Homework #13b

For Friday, April 29

1. Do problem 5 on page 117.

⋆ 2. Let $(0,1)$ denote the open interval of real numbers between 0 and 1:

$$(0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}.$$

Let $[0,1]$ denote the closed interval

$$[0,1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}.$$

Let $(0,\infty)$ denote the positive real numbers,

$$(0,\infty) = \{x \in \mathbb{R} \mid x > 0\}.$$

a. Show that $\langle (0,1), < \rangle$ is isomorphic to $\langle (0,1), > \rangle$, by exhibiting a bijective function from $(0,1)$ to $(0,1)$ and proving that it is an isomorphism of the two structures. Note that the underlying language has a single binary relation r that is interpreted as $<$ in the first structure and $>$ in the second.

b. Show that $\langle (0,1), < \rangle$ is isomorphic to $\langle (0,\infty), < \rangle$. (Hint: consider the function $f(x) = \frac{x}{1-x}$.)

c. Show that $\langle (0,1), < \rangle$ is *not* isomorphic to $\langle [0,1], < \rangle$. (Hint: use Lemma 3.3.3 in van Dalen, and find a sentence that is true in one structure but false in the other.)

⋆ 3. Let $\mathcal{P} = \langle P, < \rangle$ be a linear ordering. $\mathcal{P}$ is said to be a *well-ordering* if every nonempty subset of P has a least (minimum) element. Note that $\langle \mathbb{N}, < \rangle$ has this property, so you can think of elements of a well-ordering as "generalized numbers" (a.k.a. "ordinals").

a. Show that the structure $\mathcal{B}$ in exercise 14 on page 91 of van Dalen is a well-ordering. Note that in this structure, the natural numbers are ordered so that all the even numbers come first, followed by the odd numbers:

$$0, 2, 4, 6, \ldots, 1, 3, 5, 7, 9 \ldots$$

b. Do problem 6 on page 117. In other words, use the suggestion to show that there is no set of sentences Γ such that the models of Γ are exactly the well-orderings.

4. Do problems 7–10 on page 117.

$\star$ 5. Do problem 13 on page 117. (Note that saying that $Mod(T_1 \cup T_2) = \emptyset$ is equivalent to saying that $T_1 \cup T_2$ is inconsistent. Use the compactness theorem, or, equivalently, the fact that any natural deduction proof has only finitely many hypotheses.)

$\star$ 6. Show that if T_1 and T_2 are consistent theories, and $T_1 \neq T_2$, then $Mod(T_1) \neq Mod(T_2)$. In other words, if $T_1 \neq T_2$, then there is a structure that is a model of one but not the other. (Hint: show that if $T_1 \neq T_2$, there is a sentence φ in one but not the other. Without loss of generality, say φ is in T_1 but not T_2. Using the fact that T_2 is a theory, show $T_2 \cup \{\neg\varphi\}$ is consistent.)

$\circ$ 7. Do problem 1 on page 132. This is a nice application of compactness.

8. Do problem 2 on page 132.