full detail.

TAt $Tp \Vdash \varphi$ | $Tp' \Vdash \varphi$ for any $p' \geq p$, φ atomic		FAt $Fp \Vdash \varphi$ φ atomic	
T$\vee$ $Tp \Vdash \varphi \vee \psi$ / \\ $Tp \Vdash \varphi$ $Tp \Vdash \psi$	F$\vee$ $Fp \Vdash \varphi \vee \psi$ | $Fp \Vdash \varphi$ | $Fp \Vdash \psi$	T$\wedge$ $Tp \Vdash \varphi \wedge \psi$ | $Tp \Vdash \varphi$ | $Tp \Vdash \psi$	F$\wedge$ $Fp \Vdash \varphi \wedge \psi$ / \\ $Fp \Vdash \varphi$ $Fp \Vdash \psi$
T$\rightarrow$ $Tp \Vdash \varphi \rightarrow \psi$ / \\ $Fp' \Vdash \varphi$ $Tp' \Vdash \psi$ for any $p' \geq p$	F$\rightarrow$ $Fp \Vdash \varphi \rightarrow \psi$ | $Tp' \Vdash \varphi$ | $Fp' \Vdash \psi$ for some new $p' \geq p$	T$\neg$ $Tp \Vdash \neg\varphi$ | $Fp' \Vdash \varphi$ for any $p' \geq p$	F$\neg$ $Fp \Vdash \neg\varphi$ | $Tp' \Vdash \varphi$ for some new $p' \geq p$
T$\exists$ $Tp \Vdash (\exists x)\varphi(x)$ | $Tp \Vdash \varphi(c)$ for some new c	F$\exists$ $Fp \Vdash (\exists x)\varphi(x)$ | $Fp \Vdash \varphi(c)$ for any appropriate c	T$\forall$ $Tp \Vdash (\forall x)\varphi(x)$ | $Tp' \Vdash \varphi(c)$ for any $p' \geq p$, any appropriate c	F$\forall$ $Fp \Vdash (\forall x)\varphi(x)$ | $Fp' \Vdash \varphi(c)$ for some new $p' \geq p$, and new c

FIGURE 55.

Definition 3.2: We continue to use our fixed language $\mathcal{L}$ and extension by constants $\mathcal{L}_C$. We also fix a set $S = \{p_i | \ i \in \mathcal{N}\}$ of potential candidates for the p's and q's in our forcing assertions. An *intuitionistic tableau* (for $\mathcal{L}$) is a binary tree labeled with signed forcing assertions which are called the *entries* of the tableau. The class of all intuitionistic tableaux (for $\mathcal{L}$) is defined by induction. We simultaneously define, for each tableau τ, an ordering $\leq_\tau$ on the elements of S appearing in τ.

(i) Each atomic tableau τ is a tableau. The requirement that c be new in cases (T$\exists$) and (F$\forall$) here simply means that c is one of the constants c_i added on to $\mathcal{L}$ to get $\mathcal{L}_C$ which does not appear in φ. The phrase "any c" in (F$\exists$) and (T$\forall$) means any constant in $\mathcal{L}$ or in φ. The requirement that p' be new in (F$\rightarrow$), (F$\neg$) and (F$\forall$) here means that p' is any of the p_i other than p. We also declare p' to be larger than p in the associated ordering. The phrase "any $p' \geq p$" in (T$\rightarrow$), (T$\neg$), (T$\forall$) and (TAt) in this case simply means that p' is p. (Of course we always declare $p \leq p$ for every p in every ordering we define.)

(ii) If τ is a finite tableau, P a path on τ, E an entry of τ occurring on P and τ' is obtained from τ by adjoining an atomic tableau with root entry E to τ at the end of the path P, then τ' is also a tableau. The ordering $\leq_{\tau'}$ agrees with $\leq_\tau$ on the p_i appearing in τ. Its behavior on any new element is defined below when we explain the meaning of the restrictions on p' in the atomic tableaux for cases (F$\rightarrow$), (F$\neg$) and (F$\forall$).

The requirement that c be new in cases (T$\exists$) and (F$\forall$) here means that it is one of the c_i (and so not in $\mathcal{L}$) that do not appear in any entry on τ. The phrase "any c" in (F$\exists$) and (T$\forall$) here means any c in $\mathcal{L}$ or appearing in an entry on P of the form $Tq \Vdash \psi$ or $Fq \Vdash \psi$ with $q \leq_\tau p$.

In (F$\rightarrow$), (F$\neg$) and (F$\forall$) the requirement that $p' \geq p$ be new means that we choose a p_i not appearing in τ as p' and we declare that it is larger than p in $\leq_{\tau'}$. (Of course, we ensure transitivity by declaring that $q \leq_\tau p'$ for every $q \leq_\tau p$.) The phrase "any $p' \geq p$" in (T$\rightarrow$), (T$\neg$), (T$\forall$) and (TAt) means we can choose any p' that appears in an entry on P and has already been declared greater than or equal to p in $\leq_\tau$.

(iii) If $\tau_0, \tau_1, \ldots, \tau_n, \ldots$ is a sequence of finite tableaux such that, for every $n \geq 0$, τ_{n+1} is constructed from τ_n by an application of (ii), then $\tau = \cup \tau_n$ is also a tableau.

As in predicate logic, we insist that the entry E in Clause (ii) formally be repeated when the corresponding atomic tableau is added on to P. This is again crucial to the properties corresponding to a classical tableau being finished. In our examples below, however, we typically omit them purely as a notational convenience.

Note that if we do not declare that either $p \leq p'$ or $p' \leq p$ in our definition of $\leq_\tau$, then p and p' are incomparable in $\leq_\tau$.

We make good on our previous remark about the relation of the ordering $\leq_\tau$ to paths through the tableau τ with the following lemma.

Lemma 3.3: *For any intuitionistic tableau τ with associated ordering $\leq_\tau$, if $p' \leq_\tau p$, then p and p' both appear on some common path through τ.*

Proof: The proof proceeds by an induction on the definition of τ and $\leq_\tau$. We leave it as Exercise 31. □

Definition 3.4: (*Intuitionistic tableau proofs*) Let τ be a intuitionistic tableau and P a path in τ.

(i) P is *contradictory* if, for some forcing assertion $p \Vdash \varphi$, both $Tp \Vdash \varphi$ and $Fp \Vdash \varphi$ appear as entries on P.

(ii) τ is *contradictory* if every path through τ is contradictory.

(iii) τ is an *intuitionistic proof of φ* if τ is a finite contradictory intuitionistic tableau with its root node labeled $Fp \Vdash \varphi$ for some $p \in S$. φ is *intuitionistically provable*, $\vdash \varphi$, if there is an intuitionistic proof of φ.

Note that, as in classical logic, if there is any contradictory tableau with root node $Fp \Vdash \varphi$, then there is one which is finite, i.e., a proof of φ : Just terminate each path when it becomes contradictory. As each path is now finite, the whole tree is finite by König's lemma. Thus, the added requirement that proofs be finite (tableaux) has no affect on the existence of proofs for any sentence. Another point of view is that we could have required the path P in Clause (ii) of the definition of tableaux be noncontradictory without affecting the existence of proofs. Thus, in practice, when attempting to construct proofs we mark any contradictory path with the symbol $\otimes$ and terminate the development of the tableau along that path.

Before dealing with the soundness and completeness of the tableau method for intuitionistic logic we look at some examples of intuitionistic tableau proofs. Remember that we are abbreviating the tableaux by not repeating the entry that we are developing. We also number the levels of the tableau on the left and indicate on the right the level of the atomic tableau whose development produced the line. In all our examples, the set S from which we choose the domain of our partial order is the set of finite binary sequences. The declarations of ordering relations are dictated by the atomic tableau added on at each step and so can also be omitted. In fact, we always choose our p's and q's so as to define our orderings to agree with the usual ordering of inclusion on binary sequences.

1	$F\emptyset \Vdash \varphi \to (\psi \to \varphi)$	
2	$T0 \Vdash \varphi$	by 1
3	$F0 \Vdash \psi \to \varphi$	by 1
4	$T00 \Vdash \psi$	by 3
5	$F00 \Vdash \varphi$	by 3
6	$T00 \Vdash \psi$	by 2
	$\otimes$	

FIGURE 56.

Example 3.5: Let φ and ψ be any atomic sentences of $\mathcal{L}$. Figure 56 provides an intuitionistic proof of $\varphi \to (\psi \to \varphi)$.

In this proof the first three lines are an instance of (F→) from the list of atomic tableaux. Lines 4 and 5 are introduced by developing line 3 in accordance with (F→) again. Line 6, which, together with line 5, provides our contradiction, follows from line 2 by atomic tableau (TA t).

Example 3.6: Any sentence of $\mathcal{L}$ of the following form is intuitionistically provable:

$$(\exists x)(\varphi(x) \vee \psi(x)) \to (\exists x)\varphi(x) \vee (\exists x)\psi(x).$$

In this proof (Figure 57) the first three lines are an instance of (F→). Line 4 follows by applying (T∃) to line 2. Lines 5 and 6 follow by applying (F∨) to line 3. Lines 7 and 8 are applications of (F∨) to lines 5 and 6, respectively. Line 9, which supplies the contradictions to lines 7 or 8 on its two branches, is an application of (T∨) to line 4.

Example 3.7: Consider $(\forall x)(\varphi(x) \wedge \psi(x)) \to (\forall x)\varphi(x) \wedge (\forall x)\psi(x)$.

Note here (Figure 58) that we develop both sides of the branching at line 4 and write the parallel developments side by side. Also of note is the use of (T∀) applied to line 2 to get line 6. We took advantage of the ability to choose both the constants c and d and the elements of the ordering 00 and 01.

1	$F\emptyset \Vdash (\exists x)(\varphi(x) \vee \psi(x)) \rightarrow (\exists x)\varphi(x) \vee (\exists x)\psi(x)$		
2	$T0 \Vdash (\exists x)(\varphi(x) \vee \psi(x))$		by 1
3	$F0 \Vdash (\exists x)\varphi(x) \vee (\exists x)\psi(x)$		by 1
4	$T0 \Vdash \varphi(c) \vee \psi(c)$		by 2
5	$F0 \Vdash (\exists x)\varphi(x)$		by 3
6	$F0 \Vdash (\exists x)\psi(x)$		by 3
7	$F0 \Vdash \varphi(c)$		by 5
8	$F0 \Vdash \psi(c)$		by 6
9	$T0 \Vdash \varphi(c)$	$T0 \Vdash \psi(c)$	by 4
	$\otimes$	$\otimes$	by 7, 8, 9

FIGURE 57.

1	$F\emptyset \Vdash (\forall x)(\varphi(x) \wedge \psi(x)) \rightarrow (\forall x)\varphi(x) \wedge (\forall x)\psi(x)$		
2	$T0 \Vdash (\forall x)(\varphi(x) \wedge \psi(x))$		by 1
3	$F0 \Vdash (\forall x)\varphi(x) \wedge (\forall x)\psi(x)$		by 1
4	$F0 \Vdash (\forall x)\varphi(x)$	$F0 \Vdash (\forall x)\psi(x)$	by 3
5	$F00 \Vdash \varphi(c)$	$F01 \Vdash \psi(d)$	by 4
6	$T00 \Vdash \varphi(c) \wedge \psi(c)$	$T01 \Vdash \varphi(d) \wedge \psi(d)$	by 2
7	$T00 \Vdash \varphi(c)$	$T01 \Vdash \varphi(d)$	by 6
8	$T00 \Vdash \psi(c)$	$T01 \Vdash \psi(d)$	by 6
	$\otimes$	$\otimes$	by 5, 7 and 5, 8

FIGURE 58.