Homework #6

For Friday, March 4

1. A *binary truth function* is a function $f(x, y)$ that takes values of x and y in the set $\{0, 1\}$ to a value in the set $\{0, 1\}$. Note that a binary truth function is defined uniquely by its truth table.

   a. How many different binary truth functions are there?

   b. Two binary truth functions don't depend on any of their arguments: the constant 0 function and the constant 1 function. How many binary truth functions depend only on one of their two arguments?

   c. We've already seen a number of binary truth functions that depend on both arguments, namely those corresponding to the connectives $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\oplus$ (exclusive or), $|$ (nand, or the sheffer stroke), and $\downarrow$ (nor). (The last three are defined by $p \oplus q \equiv \neg(p \leftrightarrow q)$, $p|q \equiv \neg(p \wedge q)$, $p \downarrow q \equiv \neg(p \vee q)$.)

      What are the remaining ones? You can define them in words, in terms of the other connectives, or with truth tables.

$\star$ 2. Do problems 4 and 5 on page 28 of van Dalen. In other words, if $\varphi \mid \psi$, read "φ nand ψ," means that φ and ψ are not both true, and $\varphi \downarrow \psi$, read "φ nor ψ," means that neither φ nor ψ is true, show that $\{|\}$ and $\{\downarrow\}$ are complete sets of connectives.

3. Do problem 6 on page 28. In other words, show that these are the only two binary connectives that have this property.

4. Show that $\{\rightarrow, \bot\}$ is a complete set of connectives.

$\star$ 5.

   a. Show that $\{\rightarrow, \vee, \wedge\}$ is not a complete set of connectives. (Hint: show that any formula involving only these connectives is true when all the variables are true.)

   b. Conclude that $\{\rightarrow, \vee, \wedge, \leftrightarrow, \top\}$ is not a complete set of connectives. (Hint: define the last two in terms of the others.)

6. a. Show that $\{\bot, \leftrightarrow\}$ is not a complete set of connectives. (Hint: show that any formula involving only these connectives and the variables p_0 and p_1 is equivalent to one of the following: $\bot$, $\top$, p_0, p_1, $\neg p_0$, $\neg p_1$, $p_0 \leftrightarrow p_1$, or $p_0 \oplus p_1$.)

b. Conclude that $\{\bot, \top, \neg, \leftrightarrow, \oplus\}$ is not complete. (Hint: see the previous problem.)

∘ 7. How many ternary (3-ary) complete connectives are there?

∘ 8. Do problem 7 on page 28.

9. Do problem 8 on page 28. (Hint: it might help to read problem 7.)

10. Make up a truth table for a ternary connective, and then find a formula that represents it.

11. Do problems 9 and 10 on page 28.

12. Using the property $\varphi \vee (\psi \wedge \theta) \approx (\varphi \vee \psi) \wedge (\varphi \vee \theta)$, and the dual statement with $\wedge$ and $\vee$ switched, put

$$(p_1 \wedge p_2) \vee (q_1 \wedge q_2) \vee (r_1 \wedge r_2)$$

in conjunctive normal form. (Hint: try it with $(p_1 \wedge p_2) \vee (q_1 \wedge q_2)$ first.)

⋆ 13. Do problem 1 on page 39 of van Dalen. Remember that we are taking $\varphi \leftrightarrow \psi$ to abbreviate $(\varphi \to \psi) \wedge (\psi \to \varphi)$.

⋆ 14. Do problem 2 on page 39.

There is a parenthesis missing in part (b); it should read $[\varphi \to (\psi \to \sigma)] \leftrightarrow [\psi \to (\varphi \to \sigma)]$. Here the square brackets are only used to make the formula more readable; they are no different from parentheses.